Sparse time-frequency representation via atomic norm minimization

#3192

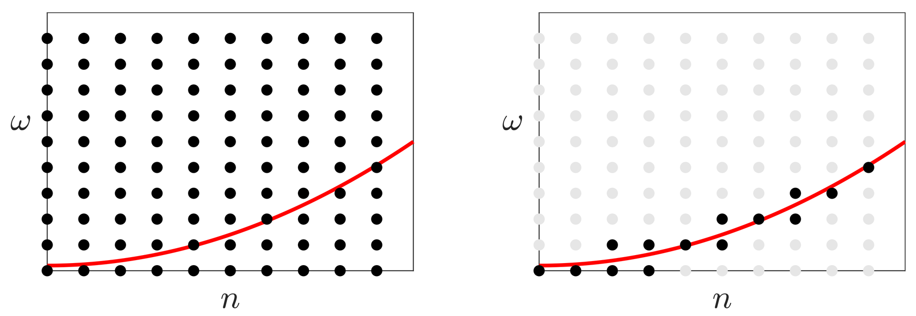
Tsubasa Kusano, Kohei Yatabe, Yasuhiro Oikawa (Waseda University, Japan)

WASEDA University 早稲田大学

Introduction

Aim: Estimate a well-localized time-frequency representation

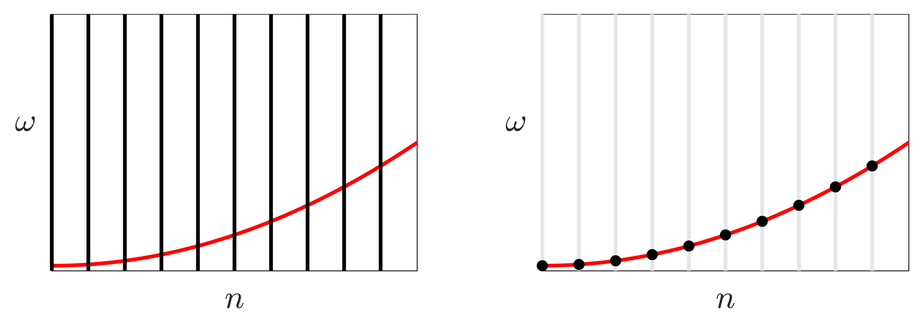
- Nonstationary signals are commonly analyzed and processed in the time-frequency (T-F) domain.
- A T-F representation obtained by the discrete Gabor transform (DGT) is spread due to windowing of an analyzed signal.
- Sparse estimation using the ℓ_1-norm needs to discretize a continuous parameter onto a grid [1].
  → may degrade the performance due to a model mismatch.



Estimating T-F representation using ℓ_1-norm

Proposal: Sparse T-F representation via atomic norm minimization

- Atomic norm [2]
  - Sparse optimization technique without discritization of continuous parameters
  - Corresponding to an infinite-dimensional dictionary of ℓ_1-norm
- Introducing atomic norm into sparse T-F estimation avoids the effect of the grid mismatch.



Estimating T-F representation using the proposed method

Sparse T-F representation

Gabor system and discrete Gabor transform

- A Gabor system is defined as a collection of sinusoids modulated by $\mathbf{g}$.

$$\mathcal{G}(\mathbf{g}, a, M) = \{\mathbf{g}_{m,n}\}_{m=0,\dots,M-1,\, n=0,\dots,N-1},$$

$$\mathbf{g}_{m,n}[l] = \mathrm{e}^{\mathrm{i}\frac{2\pi m(l-an)}{M}}\mathbf{g}[l-an]$$

- DGT and the inverse DGT with respect to the Gabor system $\mathcal{G}(\mathbf{g}, a, M)$:

$$(\mathbf{G}_{\mathbf{g}}^*\mathbf{f})[m+nM] = \langle \mathbf{f}, \mathbf{g}_{m,n}\rangle, \qquad \mathbf{G}_{\mathbf{g}}\mathbf{c} = \sum_{m,n}\mathbf{c}[m+nM]\mathbf{g}_{m,n}$$

- A T-F representation $\mathbf{c} \in \mathbb{C}^{MN}$ satisfying $\mathbf{f} = \mathbf{G}_{\mathbf{g}}\mathbf{c}$ can be obtained by DGT with a dual window $\mathbf{h}$ associated with $\mathbf{g}$.

