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Lossy Compression of Gaussian Source Using Low Density Generator Matrix Codes

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Problem Statement

➢ Gaussian Source:

- Consider a **discrete-time source** $X_1, X_2, \cdots, X_k$, where $X_i, 1 \leq i \leq k$, are **independent and identically distributed (i. i. d.) according to the Gaussian distribution** with zero mean and unit variance, i.e., $X_i \sim \mathcal{N}(0,1)$.

➢ Dead-zone Scalar Quantization:

- With a dead-zone uniform scalar quantizer of P levels, a sequence of Gaussian random variables $X_t, t \geq 1$, can be represented by a sequence of vectors $\boldsymbol{U}_t = (\boldsymbol{U}_{t,1}, \boldsymbol{U}_{t,2}, \cdots, \boldsymbol{U}_{t,P})$, where $\boldsymbol{U}_{t,i} \in \mathbb{F}_3, \mathbf{1} \leq \boldsymbol{i} \leq \boldsymbol{P}$.

Problem Statement

Rate-distortion code:

- Consider a ternary $(3^{kR}, k)$-rate distortion code:
  - encoding function: $f_k\colon\ \mathcal{X}^k \to \{1, 2, \dots, 3^{kR}\}$
  - decoding (reproduction) function: $g_k\colon\ \{1, 2, \dots, 3^{kR}\} \to \mathcal{X}^k$
  - the rate of this code is R
  - the distortion measured by the mean square error (MSE) :

$$D_k = \frac{1}{k}\,\mathrm{E}\left\|X^k - \hat{X}^k\right\|^2, \text{ where } \hat{X}^k = g_k\left(f_k(X^k)\right).$$

Problem Statement

➢ Ternary Source & Entropy:

- **A ternary discrete memoryless source, denoted as**

$$\boldsymbol{U} = \boldsymbol{U_1}, \boldsymbol{U_2}, \ldots, \quad \text{where } U_t \in \mathbb{F}_3 \triangleq \{0, 1, 2\} \text{ for t} \geq 1,$$

is i.i.d. according to a probability mass function (PMF)

$$(\boldsymbol{p_0}, \boldsymbol{p_1}, \boldsymbol{p_2}) \triangleq (\mathbf{Pr}(\boldsymbol{U_t} = 0), \mathbf{Pr}(\boldsymbol{U_t} = 1), \mathbf{Pr}(\boldsymbol{U_t} = 2)).$$

- **We focus on the partial symmetric cases with $\boldsymbol{p_1} = \boldsymbol{p_2} = \frac{\boldsymbol{\theta}}{2}$ and $\boldsymbol{p_0} = 1 - \boldsymbol{\theta}$ for some positive θ < 1.**
- **The entropy of the source is defined by** $\boldsymbol{H_3(U)} = -\sum_{\boldsymbol{i}} \boldsymbol{p_i} \log_3 \boldsymbol{p_i}$.

Problem Statement

➢ Source Coding Theorem (Lossless Compression):

- Let code rate $R > H_3(U)$. Then there exist **fixed-length codes** (ϕ_n, ψ_n) such that $R_n \leq R$ but the **symbol-error rate (SER)** $\to 0$. In the case when **variable-length** codes are allowed, we can make $\mathrm{SER} = 0$.

- This can be proved by at least three methods#:

  - ☐ Typical Set
  - ☐ Method of Types
  - ☐ Random Binning

#As proved in: T. M. Cover and J. A. Thomas, *Elements of Information Theory(Second edition),* John Wiley & Sons, Inc., Hoboken, New Jersey, 2006.

Ternary Linear Block LDGM Codes

➢ Definition:

A ternary linear block code ensemble is a **(biased) random code** defined by its **generator matrix** of the form

$$\mathbf{G} = \begin{pmatrix} G_{1,1} & G_{1,2} & \cdots & G_{1,n} \\ G_{2,1} & G_{2,2} & \cdots & G_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ G_{k,1} & G_{k,2} & \cdots & G_{k,n} \end{pmatrix}$$

where $G_{i,j}(1 \leq i \leq k, 1 \leq j \leq n)$ is generated independently according to the distribution with PMF $\Pr\{G_{i,j} = 1\} = \Pr\{G_{i,j} = 2\} = \frac{\rho}{2}, \Pr\{G_{i,j} = 0\} = 1 - \rho,\ 0 < \rho < 1/2.$

- The compression rate is defined as the code rate $R \triangleq \frac{n}{k}$

➢ Theorem:

- For any given positive number $\rho \leq 1/2$, the code ensemble is ***universal*** in terms of the SER for ternary sources. That is, for any source with $H_3(U) < R$, SER $\to 0$ as $k \to \infty$.

