COMMUNICATION OVER BLOCK FADING CHANNELS – AN ALGORITHMIC PERSPECTIVE ON OPTIMAL TRANSMISSION SCHEMES

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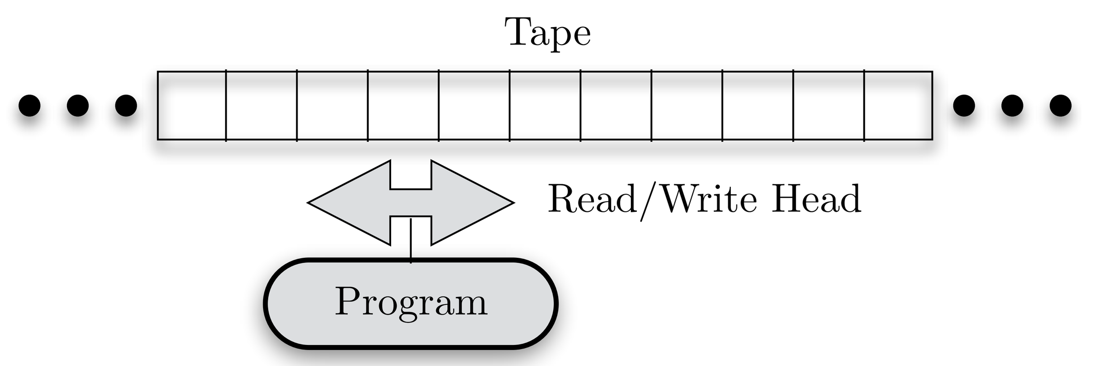
Turing Machine



Mathematical model of an abstract machine that manipulates symbols on a strip of tape according to certain given rules

- Turing machines can simulate any given algorithm and therewith provide a simple but very powerful model of computation
- **No** limitations on computational complexity, unlimited computing capacity and storage, and execute programs completely error-free

$\Rightarrow$ **Fundamental performance limits for today's digital computers**

A. M. Turing, "On computable numbers, with an application to the Entscheidungsproblem," *Proc. London Math. Soc.*, vol. 2, no. 42, pp. 230–265, 1936

A. M. Turing, "On computable numbers, with an application to the Entscheidungsproblem. A correction," *Proc. London Math. Soc.*, vol. 2, no. 43, pp. 544–546, 1937

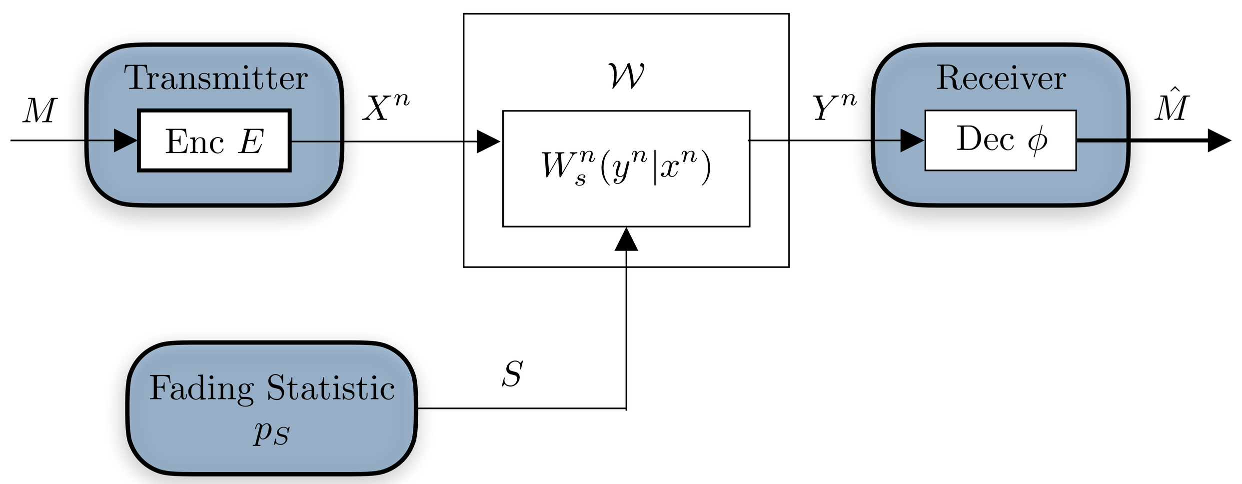
Block Fading Channel

- Provision of accurate CSI is a major challenge in wireless systems due to
  - dynamic nature of the wireless channel
  - estimation inaccuracy
  - limited feedback
  - ...

