Statistical Properties of a Modified Welch Method That Uses Sample Percentiles

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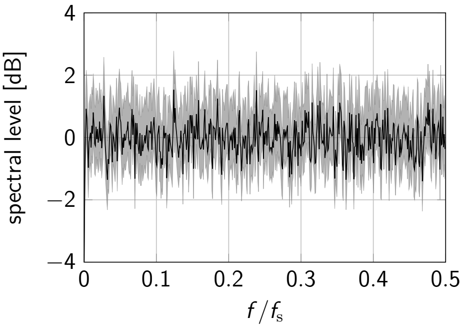
Research Effort

- **Common problem:** estimate power spectral density of noisy signals that are compromised by outliers
- **Common approach** (ocean acoustics, astrophysics, ...): use Welch's method and replace the mean averaging by a median averaging



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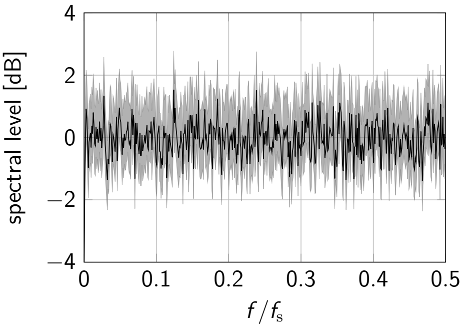
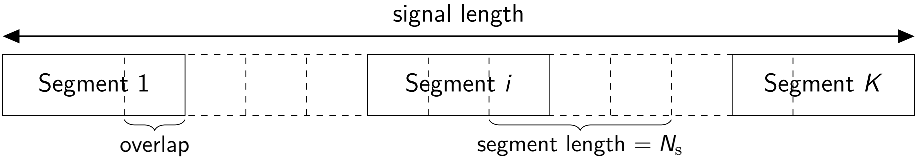
- **Common approach** (ocean acoustics, astrophysics, ...): use Welch's method and replace the mean averaging by a median averaging
- **Our work:**
  - extend Welch median idea to a more general Welch percentile estimator
  - derive its statistical properties
  - use results to derive confidence intervals
  - analyze effect of outliers on percentile estimator



Welch's Overlapped Segment Averaging (WOSA) Method



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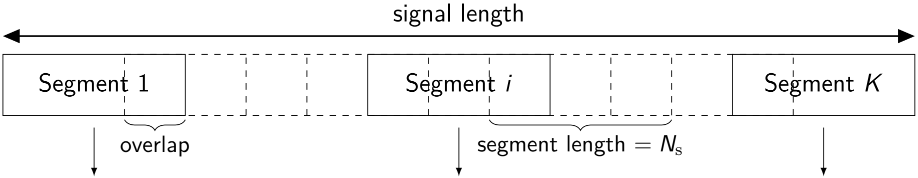

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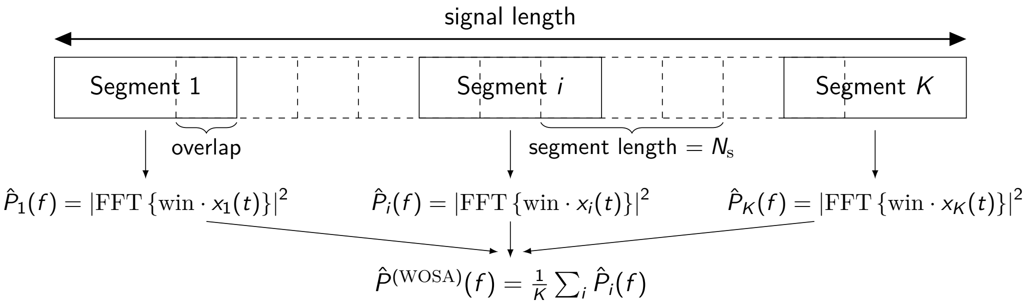

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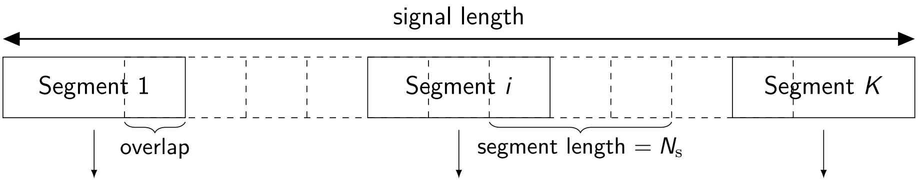

