Paper ID: 1116

Low-Rank Tensor Regression for X-Ray Tomography

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Introduction

Motivation:

- Perform tomographic reconstruction in ill-posed problem due to limited data.
- Apply tensor-based regression model to exploit the natural tensor form of the tomography while preserving its spatial correlation structure.
- Regularization for small-sample-large-parameters challenge and to stabilize the estimates.

Existing approaches:

- Analytical reconstruction techniques (e.g., filtered back projection)
- Algebraic reconstruction techniques
- Statistical algorithms (e.g., expectation maximisation)

Our contribution:

- we propose to apply the tensor regression model for tomographic reconstruction.
- Regularized version is proposed to overcome the ill-conditioned nature of tomography.

Mathematical Forward Model

- $W \in \mathbb{R}^{K \times K}$: 2D discretized object
- $\Theta, \mathcal{T}$: complete collection of $|\Theta|$ angles and $|\mathcal{T}|$ beamlets
- $\mathbb{L} = [L_{ij}^{\theta,\tau}] \in \mathbb{R}^{|\Theta||\mathcal{T}| \times K \times K}$: 3D tensor of intersection length of the beam (θ, τ) with the pixel (i, j)
- $s \in \mathbb{R}^{|\Theta||\mathcal{T}| \times 1}$: the lexicographical reordered vector of 2D measurement data (i.e., sinogram)
- $\langle \mathbb{L}, W \rangle_2$: discrete Radon projection of the object W
- Goal: reconstruct W by minimising loss function:

$$\phi(W) = \|\langle \mathbb{L}, W \rangle_2 - s\|^2$$

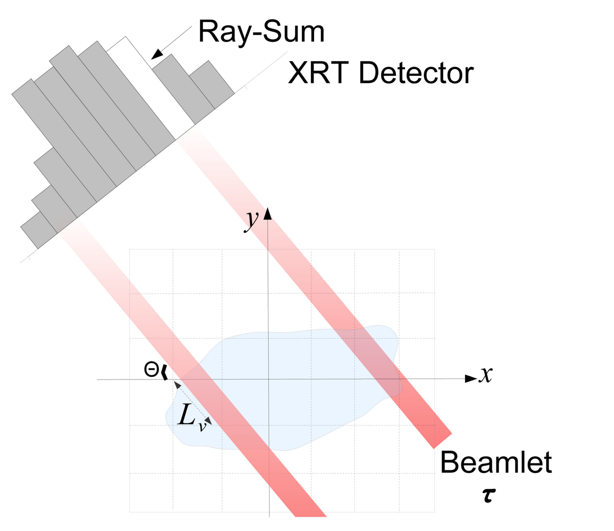

Figure 1. Discrete XRT projection geometry

Low-Rank Tensor Regression

- Vectorized approach: large-scale optimization problem due to the K^2 unknown parameters. E.g. to reconstruct 128×128 2D object requires $128^2 = 16384$ parameters.
- W with low-rank structure has rank-$\tilde{R}$ CP decomposition:

$$W = [\![\widetilde{W}_1, \widetilde{W}_2]\!] = \sum_{r=1}^{\tilde{R}} w_1^{(r)} \circ w_2^{(r)}.$$

- Low-rank approximation with $R < \tilde{R}$:

$$W \approx [\![W_1, W_2]\!] = \sum_{r=1}^{R} w_1^{(r)} \circ w_2^{(r)}, \tag{1}$$

where $W_1, W_2 \in \mathbb{R}^{K \times R}$, and $w_1^{(r)}, w_2^{(r)} \in \mathbb{R}^{K \times 1}$.

- Maximum likelihood framework using Gaussian distribution for prior model leads to tensor-based loss function:

$$\phi(W_1, W_2) = \left\| \left\langle \mathbb{L}, \sum_{r=1}^{R} w_1^{(r)} \circ w_2^{(r)} \right\rangle_2 - s \right\|^2.$$

- Alternating least squares is used to solve the resulting decomposed components

Regularized Low Rank Tensor Regression

- Tomographic reconstruction: often an ill-posed problem due to limited data.
- Regularization using prior knowledge leads to optimization of the following regularized least squares function

$$l(W_1, W_2) = \phi(W_1, W_2) + \sum_{d=1}^{2} \sum_{r=1}^{R} P_\lambda(w_d^{(r)}, \rho),$$

where P_λ is any regularization function, ρ tunes the weight applied to the regularization, and λ determines the weight of specific penalty type.

