Solving Fourier Phase Retrieval with a Reference Image as a Sequence of Linear Inverse Problems

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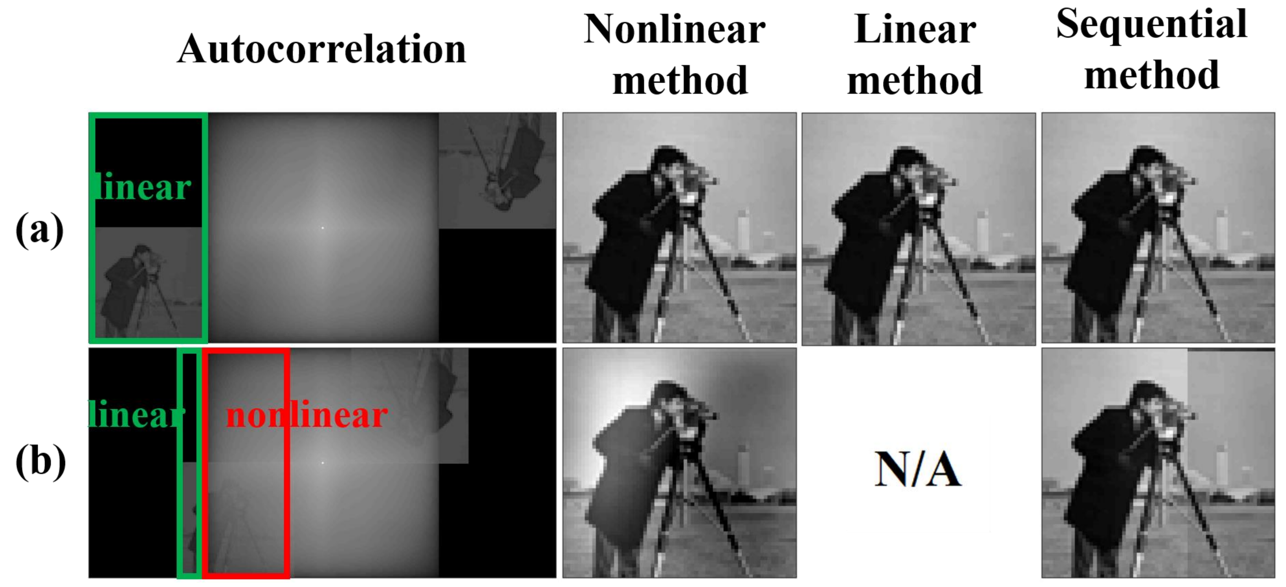
Introduction

Fourier phase retrieval problem is recovering a signal from its Fourier amplitude measurements which is also equivalent to recovering a signal from its autocorrelation. It is a nonlinear and nonconvex optimization problem and robust recovery needs either good initialization or prior information about the sparsity and support of the target signal.

Adding a known reference signal has been shown to improve the Fourier phase retrieval performance. Our work builds upon recent work on holographic phase retrieval in which the presence of a known reference signal in the image makes the recovery problem tractable [1]. In our previous work, we solved the problem using a completely nonlinear approach [2]. In this paper, we propose to solve Fourier phase retrieval with side information as a sequence of deconvolution problems assuming that some parts of the image are known as a reference and can be used to estimate the unknown parts.



Problem Formulation

- Fourier phase retrieval can be written as the following nonlinear deconvolution problem where X is the unknown image, R is the known 2D observed autocorrelation measurements of X and the operator $\star$ denotes 2D cross-correlation operator

$$\min_{\mathbf{X}} \ \|\mathbf{R} - \mathbf{X} \star \mathbf{X}\|_2^2,$$

The vectorized version can be written as $\text{vec}(\mathbf{R}) = C_{\mathbf{X}} \text{vec}(\mathbf{X}),$

Then, we have

$$\begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \\ \vdots \\ \mathbf{r}_q \end{bmatrix} = \begin{bmatrix} C_{\mathbf{x}_q} & 0 & 0 & \dots & 0 \\ C_{\mathbf{x}_{q-1}} & C_{\mathbf{x}_q} & 0 & \dots & 0 \\ C_{\mathbf{x}_{q-2}} & C_{\mathbf{x}_{q-1}} & C_{\mathbf{x}_q} & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ C_{\mathbf{x}_1} & C_{\mathbf{x}_2} & C_{\mathbf{x}_3} & \dots & C_{\mathbf{x}_q} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \\ \vdots \\ \mathbf{x}_q \end{bmatrix},$$

