Solving Fourier Phase Retrieval with a Reference Image as a Sequence of Linear Inverse Problems

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Fourier Phase Retrieval Problem

- Recovering a signal from its Fourier amplitude measurements
- Equivalent to recovering a signal from its autocorrelation
- Nonlinear and nonconvex optimization problem
- Robust recovery
  - Good initialization
  - Prior information about the sparsity or support of the target signal
- **Applications:**
  - X-ray crystallography [1], diffraction imaging [2], ptychography [3]

[1] J. Miao, T. Ishikawa, I. Robinson, and M. Murnane, *"Beyond crystallography: Diffractive imaging using coherent x-ray light sources ,"* Science (New York, N.Y.), vol.348, pp. 530–5, 05 2015
[2] J. Miao, R. L. Sandberg, and C. Song, *"Coherent x-ray diffraction imaging ,"* IEEE Journal of Selected Topics in Quantum Electronics, vol. 18, no. 1, pp. 399–410, Jan 2012.
[3] G. Zheng, R. Horstmeyer, and Ch. Yang, *"Wide-field, high-resolution Fourier ptychographic microscopy," in* Nature photonics, 2013

Phase Retrieval Methods

- Classical phase retrieval methods
  - Error reduction and alternating minimization algorithms (Fienup HIO [4] and Gerchberg-Saxton [5])
  - Exploit prior knowledge about the **non-negativity** and **support** of the target object in the scene
- Convex relaxation methods
  - PhaseLift [6] and PhaseCut [7]
  - Lifting the problem to a high-dimensional space
  - Theoretical recovery guarantees only exist for **random measurements**
- Directly solving the non-convex optimization problem
  - Wirtinger Flow [8], Amplitude Flow [9] and their variants
  - Require **careful initialization** to avoid local minima

[4] J. R. Fienup, *"Reconstruction of an object from the modulus of its Fourier transform,"* Opt. Lett. , Jul 1978
[5] R. W. Gerchberg and W. Saxton, *"A practical algorithm for the determination of phase from image and diffraction plane pictures,"* Optik, 1972
[6] E. J. Candes, et. al, *"Phaselift: Exact and stable signal recovery from magnitude measurements via convex programming,"* CPAM, 2012.
[7] I. Waldspurger, A. Aspremont, and S. Mallat, *"Phase recovery, maxcut and complex semidefinite programming,"* Mathematical Programming, Feb 2015
[8] E. J. Candes, X. Li, and M. Soltanolkotabi, "Phase retrieval via wirtinger flow: Theory and algorithms ," IEEE Transactions on Information Theory, April 2015.
[9] G. Wang, et. al *"Solving systems of random quadratic equations via truncated amplitude flow,"* IEEE Trans on Information Theory, Feb 2018