- Elastic net regularization:

$$P_\lambda(w, \rho) = \rho \left(\frac{\lambda - 1}{2} \|w\|_2^2 + (2 - \lambda)\|w\|_1 \right) \tag{2}$$

where $\lambda \in [1, 2]$.

- (2) promotes sparsity and smoothness through a convex combination of L_1 and L_2 penalties.
- (2) improves the recovery of sharp as well as smooth features of the object.

Optimization Algorithm Implementation

Algorithm 1 Low-rank tensor regression TR(R)($P_\lambda(w, \rho)$).

```
Input: s, 𝕃, W, R, λ, ρ, maximum number of iterations k_max (e.g., 100) and stopping criterion ε (e.g., 10^-4).
Initialize W_d^0 ∈ ℝ^{K×R}, for d = 1, 2.
for k = 1, 2, ..., k_max do
    W_1^k = min_{W_1} l(W_1, W_2^{k-1})
    W_2^k = min_{W_2} l(W_1^k, W_2)
    if |l(W_1^k, W_2^k) − l(W_1^{k-1}, W_2^{k-1})| < ε then
        break
    end if
end for
Output: construct W using (1) from final W_1, W_2.
```

Numerical Experiments: Setup

- Ground truth samples (see Fig. 2): a simple geometric shape consisting of circle and triangle and a MRI brain image.
- Experimental configuration: image resolution $K = 64$, of beamlets $|\mathcal{T}| = 91 > \sqrt{2}K$, $|\Theta| = 30$ angles evenly sampled within $[1, 2\pi]$.
- Error metric: root mean squared error (RMSE)

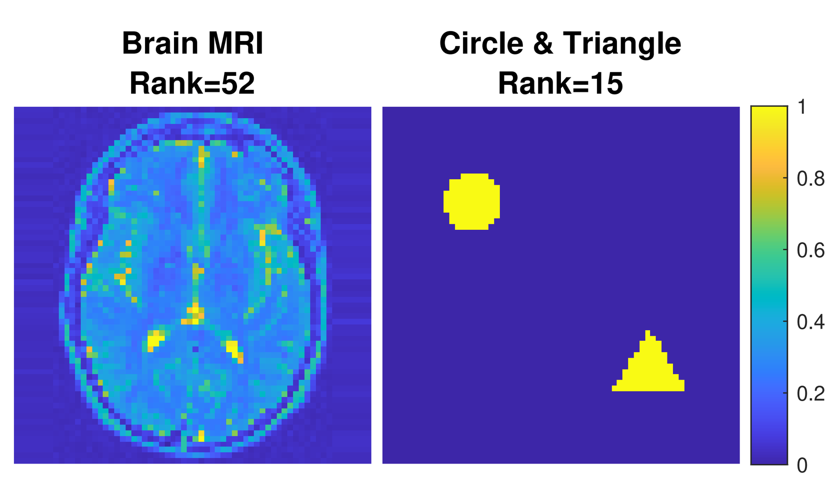

Figure 2. Test images and colormap.

References

[1] T. G. Kolda and B. W. Bader. Tensor decompositions and applications. *SIAM Review*, 51(3):455–500, 2009.

[2] H. Zhou, L. Li, and H. Zhu. Tensor regression with applications in neuroimaging data analysis. *Journal of the American Statistical Association*, 108:540–552, 2013.

