Communication Over Block Fading Channels – An Algorithmic Perspective on Optimal Transmission Schemes

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Motivation

- Provision of accurate CSI is a major challenge in wireless systems due to
  - dynamic nature of the wireless channel
  - estimation inaccuracy
  - limited feedback
  - ...

➡ Imperfect CSI must be taken into account in the system design

- We consider the general uncertainty model of *block fading channels*

➡ Capacity is known, but optimal signal processing and coding schemes remain unknown in general

- Such optimal schemes have been found only for very few specific cases and accordingly, common belief is that it is a hard problem to find them

In this work, we shed some new light upon this issue by adopting an *algorithmic perspective*

Overview Main Results

- We address this issue from a fundamental algorithmic point of view by using the concept of a *Turing machine* and the corresponding *computability framework*

➠ We study algorithmic computability of the capacity

Perfect CSI

Capacity of *discrete memoryless channels (DMCs)* is computable:

$$\boldsymbol{C(W) \in \mathbb{R}_c}$$

for computable $W \in \mathcal{CH}_c(\mathcal{X};\mathcal{Y})$.

Imperfect CSI

Capacity of *averaged channels (ACs)* is in general non-computable:

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Birth of Information Age

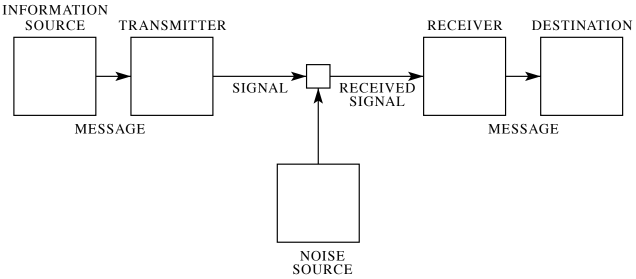
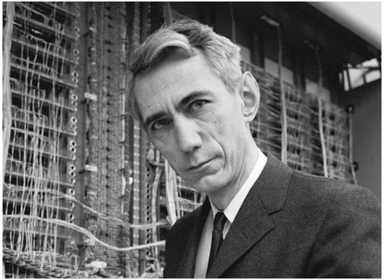


Fig. 1 — Schematic diagram of a general communication system.



- Claude Shannon laid the theoretical foundations for information theory, a mathematical communication model
  - **A mathematical theory of communication**

C. E. Shannon, "A mathematical theory of communication," *Bell Syst. Tech. J.*, vol. 27, no. 3, pp. 379–423, Jul. 1948

Perfect Channel State Information

- Discrete memoryless channels (DMCs)
- Let $\mathcal{X}$ and $\mathcal{Y}$ with $|\mathcal{X}| < \infty$ and $|\mathcal{Y}| < \infty$ be finite input and output alphabets
- Probability law for DMCs is specified by the channel

$$W^n(y^n|x^n) = \prod_{i=1}^{n} W(y_i|x_i)$$

➠ Belong to the class of independent and identically distributed (i.i.d.) channels which represent the most tractable class of channel laws

The *capacity* $C(W)$ of a discrete memoryless channel (DMC) W is

$$\boldsymbol{C(W) = \max_{X} I(X;Y) = \max_{p \in \mathcal{P}(\mathcal{X})} I(p, W)}$$

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- *Entropic quantities*
- *Single-letter*
- *Convex optimization problem*
- Of particular relevance as it allows to compute the capacity $C(W)$ as a function of the channel W given by a convex optimization problem

Can we compute the capacity algorithmically?

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1936: Birth of Computer Science

- Alan M. Turing was the first to study this kind of problems systematically
- He developed a computing model
  - ➠ **Turing machine**
- Object of interest: real numbers

A. M. Turing, "On computable numbers, with an application to the Entscheidungsproblem," *Proc. London Math. Soc.*, vol. 2, no. 42, pp. 230–265, 1936

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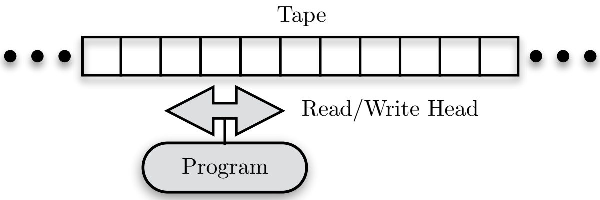
Turing Machine: The Most Powerful Computation Model



Mathematical model of an abstract machine that manipulates symbols on a strip of tape according to certain given rules

A. M. Turing, "On computable numbers, with an application to the Entscheidungsproblem," *Proc. London Math. Soc.*, vol. 2, no. 42, pp. 230–265, 1936

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Turing Machine (2)

Turing machines can simulate any given algorithm and therewith provide a simple but very powerful model of computation.

