Paper Number: 3800

TIME-DOMAIN CONCENTRATION AND APPROXIMATION OF COMPUTABLE BANDLIMITED SIGNALS

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Introduction

- Bandlimited signals play a crucial role in signal processing.
- Bandlimited signals: perfect concentration in the frequency domain; cannot simultaneously be perfectly concentrated in the time-domain.
- We consider the Bernstein spaces $\mathcal{B}_\pi^p$: bandlimited signals with finite L^p-norm as characteristic time-domain behavior.

- Most signal processing is done on digital hardware (e.g. FPGAs, DSPs, CPUs), and hence questions of computability arise.
- We study the time-domain concentration of bandlimited signals from a computational point of view → Concept of Turing computability.
- One of the key concepts of computability: effective, i.e., algorithmic control of the approximation error.

Turing Machines

- A Turing machine is an abstract device that manipulates symbols on a strip of tape according to certain rules.
- Although the concept is very simple, a Turing machine is capable of simulating any given algorithm.
- Turing machines have no limitations with respect to memory or computing time, and hence provide a theoretical model that describes the fundamental limits of any practically realizable digital computer.

Computability Basics

A sequence of rational numbers $\{r_n\}_{n\in\mathbb{N}}$ is called computable sequence if there exist recursive functions a, b, s from $\mathbb{N}$ to $\mathbb{N}$ such that $b(n) \neq 0$ for all $n \in \mathbb{N}$ and

$$r_n = (-1)^{s(n)}\frac{a(n)}{b(n)}, \qquad n \in \mathbb{N}.$$

A recursive function is a function, mapping natural numbers into natural numbers, that is built of simple computable functions and recursions. Recursive functions are computable by a Turing machine.

A real number x is said to be computable if there exist a computable sequence of rational numbers $\{r_n\}_{n\in\mathbb{N}}$ and a recursive function $\xi\colon \mathbb{N} \to \mathbb{N}$ such that for all $M \in \mathbb{N}$ we have $|x - r_n| \leqslant 2^{-M}$ for all $n \geqslant \xi(M)$.
$\mathbb{R}_c$: set of computable real numbers.

Notation

$L^p(\mathbb{R})$, $1 \leqslant p \leqslant \infty$: the usual L^p-spaces.
Bernstein space $\mathcal{B}_\sigma^p$ ($\sigma > 0$, $1 \leqslant p \leqslant \infty$): space of all functions of exponential type at most σ, whose restriction to the real line is in $L^p(\mathbb{R})$. Norm: L^p-norm on the real line.
We call a signal in $\mathcal{B}_\sigma^p$ bandlimited signal (bandwidth σ).
$\mathcal{B}_\sigma^2$: frequently used space of bandlimited functions with bandwidth σ and finite energy.
$\mathcal{B}_{\sigma,0}^\infty$: space of all functions in $\mathcal{B}_\sigma^\infty$ that vanish at infinity.
We have $\mathcal{B}_\sigma^r \subsetneq \mathcal{B}_\sigma^s \subsetneq \mathcal{B}_{\sigma,0}^\infty$ for all $1 \leqslant r < s < \infty$.

Computable Functions

We call a function f elementary computable if there exists a natural number L and a sequence of computable numbers $\{c_k\}_{k=-L}^{L}$ such that

$$f(t) = \sum_{k=-L}^{L} c_k \frac{\sin(\pi(t-k))}{\pi(t-k)}.$$

Definition: A signal in $f \in \mathcal{B}_\pi^p$, $p \in [1,\infty) \cap \mathbb{R}_c$, is called computable in $\mathcal{B}_\pi^p$ if there exists a computable sequence of elementary computable functions $\{f_n\}_{n\in\mathbb{N}}$ and a recursive function $\xi\colon \mathbb{N} \to \mathbb{N}$ such that for all $M \in \mathbb{N}$ we have