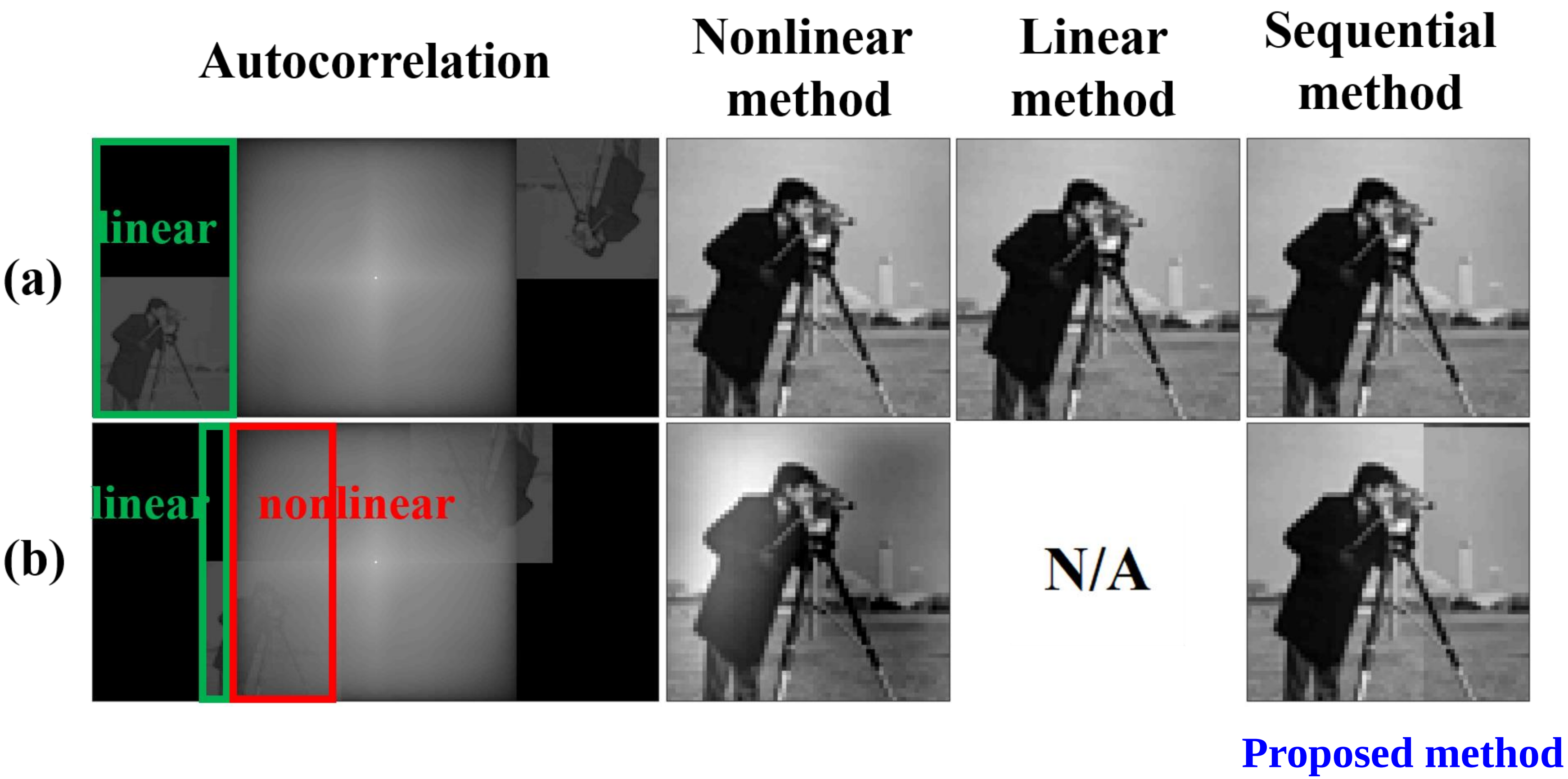
Fourier Phase Retrieval with Known Reference

- **Linear method**
  - Presence of a reference signal makes the signal recovery problem linear [10, 11, 13]
  - The existing methods only work if the support of the reference and target signals are sufficiently separated [10,11, 12, 13]
- **Nonlinear method**
  - No separation condition is needed
  - The known signal with any arbitrary size and shapes can be imposed as an image domain constraint
  - The method is a nonlinear iterative method based on Alternating Minimization and Gradient Descent [14]

[10] D. A. Barmherzig, J. Sun, E. J. Candes, T. J. Lane, and P Li, *"Holographic phase retrieval and optimal reference design ,"* Inverse Problems, 2019
[11] M. Sicairos and J. R. Fienup, *"Holography with extended reference by autocorrelation linear differential operation,"* Opt. Express , Dec 2007.
[12] M. Sicairos and J. R. Fienup, *"Direct image reconstruction from a Fourier intensity pattern using heraldo.,"* Optics letters, 2008
[13] Z. Yuan and H. Wang, *"Phase retrieval with back-ground information ,"* Inverse Problems, 2019
[14] F. Arab and M. S. Asif, *"Fourier Phase Retrieval with Arbitrary Reference Signal,"* ICASSP 2020

Fourier Phase Retrieval with Known Reference

Motivation



Problem Formulation

- Fourier phase retrieval can be written as the following nonlinear deconvolution problem

$$\min_{\mathbf{X}} \quad \|\mathbf{R} - \mathbf{X} \star \mathbf{X}\|_2^2,$$

- X is the unknown image
- R is the known 2D observed autocorrelation measurements of X
- $\star$ denotes 2D cross-correlation operator