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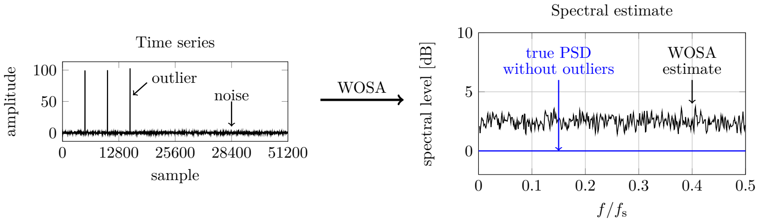
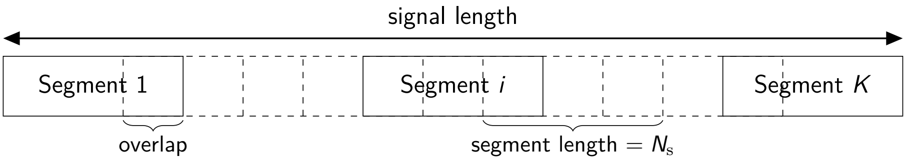


Welch Percentile (WP) Estimator



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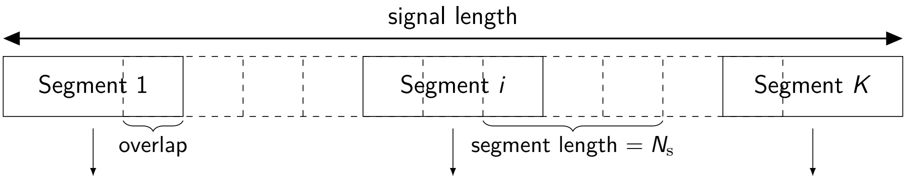

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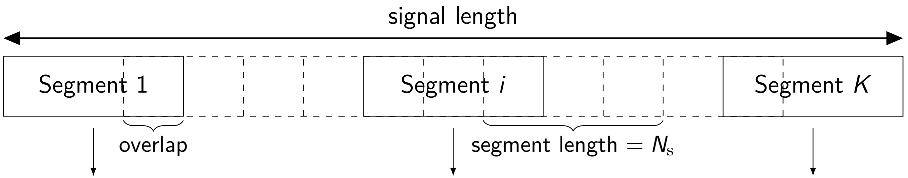

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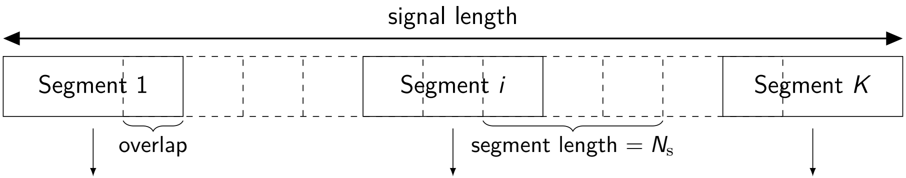

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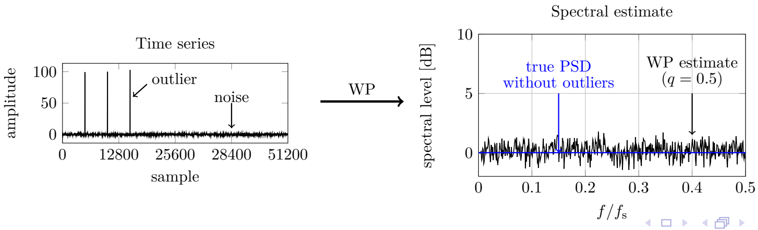

Statistical Properties of WP Estimator

- **Bias of Quantile:**

$$b = \psi(K+2) - \psi(K(1-q)+1)$$

$$\psi(n) \approx \ln(n) - \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} - \frac{1}{252n^6}$$

Variables

- b – bias
- K – number of periodograms
- q – q^{th} quantile
- ψ – digamma function
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- P – true PSD

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$$\mathrm{var}\left\{\hat{P}_q^{(\mathrm{WP})}\right\} = \frac{P^2}{b^2}\left[\psi_1(K(1-q)+1) - \psi_1(K+2)\right]$$

