# Sparsity Driven Latent Space Sampling for Generative Prior based Compressive Sensing

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## 1. Compressive Sensing using Generative Models (CSGM)

Recover $\boldsymbol{x} \in \mathbb{R}^n$ from $\boldsymbol{y} \in \mathbb{R}^m$, such that $m \ll n$. The measurement model is given as:

$$\boldsymbol{y} = \mathbf{A}\boldsymbol{x}, \quad \boldsymbol{x} = \mathbf{G}_{\boldsymbol{\theta}}(\boldsymbol{z}),$$

where $\mathbf{A} \in \mathbb{R}^{m \times n}$ satisfies Set - Restricted Eigen Value Condition(S-REC), $\mathbf{G}_{\boldsymbol{\theta}}$ is a generator model with latent input $\boldsymbol{z} \in \mathbb{R}^k$ and parameter $\boldsymbol{\theta}$.

**Definition 1** *[1] Let $\mathcal{S} \subseteq \mathbb{R}^n, \gamma > 0$, and $\delta \geq 0$. A matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ is said to satisfy the $\mathcal{S}$-REC$(\mathcal{S}, \gamma, \delta)$ if $\forall \mathbf{x}_1, \mathbf{x}_2 \in \mathcal{S}$,*

$$\|\mathbf{A}(\mathbf{x}_1 - \mathbf{x}_2)\|_2 \geq \gamma \|\mathbf{x}_1 - \mathbf{x}_2\|_2 - \delta.$$

## 2. Union-of-Submanifolds

- The latent variable $\boldsymbol{z} \in \mathbb{R}^k$ is assumed to $s$ sparse.
- Sparsity $s$ assumption on the input latent space $\mathbb{Z}$ divides the latent space into $\binom{k}{s}$ subspace $\mathcal{W}_i$.
- The generator model $\mathbf{G}_{\boldsymbol{\theta}}$ transforms each subspace $\mathcal{W}_i$ to sub-manifold $\mathcal{S}_i$.
- The domain of the reconstructed signal is a union of sub-manifolds given as:

$$\mathcal{S}_{s,\mathrm{G}_{\boldsymbol{\theta}}} = \bigcup_i \mathcal{S}_i,$$

where $\mathcal{S}_i$ is the submanifold generated by $\mathrm{G}_{\boldsymbol{\theta}}(\mathbf{z})$, with $\|\mathbf{z}\|_0 \leq s$.

## 3. Our Contributions

- We propose union-of-submanifolds to model the true data distribution.
- An optimization framework, namely, proximal meta-learning(PML) algorithm to promote sparsity into the latent variable.
- Sample complexity bounds for the proposed union-of-submanifolds model

## 4. Theoretical Results

**Theorem 1** *Let $\mathrm{G}_{\boldsymbol{\theta}} : \mathbb{R}^k \rightarrow \mathbb{R}^n$ be a generative model with $d$-layers with ReLU activation and at most $h$ neurons in each layer and $\mathbf{A} \in \mathbb{R}^{m \times n}$ be a random Gaussian matrix with IID entries such that $\mathbf{A}_{i,j} \sim \mathcal{N}(0, \frac{1}{m})$ and satisfies $\mathcal{S}$-REC$(\mathcal{S}_{s,\mathrm{G}_{\boldsymbol{\theta}}}, (1-\alpha), 0)$. For any $\mathbf{x} \in \mathbb{R}^n$, if the number of measurements given by $\mathbf{y} = \mathbf{A}\mathbf{x} \in \mathbb{R}^m$ is $\mathcal{O}(sd \log \frac{kh}{s})$, where $s$ is sparsity of the latent variable $\mathbf{z}$, then with probability $1 - \exp^{(-\Omega(m))}$:*

$$\|\mathrm{G}_{\boldsymbol{\theta}}(\hat{\mathbf{z}}) - \mathbf{x}\|_2 \leq C \min_{\mathbf{z} \in \mathbb{R}^k, \|\mathbf{z}\|_0 \leq s} \|\mathrm{G}_{\boldsymbol{\theta}}(\mathbf{z}) - \mathbf{x}\|_2 + 2\epsilon,$$