Problem Formulation

- We can vectorize the matrices and write the autocorrelation equation as

$$\mathrm{vec}(\mathbf{R}) = C_{\mathbf{X}}\mathrm{vec}(\mathbf{X}),$$

- Where C_X is the Toeplitz matrix of X, vec(R) and vec(X) are vectorized versions of R and X. We can also rewrite is as

$$C_{\mathbf{x}_i}\mathbf{z} = \mathbf{z} \star \mathbf{x}_i$$

Cross-correlation with column x_i

$$\begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \\ \vdots \\ \mathbf{r}_q \end{bmatrix} = \begin{bmatrix} C_{\mathbf{x}_q} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ C_{\mathbf{x}_{q-1}} & C_{\mathbf{x}_q} & \mathbf{0} & \dots & \mathbf{0} \\ C_{\mathbf{x}_{q-2}} & C_{\mathbf{x}_{q-1}} & C_{\mathbf{x}_q} & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ C_{\mathbf{x}_1} & C_{\mathbf{x}_2} & C_{\mathbf{x}_3} & \dots & C_{\mathbf{x}_q} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \\ \vdots \\ \mathbf{x}_q \end{bmatrix},$$

Proposed Method: Sequential Deconvolution Method

- **Reference as known columns on both sides**

Step 1:

$$\mathbf{r}_2 = C^l_{\mathbf{x}_1}\mathbf{x}_{q-1} + C_{\mathbf{x}_q}\mathbf{x}_2 = \begin{bmatrix} C^l_{\mathbf{x}_1} & C_{\mathbf{x}_q} \end{bmatrix} \begin{bmatrix} \mathbf{x}_{q-1} \\ \mathbf{x}_2 \end{bmatrix},$$

Step 2:

$$\mathbf{r}_3 - \hat{C}_{\mathbf{x}_{q-1}}\hat{\mathbf{x}}_2 \approx \begin{bmatrix} C^l_{\mathbf{x}_1} & C_{\mathbf{x}_q} \end{bmatrix} \begin{bmatrix} \mathbf{x}_{q-2} \\ \mathbf{x}_3 \end{bmatrix}.$$

Step k:

$$\mathbf{r}_{k+1} - \sum_{i=1}^{i=k-1} \hat{C}_{\mathbf{x}_{q-k+i}}\hat{\mathbf{x}}_{i+1} \approx \begin{bmatrix} C^l_{\mathbf{x}_1} & C_{\mathbf{x}_q} \end{bmatrix} \begin{bmatrix} \mathbf{x}_{q-k} \\ \mathbf{x}_{k+1} \end{bmatrix}.$$

- **Reference as known columns on one side**
  - First estimate few columns from the other sides using linear inverse problem
  - Then, everything is similar to the first scenario: estimate two columns at a time

Proposed Method

Algorithm 1 Proposed sequential recovery method

Inputs: $\mathbf{r}_2, \ldots, \mathbf{r}_{K+1}, \mathbf{x}_1, \mathbf{x}_q$, and K

for k = 1,. . . K **do**

$$\begin{bmatrix} \hat{\mathbf{x}}_{q-k} \\ \hat{\mathbf{x}}_{k+1} \end{bmatrix} = \begin{bmatrix} C^l_{\mathbf{x}_1} & C_{\mathbf{x}_q} \end{bmatrix}^{-1} \left(\mathbf{r}_{k+1} - \sum_{i=1}^{i=k-1} \hat{C}_{\mathbf{x}_{q-k+i}} \hat{\mathbf{x}}_{i+1} \right)$$

