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Mixed Monotonic Programming for Fast Global Optimization

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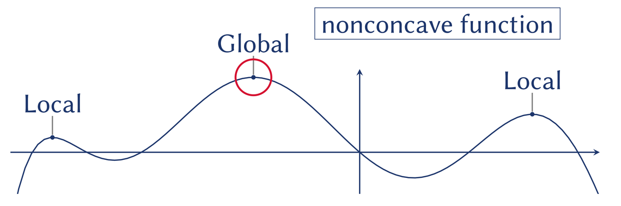
Abstract

- Global optimization framework for nonconvex optimization problems
- Exploits hidden and partial monotonicity properties
- Wide area of applications in communication systems (but not limited to this field)
- Several orders of magnitude faster than state of the art
- Published in IEEE Transactions on Signal Processing [1]

GitHub

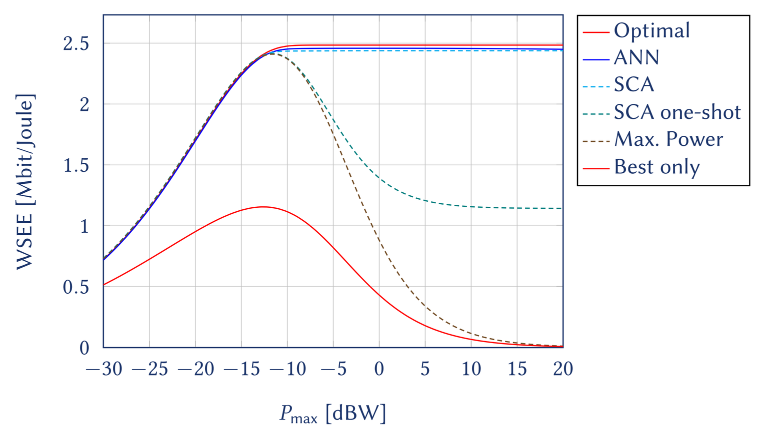
Global Optimization

- P-Time Algorithms: At most local maximum
- Convex Optimization: Local = Global
- Global Optimization: Solve multiextremal problems
- Often NP-hard → Exponential complexity



Why?

- Benchmark for fast algorithms
- Asses ultimate performance limits during system design
- Label training data for machine learning [2]



Methods [3, 4]

- Outer Approximation (Polyblock Algorithm [5])
- Branch-and-Bound

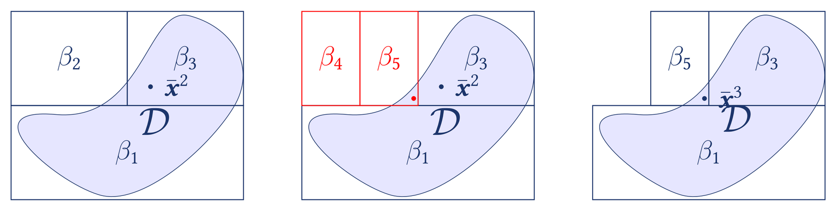
Branch-and-Bound (BB)

Optimization Problem (P)

$$\max_{\boldsymbol{x}\in\mathcal{D}} \; f(\boldsymbol{x}) \qquad \begin{array}{c} f:\mathcal{D}\to\mathbb{R} \text{ continuous} \\ \mathcal{D}\subseteq\mathbb{R}^n \text{ compact, non-empty}\end{array}$$

Algorithm (Prototype BB)