Sparse T-F representation using ℓ_1-norm [1]

- The T-F representation $\mathbf{c}$ is a redundant representation of a signal $\mathbf{f}$.
  → The T-F representation $\mathbf{c}$ satisfying $\mathbf{f} = \mathbf{G}_{\mathbf{g}}\mathbf{c}$ is not unique.
- Find a sparse $\mathbf{c}$ satisfying $\mathbf{f} = \mathbf{G}_{\mathbf{g}}\mathbf{c}$ using ℓ_1-norm.

$$\underset{\mathbf{c}}{\text{minimize}} \quad \|\mathbf{c}\|_1 \quad \text{subject to} \quad \mathbf{f} = \mathbf{G}_{\mathbf{g}}\mathbf{c}$$

- This problem is convex
  → can be efficiently solved by convex optimization algorithms.
- This formulation may provide a poor result when the signal $\mathbf{f}$ has a component whose frequency is not included in the grid.

Atomic norm

Line spectrum estimation using atomic norm

- A windowed signal at time index n is denoted as $\mathbf{f}_n = \mathbf{W}_n\mathbf{f}$.
  - $\mathbf{W}_n$ is a diagonal matrix whose diagonal elements are $\mathbf{g}[l-an]$.
- We assume that the nth windowed signal is superimposed by atoms in $\mathcal{A} = \{\mathbf{a} \in \mathbb{C}^L \,|\, \mathbf{a}[l] = \mathrm{e}^{\mathrm{i}2\pi\omega l}, \omega \in [0,1)\}$.

$$\mathbf{f}_n = \mathbf{W}_n\sum_k c_{n,k}\mathbf{a}_{n,k}, \quad \mathbf{a}_{n,k} \in \mathcal{A}.$$

- The atomic norm of $\mathbf{x}_n$ associated with a set of atoms $\mathcal{A}$ is given by

$$\|\mathbf{x}_n\|_{\mathcal{A}} = \inf\left\{\sum_k |c_{n,k}| \,\middle|\, \mathbf{x}_n = \sum_k c_{n,k}\mathbf{a}_{n,k}, \ \mathbf{a}_{n,k} \in \mathcal{A}\right\}.$$

- Line spetrum estimation using the atomic norm is formulated as

$$\underset{\mathbf{x}_n}{\text{minimize}} \quad \|\mathbf{x}_n\|_{\mathcal{A}} \quad \text{subject to} \quad \mathbf{f}_n = \mathbf{W}_n\mathbf{x}_n.$$

Semidefinite programming formulation of atomic norm

- The atomic norm is characterized by a semidefinite programming.

$$\|\mathbf{x}_n\|_{\mathcal{A}} = \min_{\mathbf{u}_n, \nu_n} \quad \frac{1}{2L}\mathrm{Tr}\,(T(\mathbf{u}_n)) + \frac{1}{2}\nu_n$$
$$\text{subject to} \quad \begin{bmatrix} T(\mathbf{u}_n) & \mathbf{x}_n \\ \mathbf{x}_n^* & \nu_n \end{bmatrix} \succeq 0$$

- $T(\mathbf{u})$ is the Hermitian Toeplitz matrix whose first row is $\mathbf{u}$.
- $\mathbf{a}_{n,k}$ can be obtained by the Vandermonde decomposition of $T(\mathbf{u})$.
- Thus, the line spetrum estimation problem is reformulated as

$$\underset{\mathbf{x}_n, \mathbf{u}_n, \nu_n}{\text{minimize}} \quad \frac{1}{2L}\mathrm{Tr}(T(\mathbf{u}_n)) + \frac{1}{2}\nu_n$$
$$\text{subject to} \quad \begin{bmatrix} T(\mathbf{u}_n) & \mathbf{x}_n \\ \mathbf{x}_n^* & \nu_n \end{bmatrix} \succeq 0, \quad \mathbf{f}_n = \mathbf{W}_n\mathbf{x}_n.$$

Proposed method

Sparse T-F representation using atomic norm

- Inducing sparsity by a sum of atomic norms under the constraint of the reconstruction of the entire signal

$$\underset{\mathbf{x}}{\text{minimize}} \quad \sum_{n=0}^{N-1}\|\mathbf{x}_n\|_{\mathcal{A}} \quad \text{subject to} \quad \mathbf{f} = \mathbf{A}_{\mathbf{g}}\mathbf{x}.$$

- $\mathbf{x} = [\mathbf{x}_0^{\mathrm{T}}, \mathbf{x}_1^{\mathrm{T}}, \dots, \mathbf{x}_{N-1}^{\mathrm{T}}]^{\mathrm{T}}$

- A sum of atomic norms for time index n

$$\min_{\mathbf{x}} \sum_{n=0}^{N-1}\|\mathbf{x}_n\|_{\mathcal{A}} = \inf\left\{\sum_{n,k}|c_{n,k}| \,\middle|\, \mathbf{x}_n = \sum_k c_{n,k}\mathbf{a}_{n,k}, \ \mathbf{a}_{n,k} \in \mathcal{A}\right\}$$

  - corresponding to the grid-less version of the ℓ_1-norm-based method.