end for

Output: $\hat{\mathbf{x}}_2, \ldots, \hat{\mathbf{x}}_{q-1}$

K: depends on the number of columns in the image

We estimate two columns at each iteration

Stability and Recovery Conditions

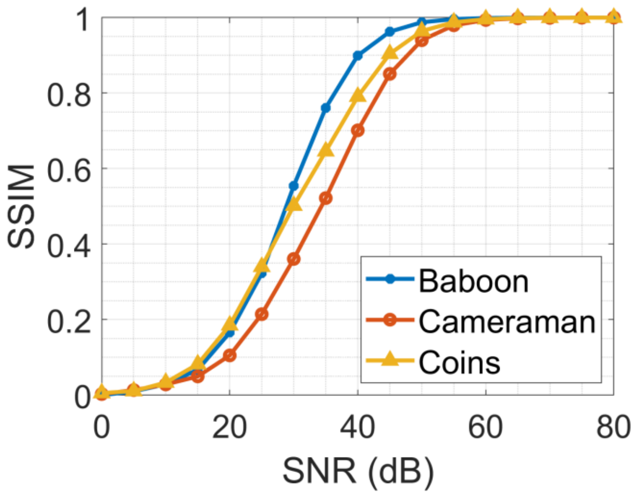
- **Stability of the algorithm at each step**
  - It depends only on the condition of matrix $H = \begin{bmatrix} C^l_{\mathbf{x}_1} & C_{\mathbf{x}_q} \end{bmatrix}$
  - For special references such as two pinholes at two sides of the image, H is full rank, for other references, we should make sure that it is full column rank
  - The Toeplitz matrices of the first and last columns need to have incoherent columns
- **Stability of the overall system**
  - The overall system, as shown in the system matrix, depends on the pixel values in the unknown image
  - This condition is much harder to evaluate
- **Stability of the proposed sequential method**
  - Presence of measurement noise and finite precision of multiplications and additions in the cross-correlation terms causes an accumulation of error
  - If the system is not well-conditioned, it will cause instability

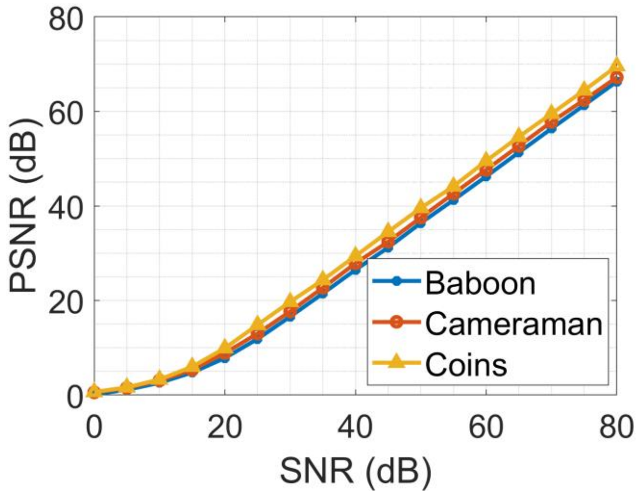
Experiments

- **Motivation**
  - The main motivation comes from **“looking around the corner problem”**
    - The reflectivity of the target objects in the scene that are hidden from the view is estimated
    - We are given the autocorrelation of the entire scene (objects within the direct line of sight and those hidden around the corner)
    - In our experiments, some parts of the scene such as the background or border around the object are known apriori
  - Another motivation is related to the **Fourier phase retrieval for a video sequence**
    - We may perfectly know parts of the scene that are static (e.g., background) and can be incorporated as side information