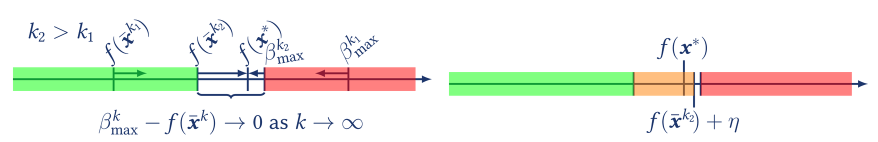
- **Step 0** Initialize outer box $\mathcal{M}_0 \supseteq \mathcal{D}$, set best known solution $\bar{\boldsymbol{x}}^0$ and value $\gamma_0 = f(\bar{\boldsymbol{x}}^0)$
- **Step 1** Select box $\mathcal{M}_k$ for branching
- **Step 2** Bisect $\mathcal{M}_k$
- **Step 3** Reduce new Boxes (optional)
- **Step 4** Compute Bound $\beta(\mathcal{M}) \geq \sup_{\boldsymbol{x}\in\mathcal{M}\cap\mathcal{D}} f(\boldsymbol{x})$ for all new boxes $\mathcal{M}$
- **Step 5** Update best known solution $\bar{\boldsymbol{x}}^k$ and value $\gamma_0 = f(\bar{\boldsymbol{x}}^k)$
- **Step 6** Delete infeasible ($\mathcal{M}\cap\mathcal{D} = \emptyset$) and suboptimal ($\beta(\mathcal{M}) \leq \gamma_k + \eta$) new boxes
- **Step 7** Terminate if no box left or $\max_{\mathcal{M}} \beta(\mathcal{M}) \leq \gamma_k$: *$\bar{\boldsymbol{x}}^k$ is global η-optimal solution*



Definition (η-Optimal Solution)

$\bar{\boldsymbol{x}}$ is η-optimal solution of (P) if

$$\forall \boldsymbol{x}\in\mathcal{D} : f(\bar{\boldsymbol{x}}) \geq f(\boldsymbol{x}) - \eta.$$



- Partition feasible set systematically
- On each partition element: Compute upper and lower bound on feasible objective values
- If branching is consistent: Upper − Lower → 0 as size(partition elements) → 0

SoA: Monotonic Optimization [5]

- Dominant global optimization framework for communication system design since 2009 [6, 7]
- Increasing function: $\boldsymbol{x} \leq \boldsymbol{x}' : f(\boldsymbol{x}) \leq f(\boldsymbol{x}')$

Monotonic Programming

$$\max_{\boldsymbol{x}\in\mathcal{G}\cap\mathcal{H}} \; f(\boldsymbol{x}) - g(\boldsymbol{x}) \qquad f, g : \text{increasing}$$

- Bounding over $\mathcal{M} = [\boldsymbol{r}, \boldsymbol{s}]$: $\max_{\boldsymbol{x}\in\mathcal{G}\cap\mathcal{H}\cap\mathcal{M}} f(\boldsymbol{x}) - g(\boldsymbol{x}) \leq \max_{\boldsymbol{x}\in\mathcal{M}} f(\boldsymbol{x}) - \max_{\boldsymbol{x}\in\mathcal{M}} g(\boldsymbol{x}) = f(\boldsymbol{s}) - g(\boldsymbol{r})$
- Example: $\log\left(1 + \frac{\boldsymbol{a}_k^T \boldsymbol{p}}{\boldsymbol{b}_k^T\boldsymbol{p} + \sigma_k^2}\right) = \log\left((\boldsymbol{a}_k + \boldsymbol{b}_k)^T\boldsymbol{p} + \sigma_k^2\right) - \log\left(\boldsymbol{b}_k^T\boldsymbol{p} + \sigma_k^2\right)$

Mixed Monotonic Programming (MMP)

Definition (MM Function)

Continuous function $F : \mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}$ such that

$$F(\boldsymbol{x},\boldsymbol{y}) \leq F(\boldsymbol{x}',\boldsymbol{y}) \quad \text{if } \boldsymbol{x}\leq\boldsymbol{x}' \quad \text{(Increasing in } \boldsymbol{x})$$
$$F(\boldsymbol{x},\boldsymbol{y}) \geq F(\boldsymbol{x},\boldsymbol{y}') \quad \text{if } \boldsymbol{y}\leq\boldsymbol{y}' \quad \text{(Decreasing in } \boldsymbol{y})$$

Then, F is called Mixed Monotonic (MM) Function.

