Independent Vector Analysis using Semi-Parametric Density Estimation via Multivariate Entropy Maximization

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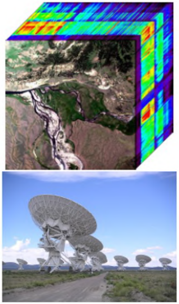
Joint blind source separation is an active area of research due to its numerous applications



The goal is to process the data

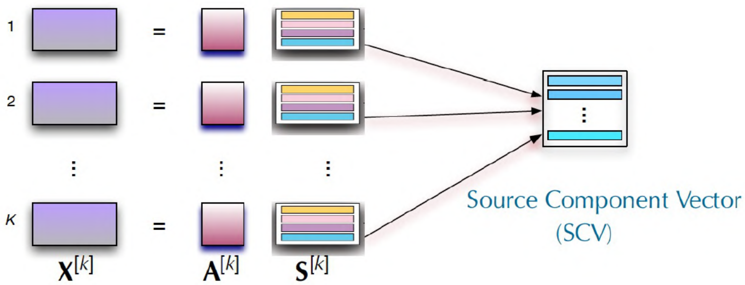
- to explore the underlying structure of multi-modal datasets, i.e., information extraction by integrating and modeling multiple modalities
- to extract meaningful information about the underlying sources collected multiple subjects

IVA is an effective solution for joint blind source separation

IVA is an extension of ICA to **multiple datasets**

ICA: $\mathbf{x} = \mathbf{A}\mathbf{s}$ $\mathbf{y} = \mathbf{W}\mathbf{x}$, $\mathbf{W} \in \mathbb{R}^{N \times N}$

IVA: $\mathbf{x}^{[k]} = \mathbf{A}^{[k]}\mathbf{s}^{[k]}$ $\mathbf{y}^{[k]} = \mathbf{W}^{[k]}\mathbf{x}^{[k]}$, $\mathbf{W}^{[k]} \in \mathbb{R}^{N \times N}$, $\mathrm{k} = 1,\ldots,\mathrm{K}$



IVA and mutual information (MI)

The goal in IVA is to estimate K **demixing matrices**, $\mathbf{W}^{[k]}$, to yield maximally independent source estimates

$$\mathbf{y}^{[k]} = \mathbf{W}^{[k]}\mathbf{x}^{[k]}$$

such that each SCV is maximally independent of all other SCVs.

MI cost function

$$J_{\text{IVA}}(\mathbf{W}^{[k]}) = \sum_{n=0}^{N} \underbrace{H(\mathbf{y}_n)}_{-E\{\log p(\mathbf{y}_n)\}} - \sum_{k=0}^{K} \log\left|\det\left(\mathbf{W}^{[k]}\right)\right| - C$$

where $\mathbf{y}_n$ is the nth estimated SCV and $H(\mathbf{y}_n)$ denotes its differential entropy.

- Instead of minimizing the MI cost function with respect to $\mathbf{W}^{[k]}$, we use a **decoupling procedure** to minimize with respect to each row of $\mathbf{W}^{[k]}$.

Decoupling MI cost function provides optimization benefits

Decoupling procedure

$$J_{\text{IVA}}(\mathbf{w}_n^{[k]}) = H[\mathbf{y}_n] - \log \left| \left(\mathbf{h}_n^{[k]}\right)^{\top} \mathbf{w}_n^{[k]} \right| - \mathbf{C}_n^{[k]}$$

for $n = 1, ..., N$, where $\mathbf{h}_n^{[k]}$ is the unit length vector is perpendicular to $\mathbf{w}_j^{[k]}, \forall j \neq n$.

Gradient

$$\frac{\partial J_{\text{IVA}}(\mathbf{w}_n^{[k]})}{\partial \mathbf{w}_n^{[k]}} = -E\left\{ \frac{\partial \log\left(p_n\left(\mathbf{y}_n\right)\right)}{\partial y_n^{[k]}} \mathbf{x}^{[k]} \right\} - \frac{\mathbf{h}_n^{[k]}}{(\mathbf{h}_n^{[k]})^{\top} \mathbf{w}_n^{[k]}}$$

where $p(\mathbf{y}_n)$ denotes its **probability density function (PDF).**

Estimation of PDF for each SCV $\mathbf{y}_n$ plays an important role in the development of IVA algorithms.