*where $\hat{\mathbf{z}}$ minimizes $\|\mathbf{y} - \mathbf{A}\mathrm{G}_{\boldsymbol{\theta}}(\mathbf{z})\|_2^2$ to within additive $\epsilon$ of the optimum, $C$ is a constant, $\Omega$ is asymptotic lower bound, and $0 < \alpha < 1$.*

## 5. Sparsity Driven Latent Space Sampling (SDLSS)

- The measurement model is defined as:

$$\boldsymbol{y} = \mathbf{A}\boldsymbol{x} \quad \text{s.t} \quad \boldsymbol{x} = \mathbf{G}_{\boldsymbol{\theta}}(\boldsymbol{z}), \quad \|\boldsymbol{z}\|_0 \leq s.$$

$\boldsymbol{x}$ is assumed to lie in the union of sub-manifolds $\mathcal{S}_{s,\mathrm{G}_{\boldsymbol{\theta}}}$.

- The optimization problem is defined as:

$$\min_{\boldsymbol{z},\boldsymbol{\theta}} \|\boldsymbol{z}\|_0 \quad \text{s.t.} \quad \|\boldsymbol{y} - \mathbf{A}\mathrm{G}_{\boldsymbol{\theta}}(\boldsymbol{z})\|_2 \leq \epsilon,$$

with $\mathbf{A}$ satisfying set-restricted eigenvalue condition (S-REC) and $\mathbf{z} \in \mathbb{R}^k$ is the latent space with sparsity at most $s$.

- The optimization cost is defined as:

$$\min_{\boldsymbol{z},\boldsymbol{\theta}} \left(\mathcal{L}_{\mathrm{G}} + \mathcal{L}_{\mathbf{A}}\right),$$

where

$$\mathcal{L}_{\mathrm{G}} = \mathbb{E}_{\boldsymbol{z}}\{E_{\boldsymbol{\theta}}(\boldsymbol{y}, \boldsymbol{z}) = \|\boldsymbol{y} - \mathbf{A}\mathrm{G}_{\boldsymbol{\theta}}(\boldsymbol{z})\|_2^2 + \|\mathbf{z}\|_0\},$$
$$\mathcal{L}_{\mathbf{A}} = \mathbb{E}_{\boldsymbol{x}_1,\boldsymbol{x}_2}\{(\|\mathbf{A}\boldsymbol{x}_1 - \boldsymbol{x}_2\|_2 + \delta - \gamma\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|_2)^2\},$$

- $\mathcal{L}_{\mathrm{G}}$ and $\mathcal{L}_{\mathbf{A}}$ represent the measurement loss and $\mathcal{S} - REC$ loss, respectively.
- The proximal step is introduced to enforce sparsity in the latent space via meta-learning. We name the learning process as proximal meta-learning(PML).

$$E_{\theta}(\mathbf{y}_i, \mathbf{z}_i) = \|\mathbf{y}_i - \mathbf{A}\mathrm{G}_{\boldsymbol{\theta}}(\mathbf{z}_i)\|_2^2 + \|\mathbf{z}_i\|_0,$$
$$\hat{\boldsymbol{z}}_i = \arg\min_{\boldsymbol{z}_i} E_{\theta}(\boldsymbol{y}_i, \boldsymbol{z}_i)$$
$$= \mathrm{P}_s\left(\boldsymbol{z}_i - \beta\nabla_{\boldsymbol{z}} f(\boldsymbol{y}_i, \boldsymbol{z}_i)\right),$$