- **No** limitation on computational complexity
- Unlimited computing capacity and storage
- Completely error-free execution of programs
- Most powerful programming languages are Turing-complete (such as C, C++, Java, etc.)
- All discrete computing models are equivalent
(von Neumann, Gödel, Minsky, . . .)

Any arbitrarily large finite-dimensional problem can be exactly solved without errors by a Turing machine

Turing Machine (3)

Turing machines are suited to study the limitations in performance of a digital computer:

Anything that is not Turing computable cannot be computed on a real digital computer, regardless of how powerful it may be

- Alan Turing introduced the concept of a computable real number in 1936, and demonstrated some principal limitations of computability
- In 1949 a computable monotonically increasing sequence which converges to a real non-computable number was constructed

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E. Specker, "Nicht konstruktiv beweisbare Sätze der Analysis," *Journal of Symbolic Logic*, vol. 14, no. 3, pp. 145–158, Sep. 1949

Computability of Numbers

Computable numbers are real numbers that are computable by Turing machines

Exact definition:

- A sequence $\{r_n\}_{n\in\mathbb{N}}$ is called a computable sequence if there exist recursive functions $a, b, s : \mathbb{N} \to \mathbb{N}$ with $b(n) \neq 0$ for all $n \in \mathbb{N}$ and

$$r_n = (-1)^{s(n)} \frac{a(n)}{b(n)}$$

- A real number x is said to be computable if there exists a computable sequence of rational numbers $\{r_n\}_{n\in\mathbb{N}}$ such that

$$|x - r_n| < 2^{-n}$$

Key idea: effective approximation

- $\mathbb{R}_c$ computable real numbers
- Commonly used constants like e and π are computable

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Computability of Distributions and Channels

- Based on this, we can define *computable probability distributions* and *computable channels*
- We define the set of computable probability distributions $\mathcal{P}_c(\mathcal{X})$ as the set of all probability distributions

$$p \in \mathcal{P}(\mathcal{X}) \text{ such that } p(x) \in \mathbb{R}_c,\ x \in \mathcal{X}$$

- Let $\mathcal{CH}_c(\mathcal{X};\mathcal{Y})$ be the set of all computable channels, i.e., for a channel

$$W : \mathcal{X} \to \mathcal{P}(\mathcal{Y}) \text{ we have } W(\cdot|x) \in \mathcal{P}_c(\mathcal{Y}) \text{ for every } x \in \mathcal{X}$$

Computability of $C(W)$

- *Warm-up:* Let's see if for a computable channel $W \in \mathcal{CH}_c$ the capacity $C(W)$ is computable...

Theorem:

Let $\mathcal{X}$ and $\mathcal{Y}$ be arbitrary finite alphabets. Then for all computable channels $W \in \mathcal{CH}_c$ we have

$$C(W) = \max_{p \in \mathcal{P}(\mathcal{X})} I(p, W) \in \mathbb{R}_c.$$

➠ The **capacity $C(W)$** for a computable channel $W \in \mathcal{CH}_c$ is computable and **can be algorithmically computed by a Turing machine**!

K. Weihrauch, *Computable Analysis - An Introduction.* Berlin, Heidelberg: Springer-Verlag, 2000

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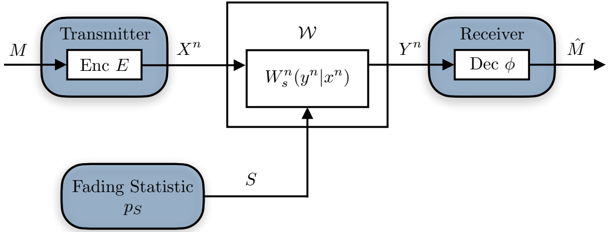
Block Fading Channel