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- **Limiting Properties ($K \to \infty$):**

$$b = -\ln(1-q)$$

$$\mathrm{var}\left\{\hat{P}_q^{(\mathrm{WP})}\right\} = \left(\frac{P}{b}\right)^2 \cdot \frac{q}{K(1-q)}$$

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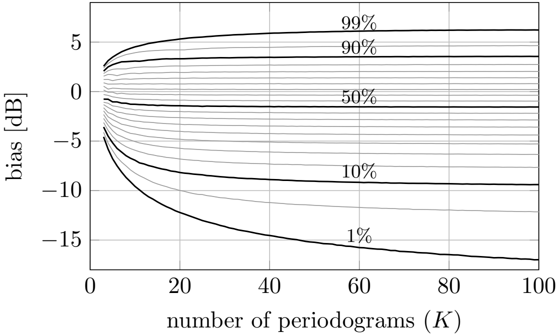
Simulation Results – Bias

no bias correction



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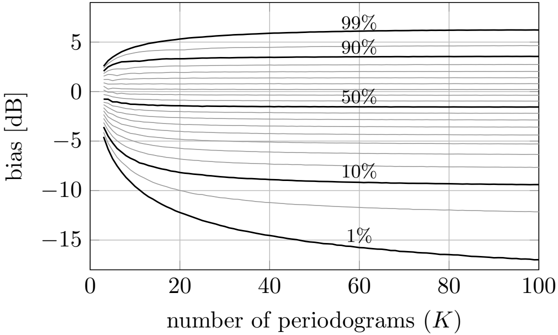
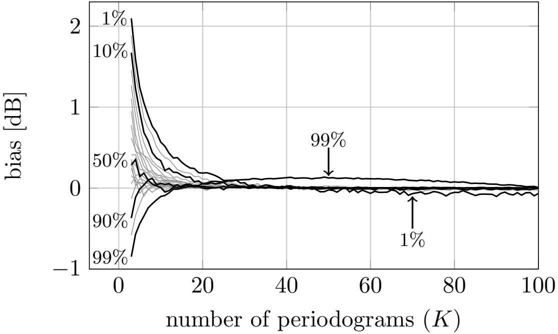
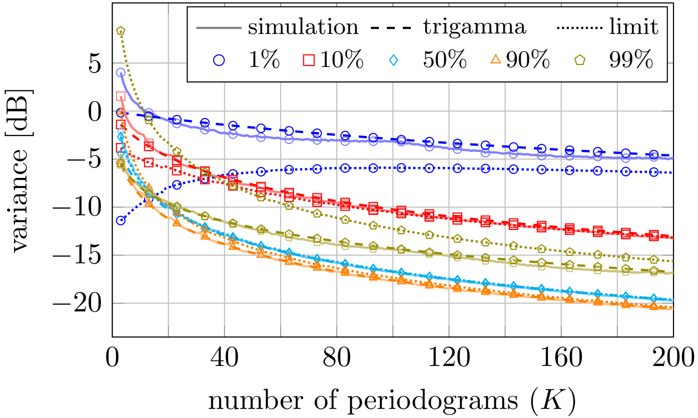


bias correction using

$$b = \psi(K+2) - \psi\left(K(1-q)+1\right)$$

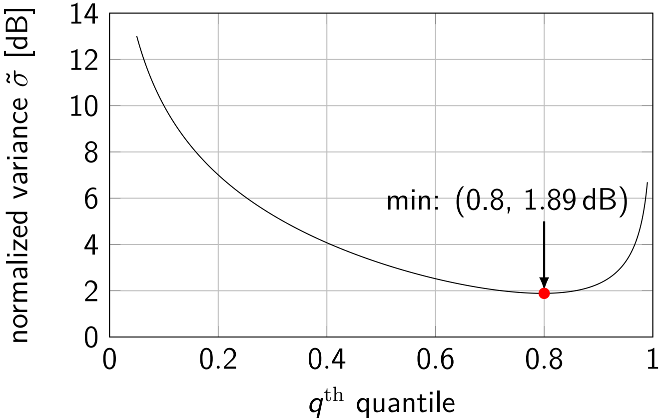


Simulation Results – Variance



- trigamma: $\text{var} = \frac{P^2}{b^2}\left[\psi_1(K(1-q)+1) - \psi_1(K+2)\right]$
- limit: $\text{var} = \left(\frac{P}{b}\right)^2 \cdot \frac{q}{K(1-q)}$