Numerical Experiments: Results

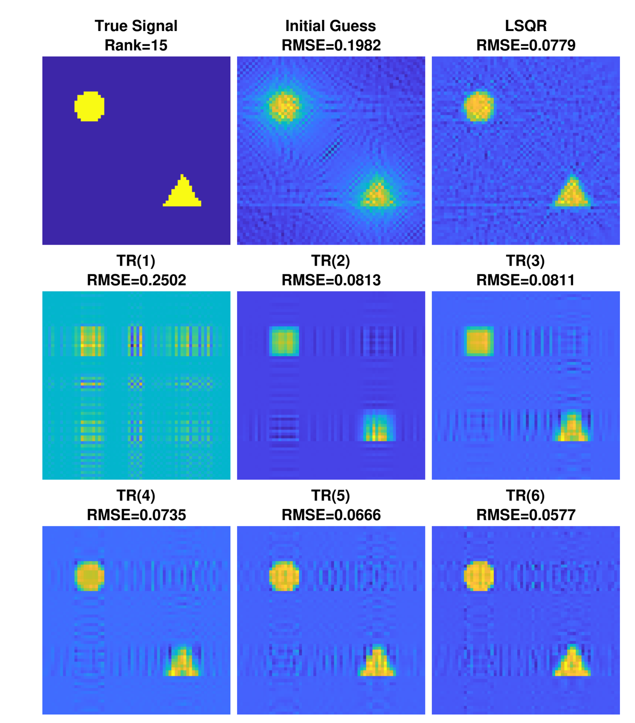

Figure 3. Reconstructed image of the simple geometric shape using unregularized TR with $R = 1, \ldots, 6$, and LSQR.

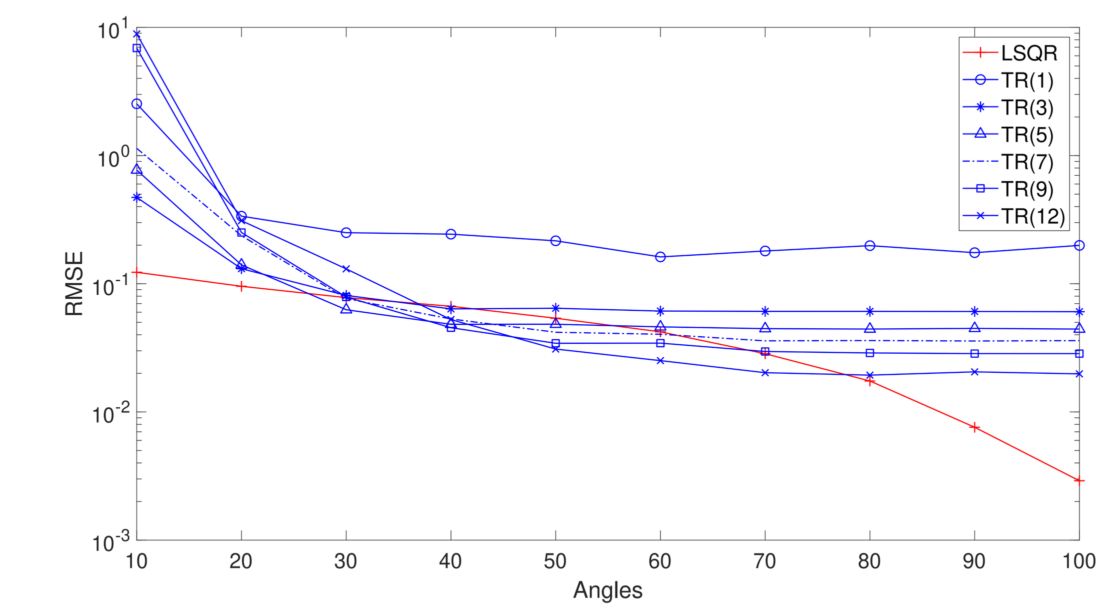

Figure 4. Comparison of RMSE with varying numbers of angles between LSQR and unregularized TR.