$\Rightarrow$ Imperfect CSI must be taken into account in the system design

- We consider the general uncertainty model of *block fading channels*

$\Rightarrow$ Capacity is known, but optimal signal processing and coding schemes remain unknown in general



- Let $\mathcal{S}$ be an arbitrary state (uncertainty) set
- State $s \in \mathcal{S}$ is unknown, but remains *constant* and follows the fading statistic $p_S \in \mathcal{P}(\mathcal{S})$

Definition: The *averaged channel (AC)*

$$\mathcal{W} := \{\{W_s \in \mathcal{CH}(\mathcal{X};\mathcal{Y})\}_{s\in\mathcal{S}}, p_S \in \mathcal{P}(\mathcal{S})\}$$

is given by the collection of all channels $W_s \in \mathcal{CH}(\mathcal{X};\mathcal{Y})$ for all states $s \in \mathcal{S}$ and additional probability distribution $p_S \in \mathcal{P}(\mathcal{S})$ on the state set $\mathcal{S}$.

Computability Framework

- A sequence of rational numbers $\{r_n\}_{n\in\mathbb{N}}$ is called *computable* if there exist recursive functions $a, b, s : \mathbb{N} \to \mathbb{N}$ with $b(n) \neq 0$ for all $n \in \mathbb{N}$ and

$$r_n = (-1)^{s(n)}\frac{a(n)}{b(n)}, \qquad n \in \mathbb{N}$$

- A *real number x is said to be computable* if there exists a computable sequence of rational numbers $\{r_n\}_{n\in\mathbb{N}}$ such that

$$|x - r_n| < 2^{-n} \qquad \text{for all } n \in \mathbb{N}$$

- $\mathbb{R}_c$ is the set of computable real numbers
- $\mathcal{P}_c(\mathcal{X})$ is the set of computable probability distributions (i.e., all $P \in \mathcal{P}(\mathcal{X})$ such that $P(x) \in \mathbb{R}_c$, $x \in \mathcal{X}$)
- $\mathcal{CH}_c(\mathcal{X};\mathcal{Y})$ is the set of all computable channels (i.e., for $W : \mathcal{X} \to \mathcal{P}(\mathcal{Y})$ we have $W(\cdot|x) \in \mathcal{P}_c(\mathcal{Y})$ for every $x \in \mathcal{X}$)

Definition: An AC $\mathcal{W} = \{\{W_s \in \mathcal{CH}_c(\mathcal{X};\mathcal{Y})\}_{s\in\mathcal{S}}, p_S \in \mathcal{P}(\mathcal{S})\}$ is said to be *computable* if there is a recursive function $\varphi : \mathcal{S} \to \mathcal{CH}_c(\mathcal{X};\mathcal{Y})$ with $\varphi(s) = W_s$ for all $s \in \mathcal{S}$ and p_S is a computable probability distribution. The set of all computable ACs is denoted by $\mathcal{AC}_c(\mathcal{X},\mathcal{S};\mathcal{Y})$.

$\Rightarrow$ The set $\mathcal{W}$ is algorithmically constructible, i.e., for every state $s \in \mathcal{S}$ the channel W_s can be constructed by an algorithm with input s

R. I. Soare, *Recursively Enumerable Sets and Degrees*. Berlin, Heidelberg: Springer-Verlag, 1987

Perfect CSI

Theorem: The *capacity $C(W)$* of a discrete memoryless channel (DMC) W is

$$C(W) = \max_X I(X;Y) = \max_{p\in\mathcal{P}(\mathcal{X})} I(p, W)$$

- *Entropic quantities*
- *Single-letter*
- *Convex optimization problem*
- Of particular relevance as it allows to **compute** the capacity $C(W)$ as a function of the channel W given by a convex optimization problem

- *Warm-up:* Let's see if for a computable channel $W \in \mathcal{CH}_c(\mathcal{X};\mathcal{Y})$ the capacity $C(W)$ is computable...

