Consensus Based Distributed Spectral Radius Estimation

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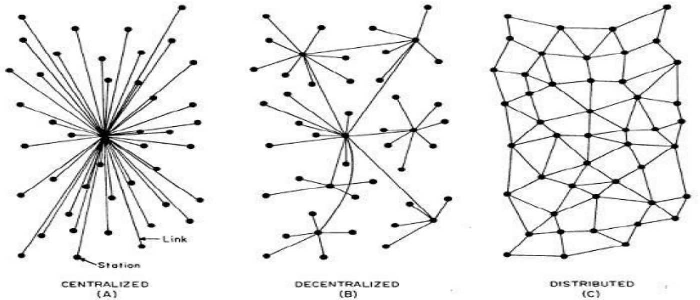
Outline

Wireless Sensor Networks

Broadly classified into: centralized, decentralized and distributed.

- Drawbacks of centralized & decentralized WSNs
  - Single or few fusion centers creates bottle neck for data aggregation
  - Vulnerable to attacks
  - Implementation and design issues (Tx/Rx power and resource management)



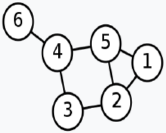
Source: Baran, Paul. *"On distributed communications networks."* IEEE transactions on Communications Systems 12, no. 1, 1964

Distributed Wireless Sensor Networks

- Characteristics / Advantages
  - No fusion center
  - Fault tolerance
  - Efficient resource management
- Challenges
  - Time synchronization
  - Noise in wireless channels
- Constraints
  - Nearest neighbor communication.
  - Additive channel noise.
  - Memory constraints.
  - Secure information transfer.
- Applications
  - Environmental monitoring
  - Habitat monitoring
  - Industrial and military applications
  - Social networks

Graph theory background

- Graph representation of distributed network
  - Distributed network with N nodes.
  - Undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, communications among neighbors.
  - Degree matrix $\mathbf{D}$: Diagonal matrix with the degrees of the nodes.
  - Adjacency matrix $\mathbf{A}$: $a_{ij} = 1$ if $\{i, j\} \in \mathcal{E}$ and $a_{ij} = 0$, otherwise.
  - Laplacian matrix $\mathbf{L} = \mathbf{D} - \mathbf{A}$ used to characterize network.
  - Spectral radius of the graph, $\rho = \lambda_{max}(\mathbf{A})$.
  - Principal eigenvector of $\mathbf{A}$ is always positive.

Labeled graph	Degree matrix	Adjacency matrix	Laplacian matrix
	$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 2 & -1 & 0 & 0 & -1 & 0 \\ -1 & 3 & -1 & 0 & -1 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 3 & -1 & -1 \\ -1 & -1 & 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}$

Source: http://kuanbutts.com/2017/10/21/spectral-cluster-berkeley/

Distributed Spectral Radius Estimation

- Digital communication setting
  - **G. Muniraju**, C. Tepedelenlioglu and A. Spanias, "Consensus Based Distributed Spectral Radius Estimation," in *IEEE Signal Processing Letters*, vol. 27, pp. 1045-1049, 2020, doi: 10.1109/LSP.2020.3003237.
- Analog communication setting
  - **G. Muniraju**, C. Tepedelenlioglu and A. Spanias, "Distributed Spectral Radius Estimation in Wireless Sensor Networks," in *2019 53rd Asilomar Conference on Signals, Systems, and Computers*, Pacific Grove, CA, USA, 2019, pp. 1506-1510, doi: 10.1109/IEEECONF44664.2019.9049018.

System Model

Assumptions on System model

- Each node has a real number which is its own initial measurement.
- Transmission power determines the communication radius.
- Nodes know their locations and can estimate their neighbors.
- Nodes broadcast their state values to their neighbors in a synchronized fashion.
- Communications:
  - Packet loss model for digital communication models.
  - Time-varying graphs: a message is received with a probability $1 - p$, in order to model the imperfect communication links.

Distributed Spectral Radius Estimation

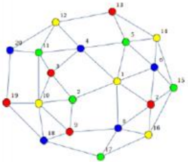
Problem Statement

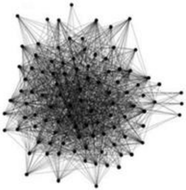
- To reach consensus at each node on $(\log(\rho))$ of the graph, using only local neighbor communications.
- Packet loss in digital models.
- Study convergence of the algorithm for fixed graphs and time-varying graphs.

Ours is the first work to address distributed spectral radius estimation in WSNs for both analog and digital communication models.

Applications

- To study graph coloring methods.
- Properties of Hamiltonian paths.
- To understand the convergence of belief propagation algorithms.
- Estimating and controlling the connectivity of the network.
- To study Mixing time of networks.
- Irregularity, sparsity and density of the networks.





Dense Graph

Sparse Graph

Distributed Spectral Radius Estimation

Digital communication model

- We consider packet loss i.e, transmitted message can be lost (failure) with a probability of p, independently for each edge.
- No analog noise.