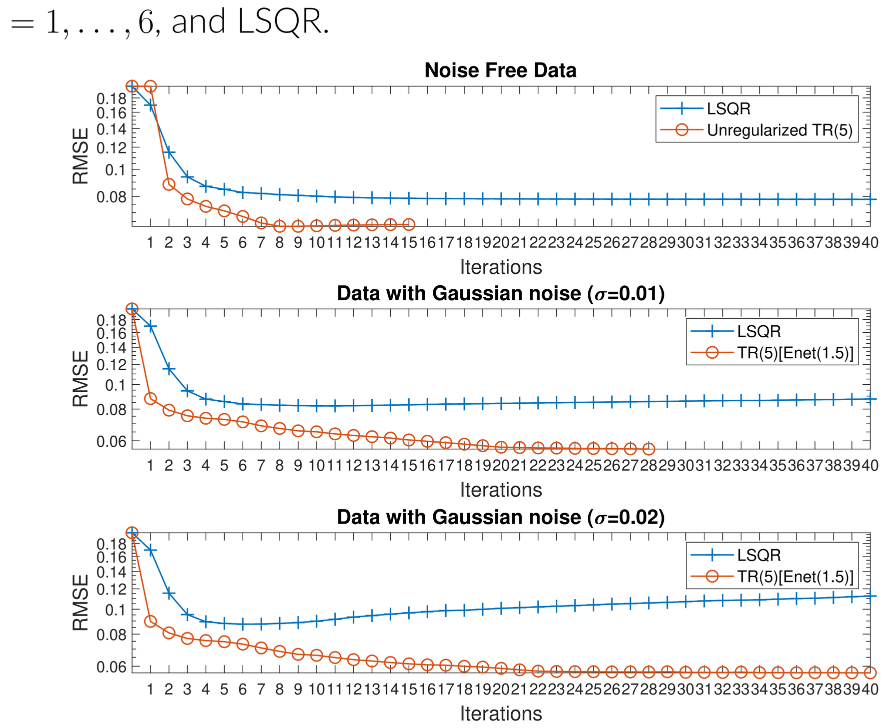

Figure 5. Iterative performance of RMSE provided by LSQR and TR(5), respectively, for recovering the simple geometric shape. Top: noise-free data; middle: 1% Gaussian noise-added data; bottom: 2% Gaussian noise-added data.

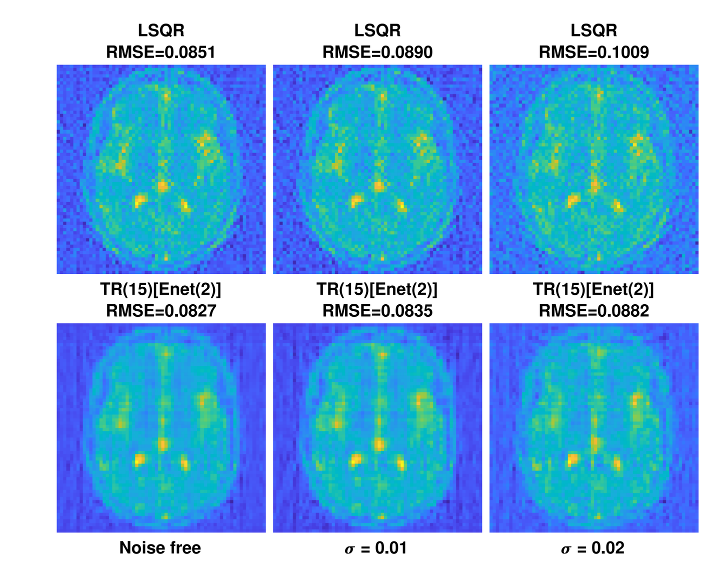

Figure 6. Reconstruction comparison of TR and LSQR for the MRI brain image. Top: LSQR; bottom: TR(15)[Enet(2)]. Left: noise-free data; middle: 1% Gaussian noise-added data; right: 2% Gaussian noise-added data.

Conclusion & Discussion

Summary:

- Exploited the underlying structure of tomography to better capture its latent multilinear structure and explored the low-rank approximation of their natural tensor form.
- Mitigated the curse of dimensionality, as well as the ill-posedness due to limited data.
- In a 2D reconstruction problem, our proposed method outperforms the linear least square solver. Further, our method is also shown to be more robust to limited number of angles and increasing levels of added noise.

Future Work

- The extension to 3D reconstruction is natural, with potentially more dramatic benefit.
- A future direction is to develop a systematic way of optimally choosing the approximation of the rank so that the computational cost and accuracy is well balanced.

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