Composition:

Functions in $\mathbb{R}$

- $F_i(\boldsymbol{x},\boldsymbol{y})$: MM
- $g(\boldsymbol{x})$: increasing
- $h(\boldsymbol{x})$: decreasing

MM functions:

- $(\boldsymbol{x},\boldsymbol{y}) \mapsto \sum_{i=1}^K F_i(\boldsymbol{x},\boldsymbol{y})$
- $(\boldsymbol{x},\boldsymbol{y}) \mapsto \max_{i=1,\dots,K} F_i(\boldsymbol{x},\boldsymbol{y})$
- $(\boldsymbol{x},\boldsymbol{y}) \mapsto \min_{i=1,\dots,K} F_i(\boldsymbol{x},\boldsymbol{y})$
- $(\boldsymbol{x},\boldsymbol{y}) \mapsto g(F_1(\boldsymbol{x},\boldsymbol{y}))$
- $(\boldsymbol{x},\boldsymbol{y}) \mapsto h(F_1(\boldsymbol{y},\boldsymbol{x}))$

If $\forall i : F_i(\boldsymbol{x},\boldsymbol{y}) \geq 0$

- $(\boldsymbol{x},\boldsymbol{y}) \mapsto \prod_{i=1}^K F_i(\boldsymbol{x},\boldsymbol{y})$ is MM

Definition (MMP)

Consider (P). Let $\mathcal{M}_0 \supseteq \mathcal{D}$ be a box. (P) is a mixed monotonic program if a MM function F on $\mathcal{M}_0$ exists such that

$$\forall \boldsymbol{x}\in\mathcal{M}_0 : F(\boldsymbol{x},\boldsymbol{x}) = f(\boldsymbol{x}).$$

Key Observations

closed-form bound!

$$\arg\max_{\boldsymbol{x}\in\mathcal{D}} f(\boldsymbol{x}) = \arg\max_{\boldsymbol{x}\in\mathcal{D}} F(\boldsymbol{x},\boldsymbol{x})$$
$$\max_{\boldsymbol{x}\in\mathcal{M}\cap\mathcal{D}} f(\boldsymbol{x}) \leq \max_{\boldsymbol{x}\in\mathcal{M}} F(\boldsymbol{x},\boldsymbol{x}) \leq \max_{\boldsymbol{x},\boldsymbol{y}\in\mathcal{M}} F(\boldsymbol{x},\boldsymbol{y}) = F(\boldsymbol{s},\boldsymbol{r}).$$

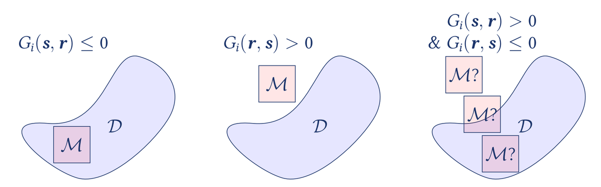
Feasibility Testing:

- Bounding independent of feasibility test
- Finite convergence: For all $\mathcal{M}$
  - Check $\mathcal{M}\cap\mathcal{D}$ conclusively
  - Find $\boldsymbol{x}\in\mathcal{M}\cap\mathcal{D}$

MM Program

$$\begin{aligned} \max_{\boldsymbol{x}} \;& F(\boldsymbol{x},\boldsymbol{x}) \\ \text{s. t. } \;& G_i(\boldsymbol{x},\boldsymbol{x}) \leq 0, \; i = 1,\dots,m \end{aligned}$$

- For $\mathcal{M} = [\boldsymbol{r},\boldsymbol{s}]$:
  $\forall i : G_i(\boldsymbol{s},\boldsymbol{r}) \leq 0 \Rightarrow \mathcal{M}\cap\mathcal{D} = \mathcal{M} \neq \emptyset$
  $\exists i : G_i(\boldsymbol{r},\boldsymbol{s}) > 0 \Rightarrow \mathcal{M}\cap\mathcal{D} = \emptyset.$
- Inconclusive for some $\mathcal{M}$



Algorithm (Mixed Monotonic Programming)