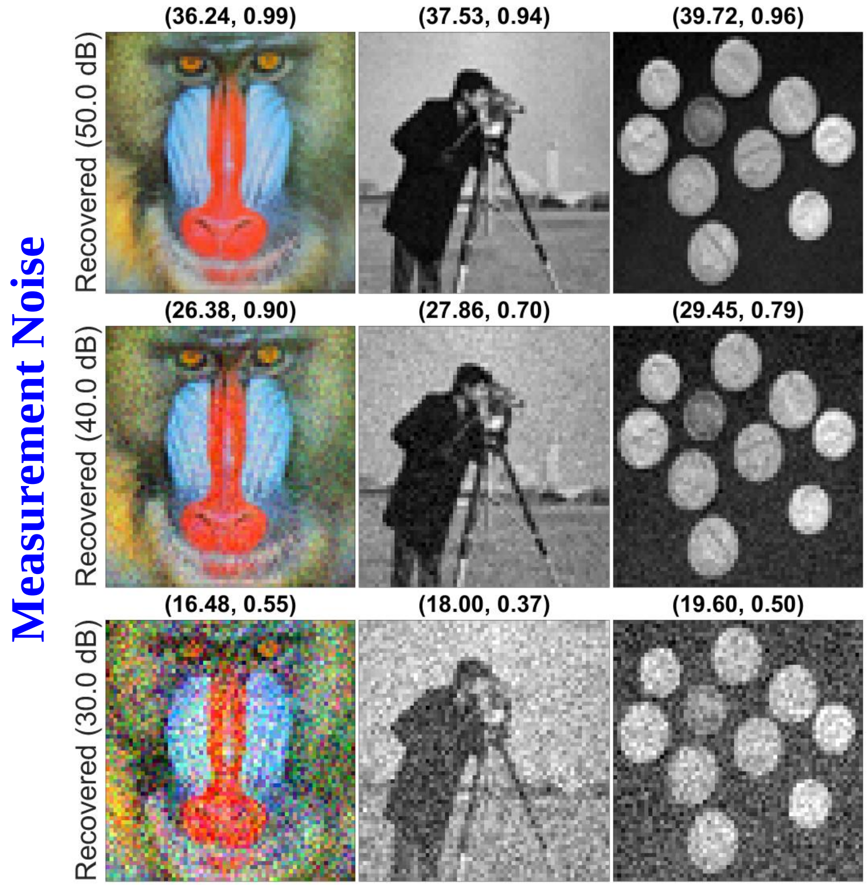
Simulation Results

- **Sequential recovery with a known border**
  - First scenario: reference border is two pinholes on both sides
  - **Dataset**
    - Sample natural images
  - **Other parameters**
    - Border width = 8 pixels
    - Different amounts of Gaussian measurement noise





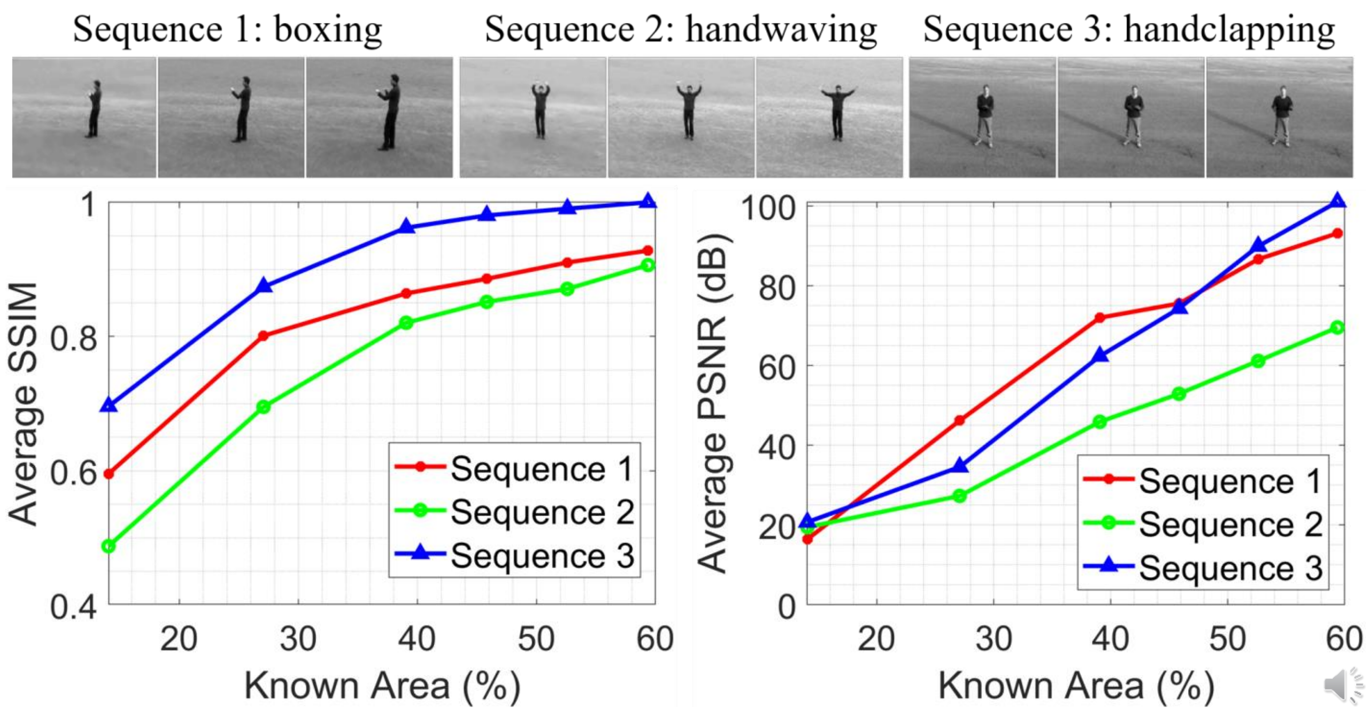
(PSNR, SSIM)