Lowest Variance Estimator



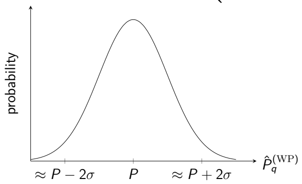
$$\tilde{\sigma} = \frac{\mathrm{var}\left\{\hat{P}_q^{(\mathrm{WP})}\right\}}{\mathrm{var}\left\{\hat{P}^{(\mathrm{WOSA})}\right\}} = \frac{q}{b^2(1-q)},$$

with $b = -\ln(1-q)$

- $\mathrm{var}\left\{\hat{P}_{0.8}^{(\mathrm{WP})}\right\} = 1.54 \cdot \mathrm{var}\left\{\hat{P}^{(\mathrm{WOSA})}\right\}$
- $\mathrm{var}\left\{\hat{P}_{0.5}^{(\mathrm{WP})}\right\} = 2.08 \cdot \mathrm{var}\left\{\hat{P}^{(\mathrm{WOSA})}\right\}$

Confidence Intervals

- $\hat{P}_q^{(\mathrm{WP})} \stackrel{\mathrm{d}}{=} P \cdot \mathcal{N}\left(1, \frac{q}{b^2 K(1-q)}\right)$



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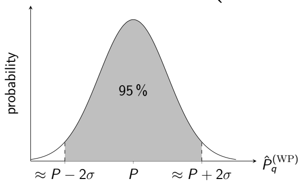

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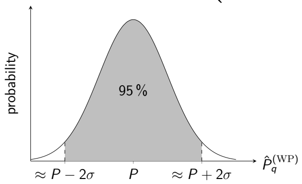
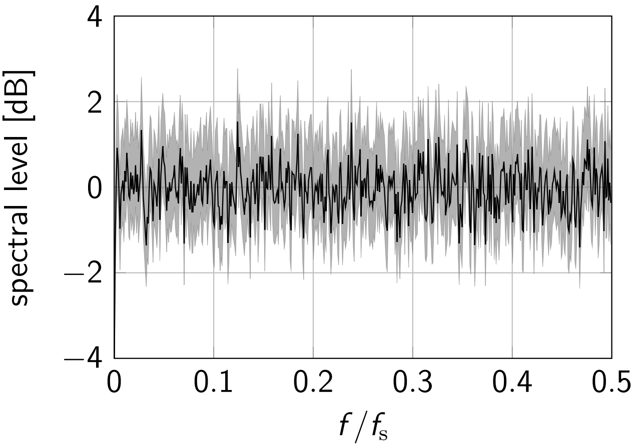


Parameters:

- $K = 100$
- $q = 0.8$
- Hann-window, 50 % overlap



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WP estimator performs well if percentage of outliers is small ($\leq 5\,\%$)

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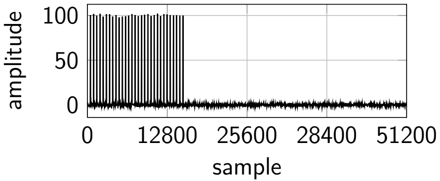
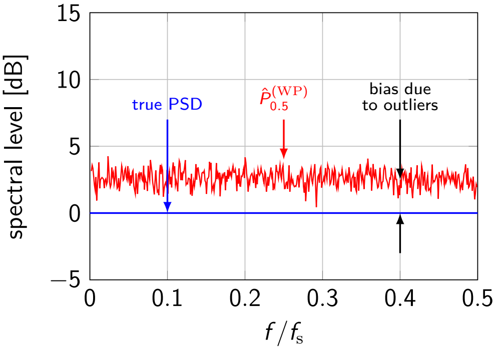
Large percentage of outliers:

- simulated white noise, $K = 100$
- $30\,\%$ of periodograms contain outliers





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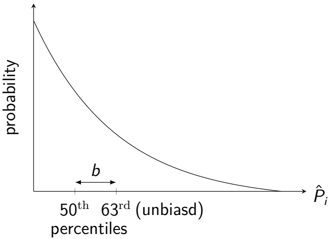