- Reconstructing the signal $\mathbf{f}$ by windowning and summing $\mathbf{x}_n$

$$\mathbf{A}_{\mathbf{g}}\mathbf{x} = \sum_{n=0}^{N-1}\mathbf{W}_n\mathbf{x}_n$$

Algorithm for solving this problem

- Semidefinite programming formulation of the proposed method

$$\underset{\mathbf{x}, \mathbf{u}, \boldsymbol{\nu}}{\text{minimize}} \quad \sum_{n=0}^{N-1}\frac{1}{2L}\mathrm{Tr}(T(\mathbf{u}_n)) + \frac{1}{2}\nu_n$$
$$\text{subject to} \quad \begin{bmatrix} T(\mathbf{u}_n) & \mathbf{x}_n \\ \mathbf{x}_n^* & \nu_n \end{bmatrix} \succeq 0, \ \text{for } n = 0, \dots, N-1$$
$$\mathbf{f} = \mathbf{A}_{\mathbf{g}}\mathbf{x}$$

- Alternating direction method of multipliers (ADMM) for solving this problem.
  - $P_C(\cdot)$: The projection onto the set $C = \{\mathbf{x} | \mathbf{A}_{\mathbf{g}}\mathbf{x} = \mathbf{f}\}$

$$P_C(\mathbf{v}) = \mathbf{v} - \mathbf{A}_{\mathbf{g}}^*(\mathbf{A}_{\mathbf{g}}\mathbf{A}_{\mathbf{g}}^*)^{-1}(\mathbf{A}_{\mathbf{g}}\mathbf{v} - \mathbf{f}).$$

  - $T^{\dagger}(\cdot)$: The pseudo-inverse operator of T

$$T^{\dagger}(\mathbf{X})[n] = \frac{1}{2(L-n)}\sum_{k=0}^{L-n-1}\left(\mathbf{X}[k, k+n] + \overline{\mathbf{X}[k+n, k]}\right).$$

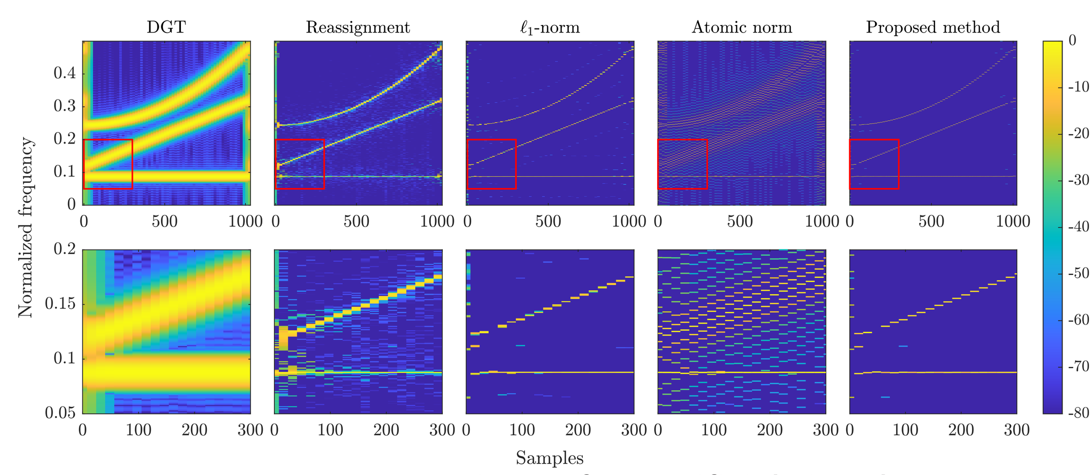
  - $P_{\mathbb{S}_+}$: The projection onto the positive semidefinite cone