Initialize $\mathscr{R}_0 = \{[\underline{\boldsymbol{x}}, \bar{\boldsymbol{x}}]\}$, $\gamma = -\infty$, $k = 0$, $\varepsilon > 0$
repeat
  Select $[\boldsymbol{r}^k, \boldsymbol{s}^k] \in \arg\max\{F(\boldsymbol{s},\boldsymbol{r}) \mid [\boldsymbol{r},\boldsymbol{s}] \in \mathscr{R}_k\}$ ▷ Branch Selection
  Select $j_k \in \arg\max_j s_j^k - r_j^k$ ▷ Dimension for Branching
  Compute $\mathscr{P}_k = \{\mathcal{M}_-^k, \mathcal{M}_+^k\}$ with $v_k = \frac{s_{j_k} + r_{j_k}}{2}$ and ▷ Subdivision
    $\mathcal{M}_-^k = \{\boldsymbol{x} : r_j^k \leq x_j \leq v_k,\; r_i^k \leq x_i \leq s_i^k \; (i \neq j)\}$
    $\mathcal{M}_+^k = \{\boldsymbol{x} : v_k \leq x_j \leq s_j^k,\; r_i^k \leq x_i \leq s_i^k \; (i \neq j)\}$
  for all $[\boldsymbol{r},\boldsymbol{s}] \in \mathscr{P}_k$ **do** ▷ Evaluate new Branches
    if $[\boldsymbol{r},\boldsymbol{s}] \cap \mathcal{F} \neq \emptyset$ **and** $F(\boldsymbol{s},\boldsymbol{r}) > \gamma + \varepsilon$ **then** ▷ Prune
      Add $[\boldsymbol{r},\boldsymbol{s}]$ to $\mathscr{S}_k$
      Find $\boldsymbol{x} \in \mathcal{F} \cap [\boldsymbol{r},\boldsymbol{s}]$ ▷ Feasible Point
      if $f(\boldsymbol{x}) > \gamma$ **then** ▷ Update CBS
        $\bar{\boldsymbol{x}} \leftarrow \boldsymbol{x}$
        $\gamma \leftarrow f(\boldsymbol{x})$
      end if
    end if
  end for
  $\mathscr{R}_{k+1} \leftarrow \mathscr{S}_k \cup \mathscr{R}_k \setminus \{[\boldsymbol{r}^k, \boldsymbol{s}^k]\}$ ▷ Replace old Branch
  $k \leftarrow k + 1$
until $\mathscr{R}_k = \emptyset$
return $\bar{\boldsymbol{x}}$ as the optimal solution

Remedies:

1) More Assumptions on $G_i(\boldsymbol{x},\boldsymbol{y})$
   - Finite Convergence
   - In each dimension: Either in- or decreasing
   - Not always possible
2) Modify BB Algorithm & accept infinite convergence
   - *Might* lead to slow algorithm
   - Most popular approach. Often works well.
3) ε-Approximate Feasibility
   - Relax constraints by $\varepsilon > 0$
   - Finite Convergence
   - Numerical Issues (wrong solution!)
4) Successive Incumbent Transcending (SIT)
   - Tighten constraints by $\varepsilon > 0$
   - Solve easier "SIT dual" problem
   - MMP-SIT: [8]

Further Information

- Tutorial @ IEEE ICASSP 2021 & IEEE ICC 2021 by Bho Matthiesen & Eduard Jorswieck
  Efficient Global Optimization and its Application to Wireless Interference Networks
- Thesis "Efficient Globally Optimal Resource Allocation in Wireless Interference Networks" [9]

Example: Gaussian Interference Channel

Interference Channel:

- Model for heterogeneous dense small-cell networks
- Treating interference as noise:

$$R_i \leq \log\left(1 + \frac{\alpha_i p_i}{\sum_{j\neq i} \beta_{ij} p_j + \sigma^2}\right)$$

- Resource allocation examples

$$\max_{\boldsymbol{p}\in[\boldsymbol{0},\boldsymbol{P}]} \; w_i \sum_i R_i$$
$$\max_{\boldsymbol{p}\in[\boldsymbol{0},\boldsymbol{P}]} \; \frac{\sum_i R_i}{\sum_i \mu_i p_i + P_c}$$

- An MM representation is all it takes to solve such an MM problem
- Weighted sum rate

$$F_{\mathrm{WSR}}(\boldsymbol{x},\boldsymbol{y}) = \sum_i w_i \log\left(1 + \frac{1 + \boldsymbol{a}_i^T\boldsymbol{x}}{\boldsymbol{b}_i^T\boldsymbol{y}}\right)$$

- Global energy efficiency

$$F_{\mathrm{GEE}}(\boldsymbol{x},\boldsymbol{y}) = \frac{F_{\mathrm{WSR}}(\boldsymbol{x},\boldsymbol{y})}{\boldsymbol{\mu}^T\boldsymbol{y} + P_c}$$