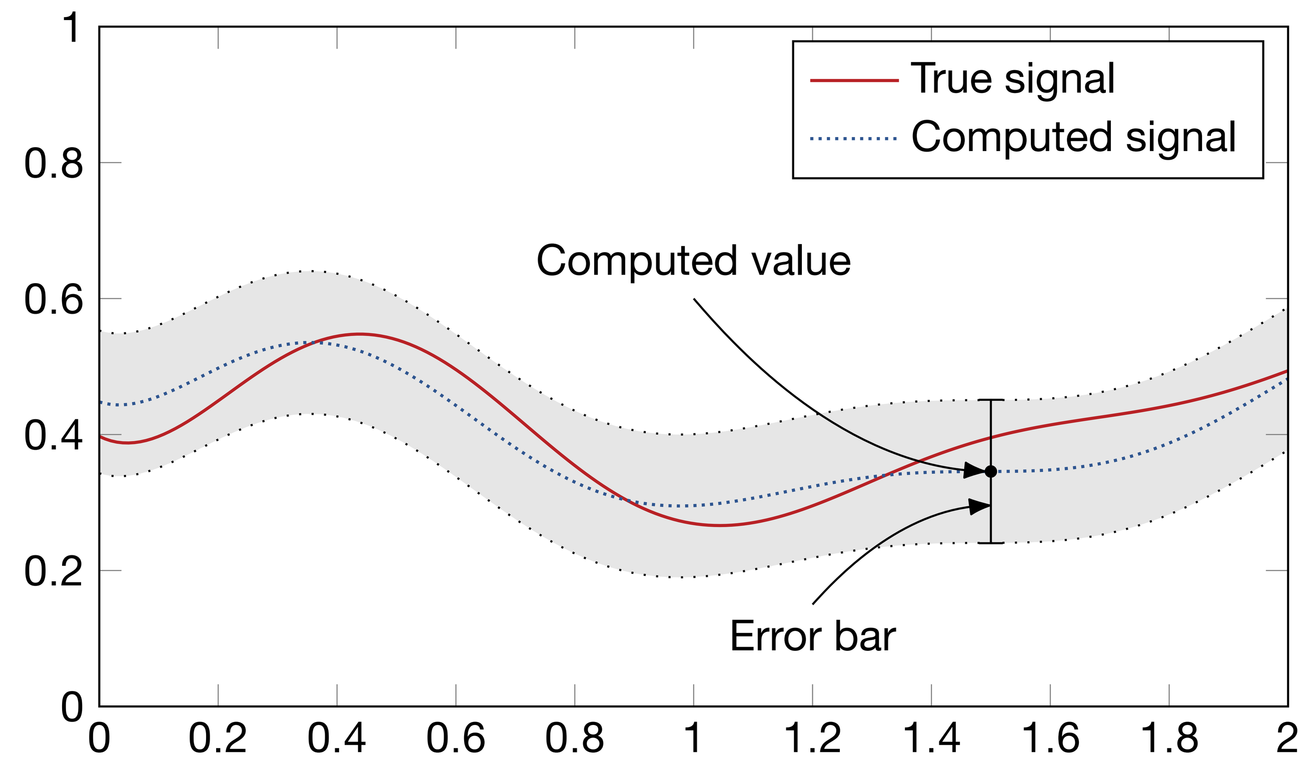
$$\|f - f_n\|_{\mathcal{B}_\pi^p} \leqslant \frac{1}{2^M}$$

for all $n \geqslant \xi(M)$.

$\mathcal{CB}_\pi^p$, $p \in [1,\infty) \cap \mathbb{R}_c$ set of all signals in $\mathcal{B}_\pi^p$ that are computable in $\mathcal{B}_\pi^p$.
$\mathcal{CB}_{\pi,0}^\infty$: set of all signals in $\mathcal{B}_{\pi,0}^\infty$ that are computable in $\mathcal{B}_{\pi,0}^\infty$.