The maximum entropy principle

The best probability distribution to model a dataset is the one with largest entropy

$$\max_{p(\mathbf{y})} H(p(\mathbf{y})) = -\int_{\mathbb{R}^K} p(\mathbf{y}) \log p(\mathbf{y}) d\mathbf{y}$$

$$\text{s.t.} \quad \int_{\mathbb{R}^K} r_i(\mathbf{y}) p(\mathbf{y}) d\mathbf{y} = \alpha_i, \ \text{for } i = 0, ..., M$$

where $\mathbf{y} \in \mathbb{R}^K$ represents the dataset and $r_i(\mathbf{y})$ are the constraints.

Lagrangian form

$$\mathcal{L}(p(\mathbf{y})) = -\int_{\mathbb{R}^K} p(\mathbf{y}) \log p(\mathbf{y}) d\mathbf{y} + \sum_{i=0}^{M} \lambda_i \int_{\mathbb{R}^K} (r_i(\mathbf{y}) - \alpha_i) p(\mathbf{y}) d\mathbf{y}$$

where λ_i are the Lagrangian multipliers.

Density function

$$p(\mathbf{y}) = e^{-1+\sum_{i=0}^{M} \lambda_i r_i(\mathbf{y})}$$

λ_i are estimated by Newton iterations.

Global and local constraints provide flexible model

Global constraints

$r_0(\mathbf{y}) = 1$ $\qquad r_2(\mathbf{y}) = y^2$

$r_1(\mathbf{y}) = y$ $\qquad r_3(\mathbf{y}) = \frac{y}{(1+y^2)}$

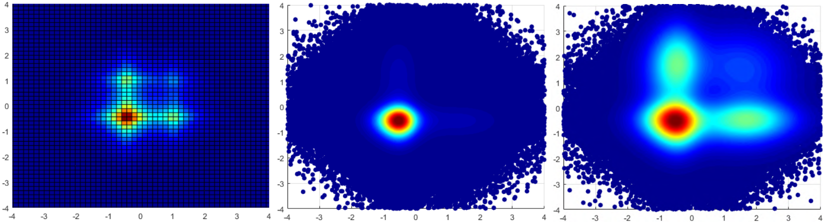
provide information about the PDF's overall statistics, such as the mean, variance, and certain higher order statistics.

Local constraint

Gaussian Kernel:

$$q(\mathbf{y}) = \frac{1}{\sqrt{|\boldsymbol{\Sigma}|(2\pi)^K}} \exp\left(-\frac{1}{2}(\mathbf{y}-\mu)\boldsymbol{\Sigma}^{-1}(\mathbf{y}-\mu)'\right)$$

provides localized information about the PDF.



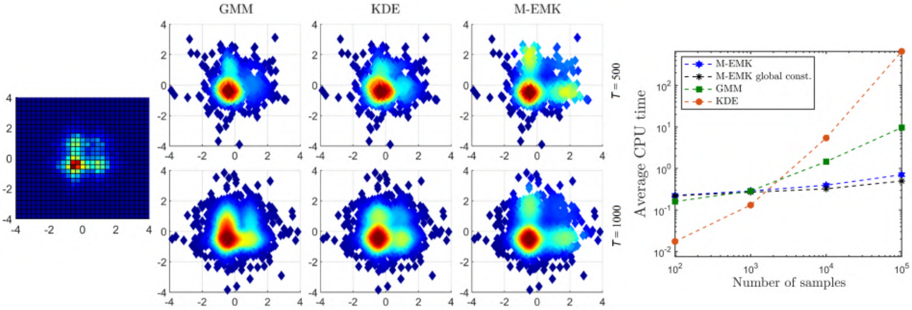
Estimation performance only using global constraints and using global and local constraints.