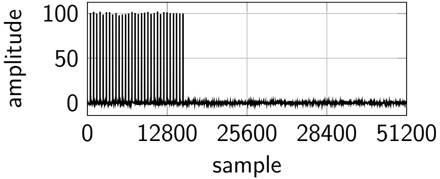

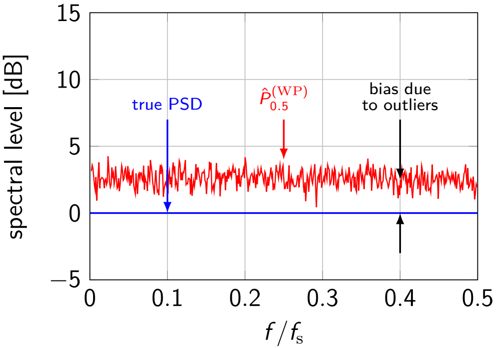

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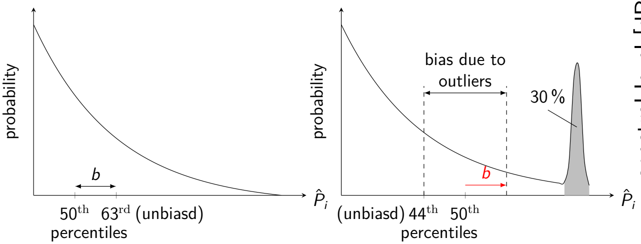
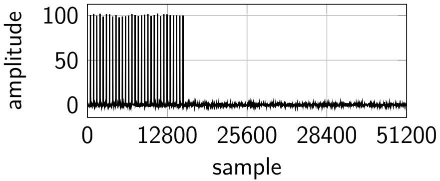
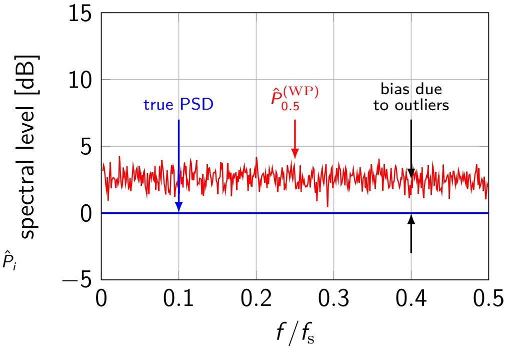
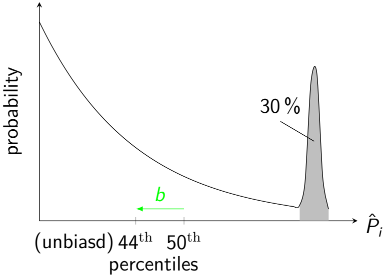






WP Estimator in Presence of Outliers (2)



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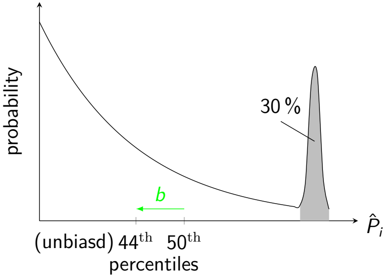

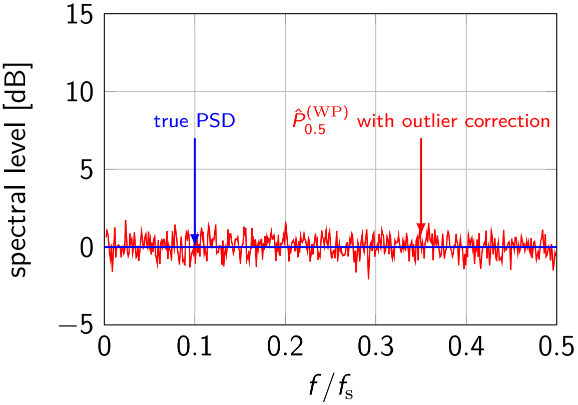

- correct for bias due to outliers by
  - $q \to \frac{q}{1-e}$, where $e =$ percentage of outliers
  - $K \to K(1-e)$

Conclusions and Future Work

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  - simple expressions for bias, variance, and limiting distribution of the Welch percentile estimator can be derived from order statistics
  - theoretical expressions show excellent agreement with simulations and can be used to derive confidence intervals
  - the 80^{th} percentile estimator shows better variance properties than the commonly used 50^{th} percentile estimator
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  - the 80^{th} percentile estimator shows better variance properties than the commonly used 50^{th} percentile estimator
  - bias and variance expressions can easily be adapted if percentage of outliers is large
- **Future Work:**
  - improve theoretical results for small number of periodograms
  - adapt estimator if percentage of outliers is unknowns