Theorem: Let $\mathcal{X}$ and $\mathcal{Y}$ be arbitrary finite alphabets. Then for all computable channels $W \in \mathcal{CH}_c$ we have

$$C(W) = \max_{p\in\mathcal{P}(\mathcal{X})} I(p, W) \in \mathbb{R}_c.$$

$\Rightarrow$ The **capacity** $C(W)$ for a computable channel $W \in \mathcal{CH}_c$ is computable and **can be algorithmically computed by a Turing machine**!

K. Weihrauch, *Computable Analysis - An Introduction*. Berlin, Heidelberg: Springer-Verlag, 2000

Imperfect CSI

Theorem: The *capacity $C(\mathcal{W})$* of an averaged channel $\mathcal{W}$ is

$$C(\mathcal{W}) = \sup_{p\in\mathcal{P}(\mathcal{X})} \inf_{s\in\mathcal{S}} I(p, W_s)$$

- *Analytically* well understood (closed-form single letter entropic expression)
- Surprisingly, not much known about its algorithmic computability and the optimal signal processing

$\Rightarrow$ Study its structure and algorithmic computability of optimal strategies

R. Ahlswede, "The weak capacity of averaged channels," *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, vol. 11, pp. 61–73, Mar. 1968

- Let's see if for a computable AC $\mathcal{W} \in \mathcal{AC}_c(\mathcal{X},\mathcal{S};\mathcal{Y})$ the capacity $C(\mathcal{W})$ is computable...

Theorem: Let $\mathcal{X}$ and $\mathcal{Y}$ be arbitrary finite alphabets. Then there is a computable averaged channel $\mathcal{W} \in \mathcal{AC}_c(\mathcal{X},\mathcal{S};\mathcal{Y})$ such that

$$C(\mathcal{W}) = \sup_{p\in\mathcal{P}(\mathcal{X})} \inf_{s\in\mathcal{S}} I(p, W_s) \notin \mathbb{R}_c.$$

$\Rightarrow$ Although the channel itself is computable, i.e., $\mathcal{W} \in \mathcal{AC}_c(\mathcal{X},\mathcal{S};\mathcal{Y})$, it is not possible to algorithmically compute $C(\mathcal{W})$!

Discussion

- *Computability framework based on Turing machines*
- *Computability of capacities*

$\Rightarrow$ Capacity value of DMCs is computable: $C(W) \in \mathbb{R}_c$

$\Rightarrow$ Capacity value of ACs is in general **not** computable: $C(\mathcal{W}) \notin \mathbb{R}_c$

$\Rightarrow$ Similarly, capacity value of compound channels is in general **not** computable! Details can be found in the journal version:

H. Boche, R. F. Schaefer, and H. V. Poor, "Communication under channel uncertainty: An algorithmic perspective and effective construction," *IEEE Trans. Signal Process.*, vol. 68, pp. 6224–6239, 2020

- *Search for capacity-achieving transmission schemes*
  - **Goal:** Turing machine $\mathfrak{T}(n) = (E_n^*, \phi_n^*)$ that outputs an optimal encoder E_n^* and optimal decoder ϕ_n^* providing the maximal possible rate while guaranteeing error probability ϵ

$\Rightarrow$ **Not** possible in general for ACs!
(Note that it is not required that the Turing machine depends recursively on the channel; it is only asked if it is possible to find such a search algorithm for a fixed and given channel and error)

$\Rightarrow$ Further studies on the algorithmic constructability of codes:

H. Boche, R. F. Schaefer, and H. V. Poor, "Turing meets Shannon: Algorithmic constructability of capacity-achieving codes," in *Proc. IEEE Int. Conf. Commun.*, Montreal, QC, Canada, Jun. 2021

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