Simulation Results

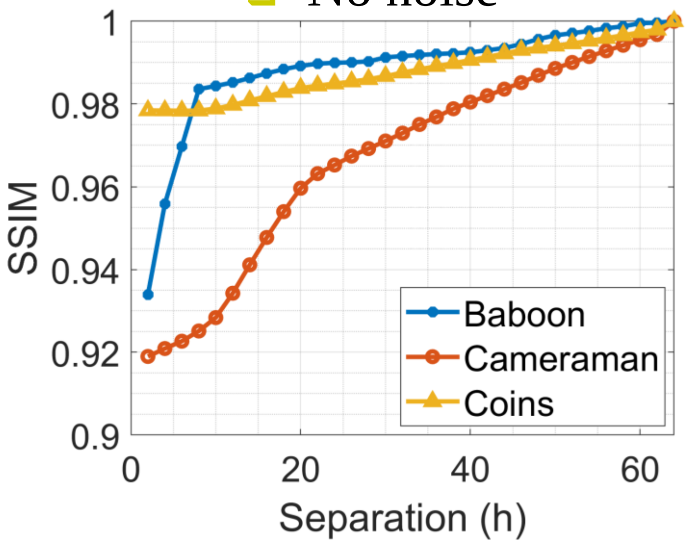
- **Sequential recovery with a known border**
  - Second scenario: a border with variable width around the images is known
  - **Dataset**
    - Frames from three sequences of KTH dataset
  - **Other parameters**
    - Border width = variable
    - No noise

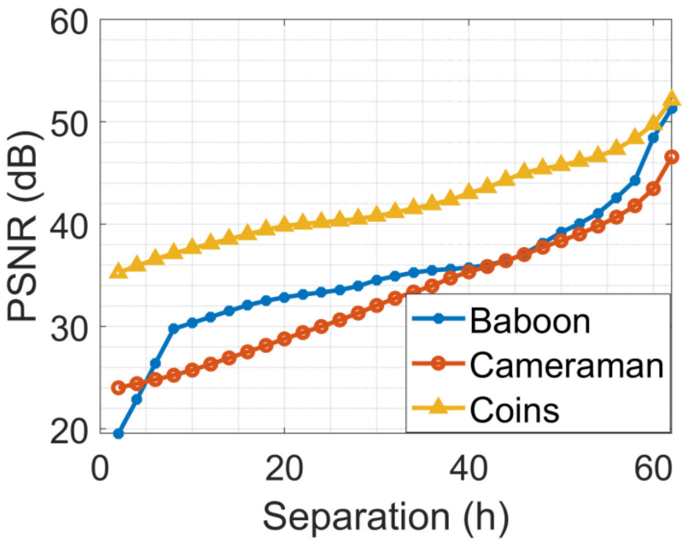
Known area = ratio of the pixels in the known border to the total number of pixels