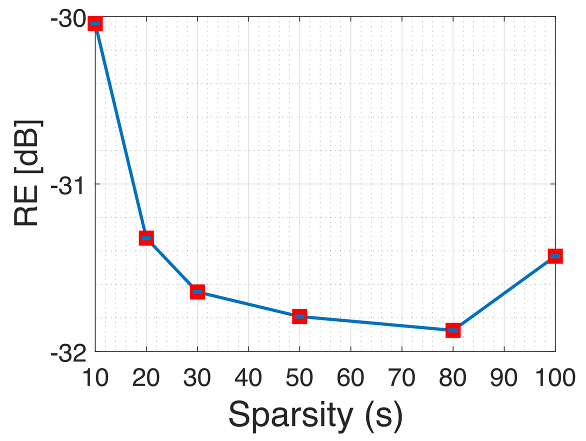
where $f(\boldsymbol{y}_i, \boldsymbol{z}_i) = \|\boldsymbol{y}_i - \mathbf{A}\mathrm{G}_{\boldsymbol{\theta}}(\boldsymbol{z}_i)\|_2^2$, $\beta$ is the learning rate, and $\mathrm{P}_s(\mathbf{u})$ is the hard-thesholding operator that sets all but the largest (in magnitude) $s$ elements of $\mathbf{u}$ to 0.

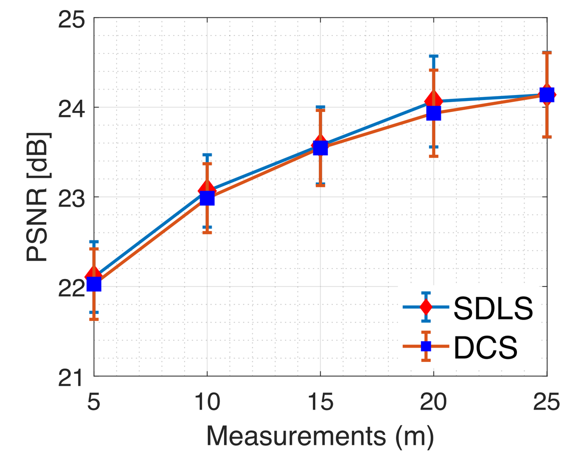
## 6. SDLSS Algorithm with PML

**Algorithm 1** : Sparsity Driven Latent Space Sampling (SDLSS) for Generative Prior

**Input:** data = $\{\mathrm{x}_i\}_{i=1}^N$, sensing matrix $\mathbf{A}$, generator $\mathrm{G}_{\boldsymbol{\theta}}$, learning rate $\alpha$, number of latent optimization steps $T$, measurement error threshold $\epsilon$, and sparsity factor $s$.
**repeat**
  Initialize generator network parameter $\boldsymbol{\theta}$.
  **for** $i = 1$ **to** $N$ **do**
    Measure the signal $\mathbf{y}_i \leftarrow \mathbf{A}\mathbf{x}_i$
    Sample $\mathbf{z} \sim \mathcal{N}(0, I)$
    **for** $t = 1$ **to** $T$ **do**
      $\hat{\mathbf{z}}_i = \mathrm{P}_s\left(\mathbf{z}_i - \beta\nabla_{\mathbf{z}} f(\mathbf{y}_i, \mathbf{z}_i)\right)$
    **end for**
  **end for**
  $\mathcal{L} = \mathcal{L}_{\mathrm{G}} + \mathcal{L}_{\mathbf{A}}$
  Update $\boldsymbol{\theta} \leftarrow \boldsymbol{\theta} - \alpha\frac{\partial\mathcal{L}}{\partial\boldsymbol{\theta}}$
**until** $\|\boldsymbol{y} - \mathbf{A}\mathrm{G}_{\boldsymbol{\theta}}(\hat{\boldsymbol{z}})\|_2 \leq \epsilon$

## 7. Results




- Fig.1 MNIST:Reconstruction error on test data as a function of sparsity; $m = 10$ and latent dimension $k = 784$.
- Fig.2 MNIST:PSNR evaluation of SDLSS and DCS as a function of measurements $m$ for the latent dimension $k$ = $100$ and sparsity $s$ = $80$.

Ground truth | DCS PSNR = 15.4 dB | SDLSS PSNR = 15.6 dB

- Reconstructed images with PSNR [dB] for $m = 20$ measurements.

## 8. Conclusions

- Sparsity in the latent space is enforced via the proposed proximal meta-learning (PML).
- The range space of the generator network, which approximates the true data distribution, is modelled as a union-of-submanifolds.
- Sample complexity bounds for the union-of-submanifolds signal model is a generalization of the result in [1].

## References

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