Where, $C_{\mathbf{x}_i}\mathbf{z} = \mathbf{z} \star \mathbf{x}_i$ denotes the cross-correlation with column x_i

Proposed Method

We call the proposed sequential phase retrieval method as sequential deconvolution. This method is an iterative method which linearizes the problem for estimation of two columns of the image at each step. Following summarizes the steps for estimation of columns where the first and last columns shown in green are known.

Step 1: $\mathbf{r}_2 = C^l_{\mathbf{x}_1}\mathbf{x}_{q-1} + C_{\mathbf{x}_q}\mathbf{x}_2 = \begin{bmatrix} C^l_{\mathbf{x}_1} & C_{\mathbf{x}_q} \end{bmatrix} \begin{bmatrix} \mathbf{x}_{q-1} \\ \mathbf{x}_2 \end{bmatrix},$

Step 2: $\mathbf{r}_3 - \hat{C}_{\mathbf{x}_{q-1}}\hat{\mathbf{x}}_2 \approx \begin{bmatrix} C^l_{\mathbf{x}_1} & C_{\mathbf{x}_q} \end{bmatrix} \begin{bmatrix} \mathbf{x}_{q-2} \\ \mathbf{x}_3 \end{bmatrix}.$

Step k: $\mathbf{r}_{k+1} - \sum_{i=1}^{i=k-1} \hat{C}_{\mathbf{x}_{q-k+i}}\hat{\mathbf{x}}_{i+1} \approx \begin{bmatrix} C^l_{\mathbf{x}_1} & C_{\mathbf{x}_q} \end{bmatrix} \begin{bmatrix} \mathbf{x}_{q-k} \\ \mathbf{x}_{k+1} \end{bmatrix}.$

Algorithm 1 Proposed sequential recovery method

Inputs: $\mathbf{r}_2, \dots, \mathbf{r}_{K+1}, \mathbf{x}_1, \mathbf{x}_q$, and K
for k = 1,... K **do**

$$\begin{bmatrix} \hat{\mathbf{x}}_{q-k} \\ \hat{\mathbf{x}}_{k+1} \end{bmatrix} = \begin{bmatrix} C^l_{\mathbf{x}_1} & C_{\mathbf{x}_q} \end{bmatrix}^{-1} \left(\mathbf{r}_{k+1} - \sum_{i=1}^{i=k-1} \hat{C}_{\mathbf{x}_{q-k+i}}\hat{\mathbf{x}}_{i+1}\right)$$