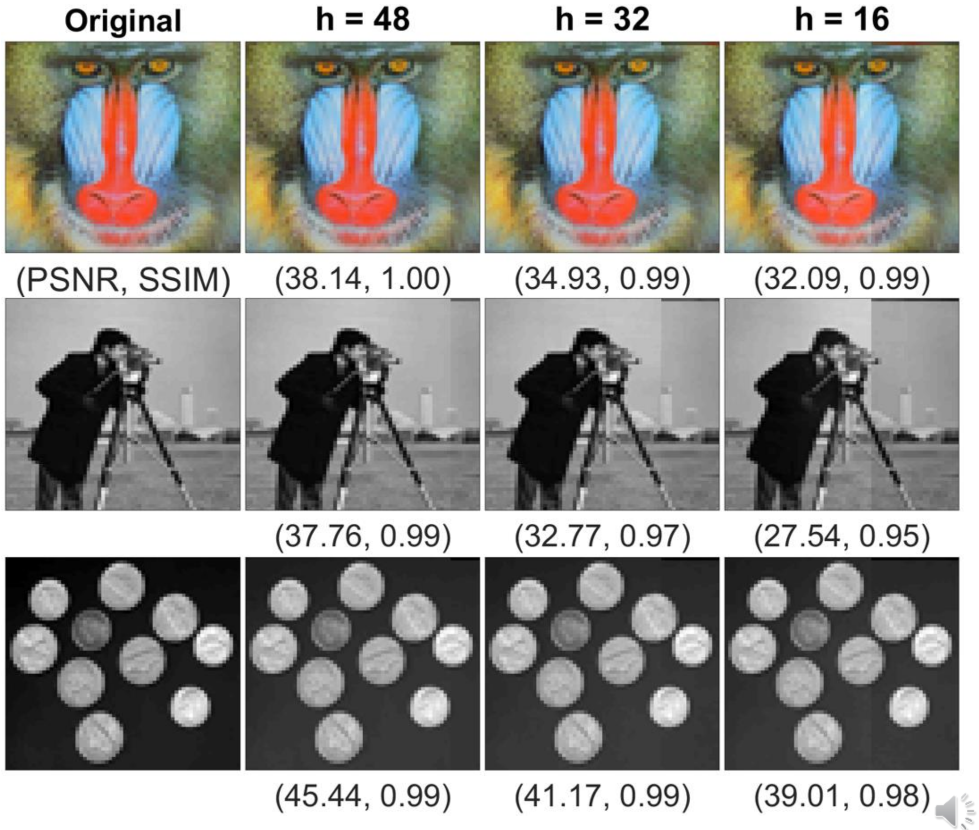
Simulation Results

- Sequential recovery with a known patch
  - **Dataset**
    - Sample natural images
  - **Other parameters**
    - A pinhole reference is added to the right side of the image
    - Different amounts of separation (h)
    - No noise





Separation between unknown object and known pinhole



Simulation Results

- **Comparison to the existing methods**
  - Linear method is not applicable in this scenario
  - **Dataset**
    - Sample natural images and video frames from KTH dataset

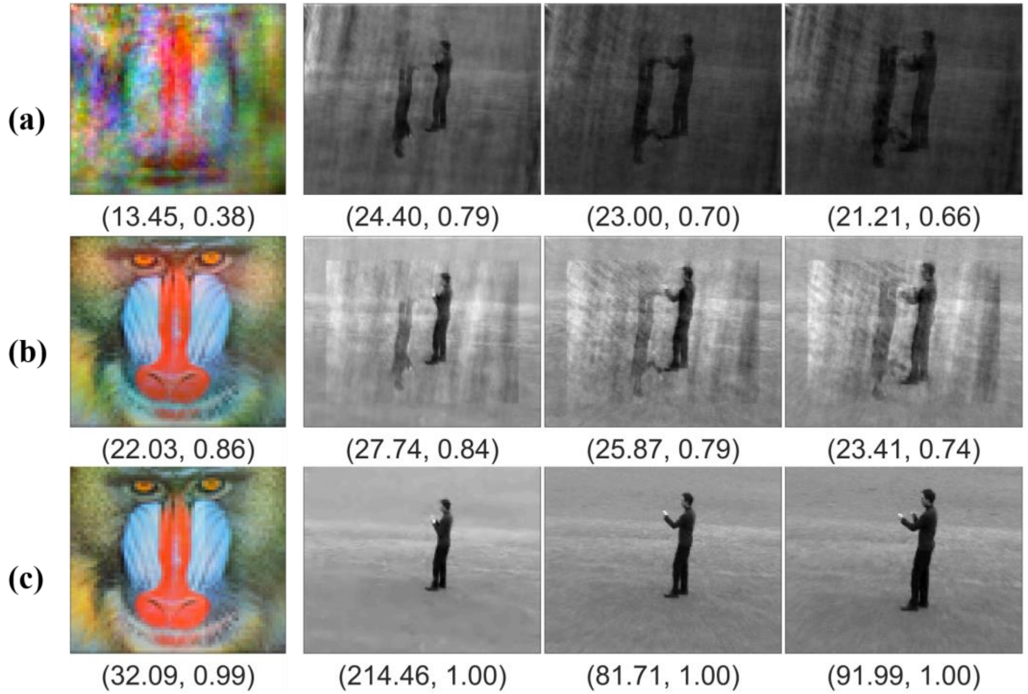
Known pinhole at h = 16

Known border: 15-pixel wide

Alternating minimization without side information

Nonlinear method

Proposed sequential method



Summary

- We proposed a sequential method to solve Fourier phase retrieval problem with a known reference
- No constraints on the separation between known reference and target image
- We solved the problem as a sequence of deconvolutions
- Simulation results showed that our method can reliably recover images from Fourier amplitude measurements under different settings for reference and measurement noise levels

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