- Main update equations:

$$\boxed{x_i(t) = \log\left(\sum_{j=1}^{N} a_{ij}\exp(x_j(t-1))\right)}, \quad \text{for } i = 1, \cdots, N.$$

$$y_i(t) = \frac{1}{t}x_i(t).$$

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Distributed Spectral Radius Estimation

Theorem

In a connected non-bipartite graph $\mathcal{G}$, with all nodes initialized to $\mathbf{x}(0) = \mathbf{0}$, we have for large t,

$$\mathbf{y}(t) = log(\rho)\mathbf{1} + \frac{1}{t} log\left[\mathbf{q}_1 ||\mathbf{q}_1||_1\right] + \mathcal{O}\left(\frac{1}{t} \; (\rho_2/\rho)^t\right)$$

where, $\mathbf{q}_1$ is the principal eigenvector of $\mathbf{A}$. In bipartite graphs,

$$\mathbf{y}(t) = log(\rho)\mathbf{1} + \frac{1}{t}\left(log\left[\mathbf{q}_1 \sum_{j=1}^{N} q_{1j} + (-1)^t \mathbf{q}_N \sum_{j=1}^{N} q_{Nj}\right]\right) + \mathcal{O}\left(\frac{1}{t}\left(\frac{\max(|\rho_2|, |\rho_{N-1}|)}{\rho}\right)^t\right)$$

where, $\mathbf{q}_N$ is the eigenvector corresponding to eigenvalue $-\rho$ of $\mathbf{A}$.

Theorem's Proof

- $\mathbf{x}(t) = \log\left[\mathbf{A}\exp[\mathbf{x}(t-1)]\right] = \log\left[\mathbf{A}\exp[\log\left[\mathbf{A}\exp[\mathbf{x}(t-2)]\right]]\right] = \log\left[\mathbf{A}^t\exp[\mathbf{x}(0)]\right].$
- EVD of $\mathbf{A}$ as $\mathbf{A} = \mathbf{Q}\boldsymbol{\Delta}\mathbf{Q}^{-1}$, where $\mathbf{Q} = [\mathbf{q}_1, \mathbf{q}_2, \cdots, \mathbf{q}_N]$
- $\mathbf{A}^t = (\mathbf{Q}\boldsymbol{\Delta}\mathbf{Q}^T)(\mathbf{Q}\boldsymbol{\Delta}\mathbf{Q}^T)\cdots(\mathbf{Q}\boldsymbol{\Delta}\mathbf{Q}^T) = \mathbf{Q}\boldsymbol{\Delta}^t\mathbf{Q}^T.$
- Since $\mathbf{A}$ is real and symmetric, $\mathbf{Q}^{-1} = \mathbf{Q}^T$, thus $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$
- $\mathbf{y}(t) = \frac{1}{t}\log[\mathbf{Q}\boldsymbol{\Delta}^t\mathbf{Q}^T\mathbf{1}].$
- $\mathbf{y}(t) = \frac{1}{t}\left(\log[\mathbf{Q}\boldsymbol{\Delta}^t\rho^{-t}\mathbf{Q}^T\mathbf{1}] + t\log(\rho)\mathbf{1}\right) = \log(\rho)\mathbf{1} + \frac{1}{t}\left(\log[\mathbf{Q}\mathbf{S}^t\mathbf{Q}^T\mathbf{1}]\right)$
- $\mathbf{Q}\mathbf{S}^t\mathbf{Q}^T\mathbf{1} = \sum_{i=1}^N \mathbf{q}_i s_i^t \sum_{j=1}^N q_{ij} = \mathbf{q}_1\sum_{j=1}^N q_{1j} + \sum_{i=2}^N \mathbf{q}_i s_i^t \sum_{j=1}^N q_{ij}.$
- $\mathbf{q}_i$ is the principal eigenvector. $\mathbf{q}_i$ is real and positive, with l_2-norm 1.
- $\mathbf{y}(t) = \log(\rho)\mathbf{1} + \frac{1}{t}\log\left[\mathbf{q}_1||\mathbf{q}_1||_1\right] + \mathcal{O}\left(\frac{1}{t}\ (\rho_2/\rho)^t\right)$

Distributed Spectral Radius Estimation

Algorithm 1 : Distributed estimation of spectral radius

1: **Input**: N, $\mathbf{A}$, $t_{\max}$
2: **Initialization**: $\mathbf{x}(0) = [0, \cdots, 0]^T$
3: **for** $t = 1, 2, \cdots, t_{\max}$
4: $\quad x_i(t) = \log\left(\sum_{j=1}^{N} a_{ij} \exp(x_j(t-1))\right)$
5: $\quad y_i(t) = \frac{1}{t} x_i(t)$
6: **end**
7: **Output**: $y_i(t_{\max})$

- For d-regular graphs, $y_i(t) = d = \rho$ for every t, and therefore has zero error. Since for d-regular graphs, $\mathbf{q}_1 = N^{-1/2}\mathbf{1}$, making the term $\frac{1}{t}\log\left[\mathbf{q}_1 ||\mathbf{q}_1||_1\right] = \mathbf{0}$.