Algorithm 1 Proposed algorithm

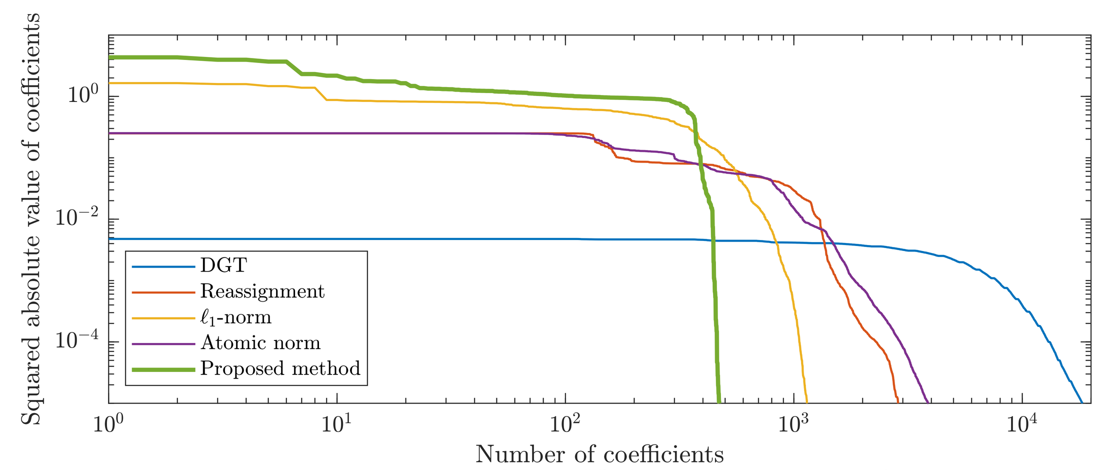
Input: $\mathbf{A}$, $\mathbf{f}$, ρ
Output: $\mathbf{x}$, $\mathbf{u}$, $\boldsymbol{\nu}$
Initialize $\mathbf{Z}_n$ and $\boldsymbol{\Lambda}_n$ for $n = 0, \dots, N-1$
for $i = 0, 1, \cdots$ **do**
  $\mathbf{x} \leftarrow P_C\left(\mathbf{z}_{\mathrm{x}} - \frac{1}{\rho}\boldsymbol{\lambda}_{\mathrm{x}}\right)$
  for $n = 1, \cdots, N$ **do**
    $\mathbf{u}_n \leftarrow T^{\dagger}\left(\mathbf{Z}_{\mathrm{T}n} - \frac{1}{\rho}\left(\boldsymbol{\Lambda}_{\mathrm{T}n} - \frac{1}{2}\mathbf{I}_L\right)\right)$
    $\nu_n \leftarrow z_{\nu n} - \frac{1}{\rho}\left(\lambda_{\nu n} - \frac{1}{2}\right)$
    $\mathbf{Z}_n \leftarrow P_{\mathbb{S}_+}\left(\begin{bmatrix} T(\mathbf{u}_n) & \mathbf{x}_n \\ \mathbf{x}_n^* & \nu_n \end{bmatrix} + \frac{1}{\rho}\boldsymbol{\Lambda}_n\right)$
    $\boldsymbol{\Lambda}_n \leftarrow \boldsymbol{\Lambda}_n + \rho\left(\begin{bmatrix} T(\mathbf{u}_n) & \mathbf{x}_n \\ \mathbf{x}_n^* & \nu_n \end{bmatrix} - \mathbf{Z}_n\right)$
  end for
end for

Numerical experiment

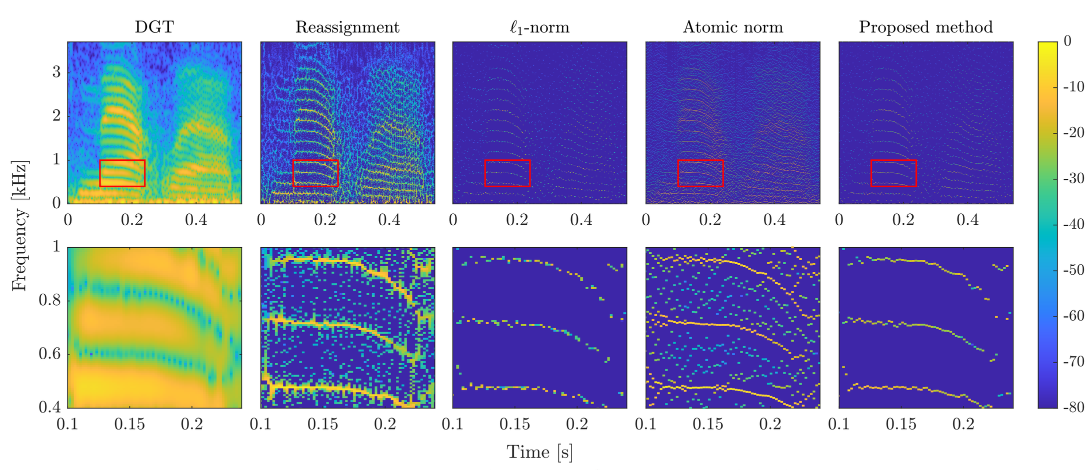
- The T-F representations obtained by the proposed method were the most-localized among these T-F representations.
  → The proposed method provides a sparser T-F representation via atomic norm minimization as a result of avoiding the grid mismatch.



T-F representations of an artificial signal



Squared absolute value of coefficients in decending order.



T-F representations of a speech signal

[1] E. Sejdić, I. Orović, and S. Stanković, "Compressive sensing meets time-frequency: An overview of recent advances in time-frequency processing of sparse signals," Digital Signal Process., vol. 77, pp. 22–35, 2018.
[2] Y. Chi and M. Ferreira Da Costa, "Harnessing sparsity over the continuum: Atomic norm minimization for superresolution," IEEE Signal Process. Mag., vol. 37, no. 2, pp. 39–57, 2020.