Multivariate entropy maximization with kernels (M-EMK)

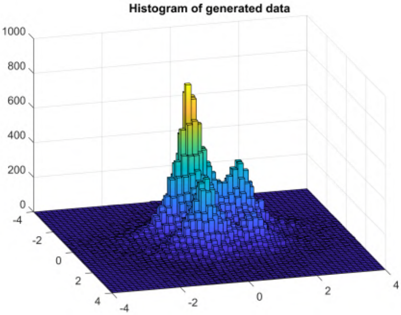
Quasi-Monte Carlo multidimensional integral approximation

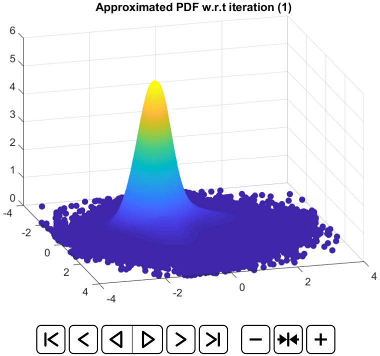
$$Q_{T,K}\left(p(\mathbf{y})\right) = \Omega\left(\frac{1}{T}\sum_{i=0}^{T-1} p(s)\right)$$

where Ω, T, p and s denote the dimensional measure of the integration region, sample size, density function and the generated quasi-random points sequence, respectively.



Density estimation performance of M-EMK





Approximation to $T = 10000$ samples generated from a mixture of multivariate generalized Gaussians.

Incorporation into the IVA model

M-EMK density function

$$p(\mathbf{y}) = \exp\left\{-1 + \sum_{i=0}^{M} \lambda_i r_i(\mathbf{y})\right\}$$

IVA-M-EMK

$$J_{\text{IVA}}(\mathbf{W}^{[k]}) = \sum_{n=0}^{N} -E\{\log p(\mathbf{y})\} - \sum_{k=0}^{K} \log\left|\det\left(\mathbf{W}^{[k]}\right)\right| - C$$

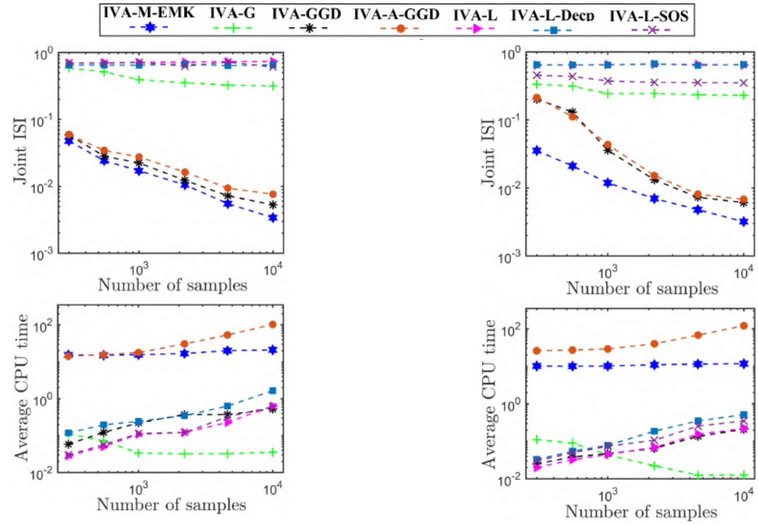
The gradient w.r.t. $\mathbf{w}_n^{[k]}$ is given by

$$\frac{\partial J_{\text{IVA}}(\mathbf{w}_n^{[k]})}{\partial \mathbf{w}_n^{[k]}} = -\sum_{i=0}^{M} \lambda_i \frac{\partial r_i(\mathbf{y}_n)}{\partial y_n^{[k]}} E\left\{\mathbf{x}^{[k]}\right\} - \frac{\mathbf{h}_n^{[k]}}{\left(\mathbf{h}_n^{[k]}\right)^{\top} \mathbf{w}_n^{[k]}}$$

λ_i are provided by M-EMK.

IVA-M-EMK takes advantage of the accurate estimation capability of M-EMK to improve source separation performance.

Experimental results using simulated sources



$K = 3$, one unimodal MGGD, and two multi-modal MGGD sources.

$K = 2$, three multi-modal MGGD sources.

Performance comparison in terms of Joint ISI and average CPU time for different number of sample size.

Summary of contributions

- Present a new multivariate probability density estimator based on the maximum entropy principle that
  - provides flexible multivariate PDFs while keeping computational complexity low;
  - provides superior performance over popular density estimation procedures;
- Derive an efficient IVA algorithm that
  - accurately separates sources from a wide range of multivariate PDFs outperforming widely used IVA algorithms;
  - enables the application of IVA to many practical applications where a flexible multivariate density modeling is needed;

Research team

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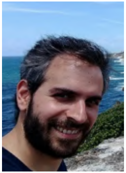

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