Global Energy Efficiency:

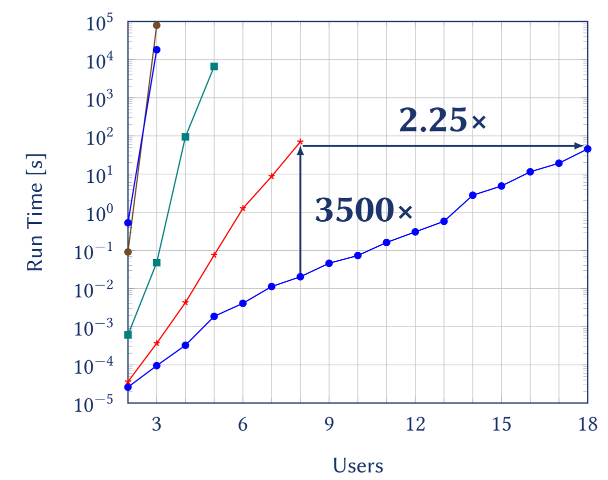
$$\max_{\boldsymbol{p}\in[\boldsymbol{r},\boldsymbol{s}]} \frac{\sum_k \log\left(1 + \frac{\boldsymbol{b}_k^T\boldsymbol{p}}{\boldsymbol{c}_k^T\boldsymbol{p} + \sigma_k^2}\right)}{\boldsymbol{\phi}^T\boldsymbol{p} + P_c} \leq \frac{\sum_k \log\left(1 + \frac{\boldsymbol{b}_k^T\boldsymbol{s}}{\boldsymbol{c}_k^T\boldsymbol{r} + \sigma_k^2}\right)}{\boldsymbol{\phi}^T\boldsymbol{r} + P_c} = F(\boldsymbol{s},\boldsymbol{r})$$

increasing; decreasing; decreasing

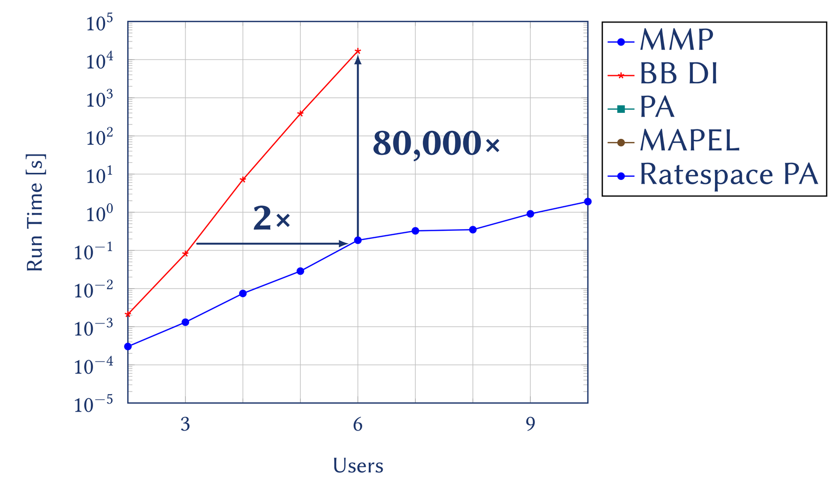
GEE Maximization with MMP (C++)

```cpp
#include "MMP.h"

template <size_t Dim>
class GEE : public MMP<Dim> {
        public:
        double alpha[Dim], Pc;
        std::array<double, Dim> beta[Dim], psi;

        GEE() : MMP<Dim>() {}
        double MMPobj(const vtypeS& x, const vtypeS& y) const override;
};

template <size_t Dim>
double GEE<Dim>::MMPobj(const vtypeS& x, const vtypeS& y) const {
        double ret = 0;

        for (size_t i = 0; i < Dim; ++i) {
                double denom = std::inner_product(y.begin(), y.end(),
                beta[i].begin(), 1.0);
                ret += std::log2(1.0 + alpha[i] * x[i] / denom);
        }

        return ret / std::inner_product(y.begin(), y.end(),
        psi.begin(), Pc);
}
```

Throughput (IFC TIN)



Global EE (IFC TIN)



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