- We can approximate any signal f by an elementary computable signal, where we have an "effective", i.e. computable control of the approximation error.

Control of the approximation error



- Advantages: intuitively clear, very general, easy to perform analytical calculations.
- Drawbacks: difficult to answer questions about the time concentration behavior, connection to the usual definition of a computable continuous function unclear.

Definition: A function $f\colon \mathbb{R} \to \mathbb{R}$ is a called computable continuous function if

1. f maps every computable sequence $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ into a computable sequence $\{f(t_n)\}_{n\in\mathbb{N}}$ of real numbers.
2. there exists a recursive function $d\colon \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ such that for all $L, M \in \mathbb{N}$ we have: $|t_1 - t_2| \leqslant 1/d(L,M)$ implies $|f(t_1) - f(t_2)| \leqslant 2^{-M}$ for all $t_1, t_2 \in [-L, L]$.

Time Concentration

"Amount" of the signal f in $[-L, L]$: $\displaystyle\int_{-L}^{L} |f(t)|^p \, dt$

Time concentration on $[-L, L]$: $\displaystyle\int_{-\infty}^{\infty} |f(t)|^p \, dt - \int_{-L}^{L} |f(t)|^p \, dt = \int_{|t|>L} |f(t)|^p \, dt$

- The smaller the value, the more concentrated is the signal.
- When is the convergence effective?

Observation: If $f \in \mathcal{CB}_\pi^p$, $p \in [1,\infty) \cap \mathbb{R}_c$, then

- $\|f\|_{\mathcal{B}_\pi^p} \in \mathbb{R}_c$.
- Since $\{\int_{|t|\leqslant L} |f(t)|^p \, dt\}_{L\in\mathbb{N}}$ is monotonically increasing, the convergence is effective.
- For $f \in \mathcal{CB}_\pi^p$ we have an algorithmic description of the time concentration behavior.

Main Result

Definition of a computable bandlimited signal using the idea of effective time concentration:

Definition: We say that a signal $f \in \mathcal{B}_\pi^p$, $p \in [1,\infty) \cap \mathbb{R}_c$ has an effectively computable time-domain concentration if

1. f is a computable continuous function, and
2. there exists a recursive function $\xi\colon \mathbb{N} \to \mathbb{N}$ such that for all $M \in \mathbb{N}$ we have
$$\left| \|f\|_{\mathcal{B}_\pi^p}^p - \int_{-L}^{L} |f(t)|^p \, dt \right| \leqslant \frac{1}{2^M}$$
for all $L \geqslant \xi(M)$.

$\mathcal{CT}_\pi^p$, $p \in [1,\infty) \cap \mathbb{R}_c$: set of such functions.
For $p = \infty$, i.e., signals $f \in \mathcal{B}_{\pi,0}^\infty$, we use an analogous definition, with $|\|f\|_{\mathcal{B}_{\pi,0}^\infty} - \max_{|t|\leqslant L} |f(t)|\, dt| \leqslant 1/2^M$.

Theorem 1: Let $p \in (1,\infty) \cap \mathbb{R}_c$. Then we have $\mathcal{CT}_\pi^p = \mathcal{CB}_\pi^p$.

- For $p \in (1,\infty) \cap \mathbb{R}_c$, the sets $\mathcal{CT}_\pi^p$ and $\mathcal{CB}_\pi^p$ coincide.
- No longer true for $p = 1$ and $p = \infty$.

Theorem 2: Let $p \in (1,\infty) \cap \mathbb{R}_c$. Then we have $f \in \mathcal{CB}_\pi^p$ if and only if $f \in \mathcal{B}_\pi^p$, f is a computable continuous function, and $\|f\|_{\mathcal{B}_\pi^p} \in \mathbb{R}_c$.

- Simple characterization of $\mathcal{CB}_\pi^p$ signals.
- No longer true for $p = 1$ and $p = \infty$.

SPTM-18: Sampling Theory, Analysis and Methods
ICASSP 2021