- $\log(\cdot)$ and $\exp(\cdot)$ can be replaced by any pair of inverse functions and appropriate modifications in Algorithm. The advantage of the $\log - \exp$ pair is that the elements of $\mathbf{x}(t)$ grow linearly with t, which ensures that $\mathbf{y}(t) = t^{-1}\mathbf{x}(t)$ converges.

Time-varying graphs

- We model the unreliable links or packet loss as a time-varying graph that has independently removed edges with probability p.
- $x_i(t+1) = \log\left(\exp(x_j(t) + \sum_{j=1}^{N} a_{ij} b_{ij}(t) \exp(x_j(t)) \right)$.
- $b_{ij}(t) \sim \mathrm{Ber}(1-p)$, and $P(b_{ij}(t) = 0) = p, i \neq j$, is independent Bernoulli random variables capturing packet loss on edges.
- $\mathbf{x}(t) = \log\left[\left(\prod_{k=1}^{t} (\mathbf{I} + \mathbf{A}_k) \right) \mathbf{1} \right]$
- $y_i(t) = \frac{1}{t} \log\left(\prod_{k=1}^{t} (1 + B_k) \right)$, where $B_k^{(i)} \sim \mathrm{Bin}(d, 1-p)$
- Regular graphs : $\mathbf{y}(t) \simeq \log(1 + d(1-p))\mathbf{1}$.
- Irregular graphs : $\mathbf{y}(t) \simeq \log(1 + \rho(1-p))\mathbf{1}$.
- Hence, $\rho \simeq \frac{\exp[\mathbf{y}(t)] - 1}{1-p}$.

Non-bipartite graphs

- $N = 75$
- $\rho = 37.1142$
- $log(\rho) = 3.614$

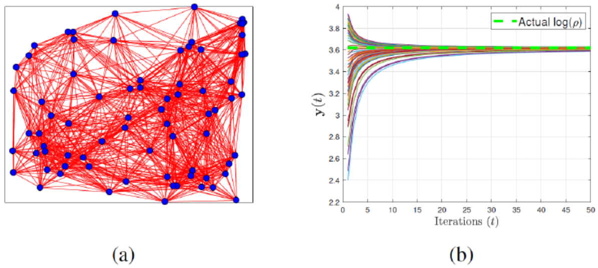

Figure: (a) Non-bipartite graph with N = 75 nodes. (b) Convergence of Algorithm 1 for the non-bipartite graph.

Bipartite graphs

- $N = 20$
- $\rho = 4.3739$
- $log(\rho) = 1.4756$

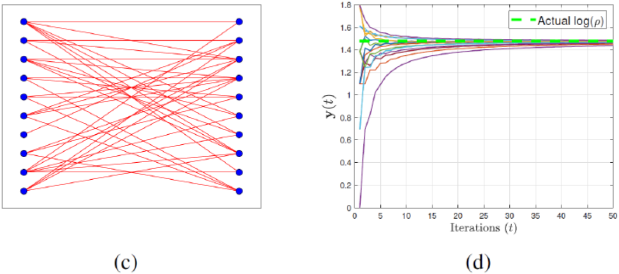

Figure: (c) Bipartite graph with N = 20 nodes. (d) Convergence of Algorithm 1 for the bipartite graph.

Regular graphs

- $N = 100$
- $d = \{10, 20, \cdots, 60\}$
- $p = \{0, 0.1, \cdots, 0.8\}$.

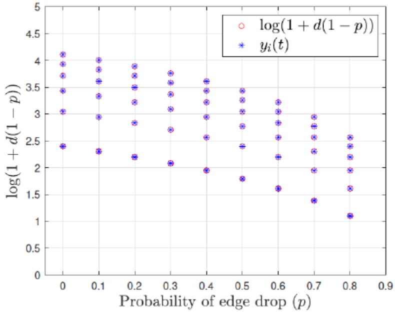

Figure: Estimated $\log(1 + d(1 - p))$ for a regular time-varying graphs with $N = 100$.

Irregular graphs

- $N = 100$
- $d = \{10, 20, \cdots, 60\}$
- $p = \{0, 0.1, \cdots, 0.8\}$.

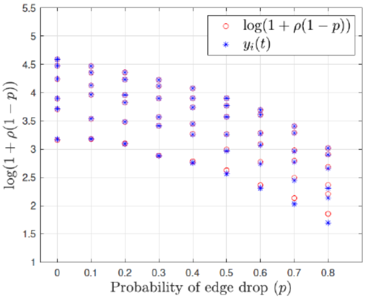

Figure: Estimated $\log(1 + \rho(1 - p))$ for a irregular time-varying graphs with $N = 100$.

Summary

- A distributed algorithm to compute ρ of the network in the presence of additive channel noise or packet loss was presented.
- Simple log-sum-exp based update to converge on ρ.
- Convergence of the algorithm and estimation error were presented, for both bipartite and non-bipartite graphs.
- The algorithm works for any connected graph structure.

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Thank you

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