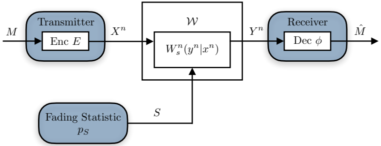
- Let $\mathcal{S}$ be an arbitrary state (uncertainty) set
- State $s \in \mathcal{S}$ is unknown, but remains *constant* and follows the statistic $p_S \in \mathcal{P}(\mathcal{S})$

The *averaged channel (AC)*

$$\boldsymbol{\mathcal{W}} := \{\{\boldsymbol{W_s} \in \boldsymbol{\mathcal{CH}}(\boldsymbol{\mathcal{X}};\boldsymbol{\mathcal{Y}})\}_{s\in\mathcal{S}}, \boldsymbol{p_S} \in \boldsymbol{\mathcal{P}}(\boldsymbol{\mathcal{S}})\}$$

is given by the collection of all channels $W_s \in \mathcal{CH}(\mathcal{X};\mathcal{Y})$ for all states $s \in \mathcal{S}$ and additional probability distribution $p_S \in \mathcal{P}(\mathcal{S})$ on the state set $\mathcal{S}$.

Averaged Channel



The *capacity* $C(\mathcal{W})$ of an averaged channel $\mathcal{W}$ is

$$C(\mathcal{W}) = \sup_{p \in \mathcal{P}(\mathcal{X})} \inf_{s \in \mathcal{S}} I(p, W_s)$$

- *Analytically* well understood (closed-form single letter entropic expression)
- Surprisingly, not much known about its algorithmic computability and the optimal signal processing

➠ Study its structure and algorithmic computability of optimal strategies

R. Ahlswede, "The weak capacity of averaged channels," *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, vol. 11, pp. 61–73, Mar. 1968

Computability of $C(\mathcal{W})$

An AC $\mathcal{W} = \{\{W_s \in \mathcal{CH}_c(\mathcal{X};\mathcal{Y})\}_{s\in\mathcal{S}}, p_S \in \mathcal{P}(\mathcal{S})\}$ is said to be *computable* if there is a recursive function $\varphi : \mathcal{S} \to \mathcal{CH}_c(\mathcal{X};\mathcal{Y})$ with $\varphi(s) = W_s$ for all $s \in \mathcal{S}$ and p_S is a computable probability distribution. The set of all computable ACs is denoted by $\mathcal{AC}_c(\mathcal{X},\mathcal{S};\mathcal{Y})$.

- The set $\mathcal{W}$ is algorithmically constructible, i.e., for every state $s \in \mathcal{S}$ the channel W_s can be constructed by an algorithm with input s

Theorem:

Let $\mathcal{X}$ and $\mathcal{Y}$ be arbitrary finite alphabets. Then there is a computable averaged channel $\mathcal{W} \in \mathcal{AC}_c(\mathcal{X},\mathcal{S};\mathcal{Y})$ such that

$$\boldsymbol{C(\mathcal{W}) = \sup_{p\in\mathcal{P}(\mathcal{X})} \inf_{s\in\mathcal{S}} I(p, W_s) \notin \mathbb{R}_c.}$$

- Although the channel itself is computable, i.e., $\mathcal{W} \in \mathcal{AC}_c(\mathcal{X},\mathcal{S};\mathcal{Y})$, it is not possible to algorithmically compute $C(\mathcal{W})$!

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Discussion

- *Computability framework based on Turing machines*
- *Computability of capacities*
  - ➠ Capacity value of DMCs is computable: $C(W) \in \mathbb{R}_c$
  - ➠ Capacity value of ACs is in general **not** computable: $C(\mathcal{W}) \notin \mathbb{R}_c$
- *Search for capacity-achieving transmission schemes*
  - **Goal:** Turing machine $\mathfrak{T}(n) = (E_n^*, \phi_n^*)$ that outputs an optimal encoder E_n^* and optimal decoder ϕ_n^* providing the maximal possible rate while guaranteeing error probability ϵ
  - ➠ **Not** possible in general for ACs!
    (Note that it is not required that the Turing machine depends recursively on the channel; it is only asked if it is possible to find such a search algorithm for a fixed and given channel and error)
  - ➠ Further studies on the algorithmic constructability of codes:

H. Boche, R. F. Schaefer, and H. V. Poor, "Turing meets Shannon: Algorithmic constructability of capacity-achieving codes," in *Proc. IEEE Int. Conf. Commun.*, Montreal, QC, Canada, Jun. 2021

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