end for
Output: $\hat{\mathbf{x}}_2, \dots, \hat{\mathbf{x}}_{q-1}$

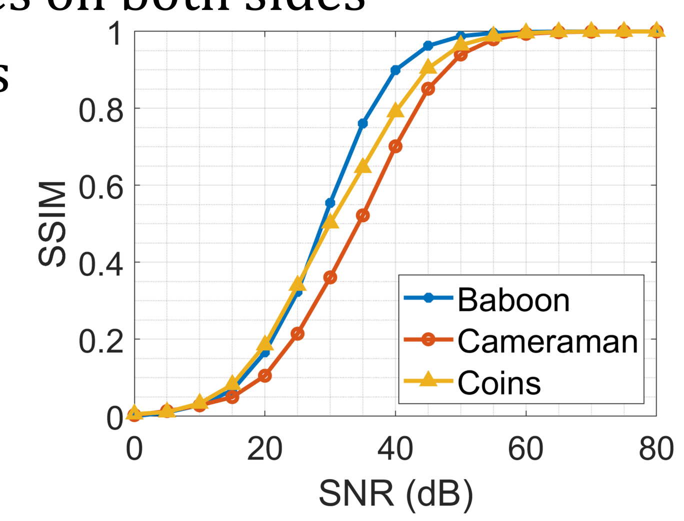
Stability and Recovery Conditions

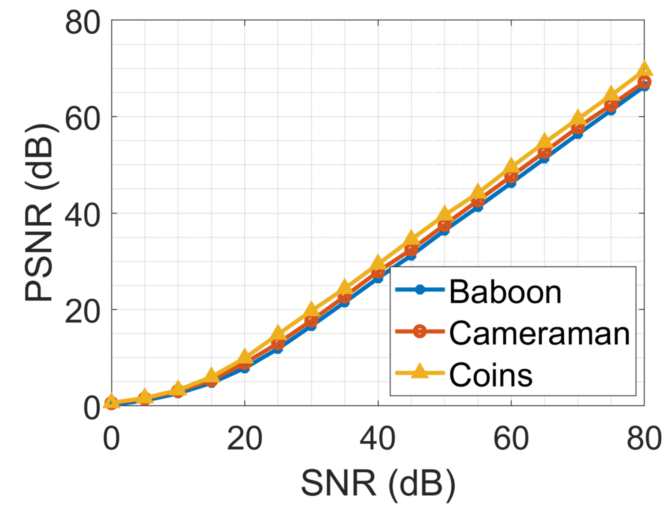
- **Stability of the algorithm at each step**
  - It depends only on the condition of matrix
  - For special references such as two pinholes at two sides of the image, H is full rank, for other references, we should make sure that it is full column rank
  - The Toeplitz matrices of the first and last columns need to have incoherent columns
- **Stability of the overall system**
  - The overall system, as shown in the system matrix, depends on the pixel values in the unknown image
  - This condition is much harder to evaluate
- **Stability of the proposed sequential method**
  - Measurement noise and finite precision of multiplications and additions in the cross-correlation terms causes an accumulation of error and if the system is not well-conditioned, it will cause instability

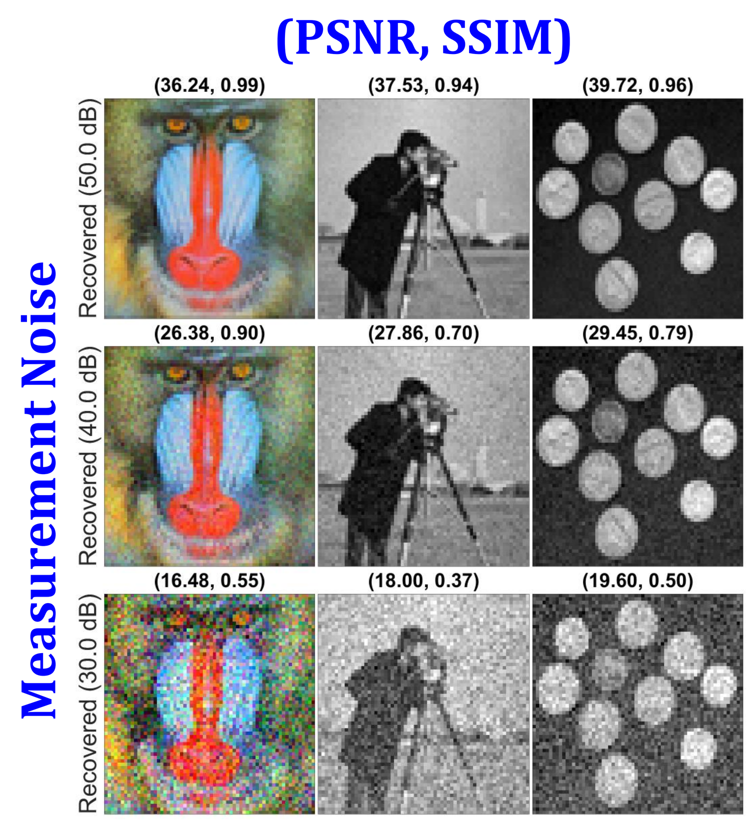
Experimental Results

The main motivation comes from "looking around the corner problem" and Fourier phase retrieval for a video sequence

- **Sequential recovery with a known border**
  - Reference border is two pinholes on both sides
  - **Dataset:** Sample natural images
  - **Other parameters**
    - Border width = 8 pixels
    - Different amounts of Gaussian measurement noise







References

[1] Barmherzig, David A. et al. "Holographic Phase Retrieval and Optimal Reference Design", Inverse problems, 2019
[2] Arab, Fahimeh .et al. "Fourier Phase Retrieval with Arbitrary Reference Signal", ICASSP 2020