■ **Proof:** It can be proved by the method of typical set and the maximum-likelihood decoding algorithm.

$$
\begin{aligned}
\mathrm{SER}(\boldsymbol{u}) &= \frac{\mathbf{E}[W_H(\hat{\boldsymbol{U}} - \boldsymbol{u})]}{k} \\
&= \sum_{\hat{\boldsymbol{u}}} \Pr\{\hat{\boldsymbol{u}} \text{ is the most likely}, \hat{\boldsymbol{u}}\mathbf{G} = \boldsymbol{u}\mathbf{G}\} \cdot W_H(\hat{\boldsymbol{u}} - \boldsymbol{u}) \\
&\leqslant \frac{T}{k} + \sum_{\hat{\boldsymbol{u}}: W_H(\hat{\boldsymbol{u}} - \boldsymbol{u}) \geqslant T} \Pr\{P(\hat{\boldsymbol{u}}) \geqslant P(\boldsymbol{u}), \hat{\boldsymbol{u}}\mathbf{G} = \boldsymbol{u}\mathbf{G}\} \\
&\leqslant \frac{T}{k} + \sum_{\substack{\hat{\boldsymbol{u}}:\ P(\hat{\boldsymbol{u}}) \geqslant P(\boldsymbol{u}) \\ W_H(\hat{\boldsymbol{u}} - \boldsymbol{u}) \geqslant T}} \Pr\{\hat{\boldsymbol{u}}\mathbf{G} = \boldsymbol{u}\mathbf{G}\} \\
&\leqslant \frac{T}{k} + 3^{k(H_3(U)+\epsilon)} \left(\frac{1}{3} + \epsilon\right)^n \\
&= \frac{T}{k} + 3^{k(R\log_3(\frac{1}{3}+\epsilon) + H_3(U) + \epsilon)} \xrightarrow[k \to \infty]{\text{converge}} \mathbf{0}
\end{aligned}
$$

Ternary Convolutional LDGM Codes

➤ Encoder:

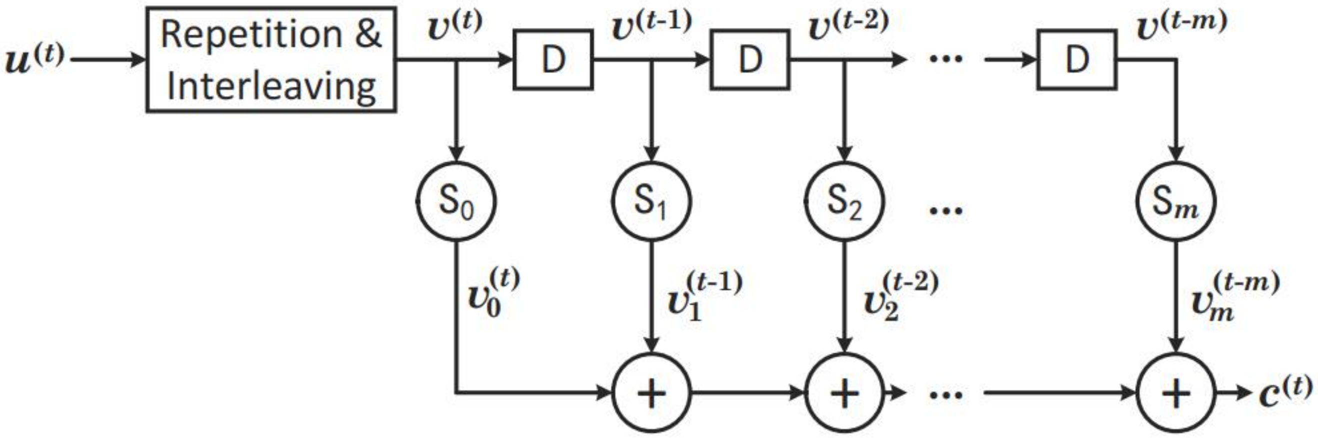

Figure: The framework of the proposed convolutional LDGM codes.

- L blocks of data for compression, $u^{(0)}, u^{(1)}, \cdots, u^{(L-1)}$
- Encoding memory $m \geq 0$
- Total code rate

$$R_L = n(L+m)/(kL) = n/k \cdot (L+m)/L \xrightarrow{L \to \infty} n/k\,.$$

Ternary Convolutional LDGM Codes

➢ Decoder:

- An iterative sliding window decoding algorithm, also viewed as an iterative message passing algorithm, is implemented for decoding.

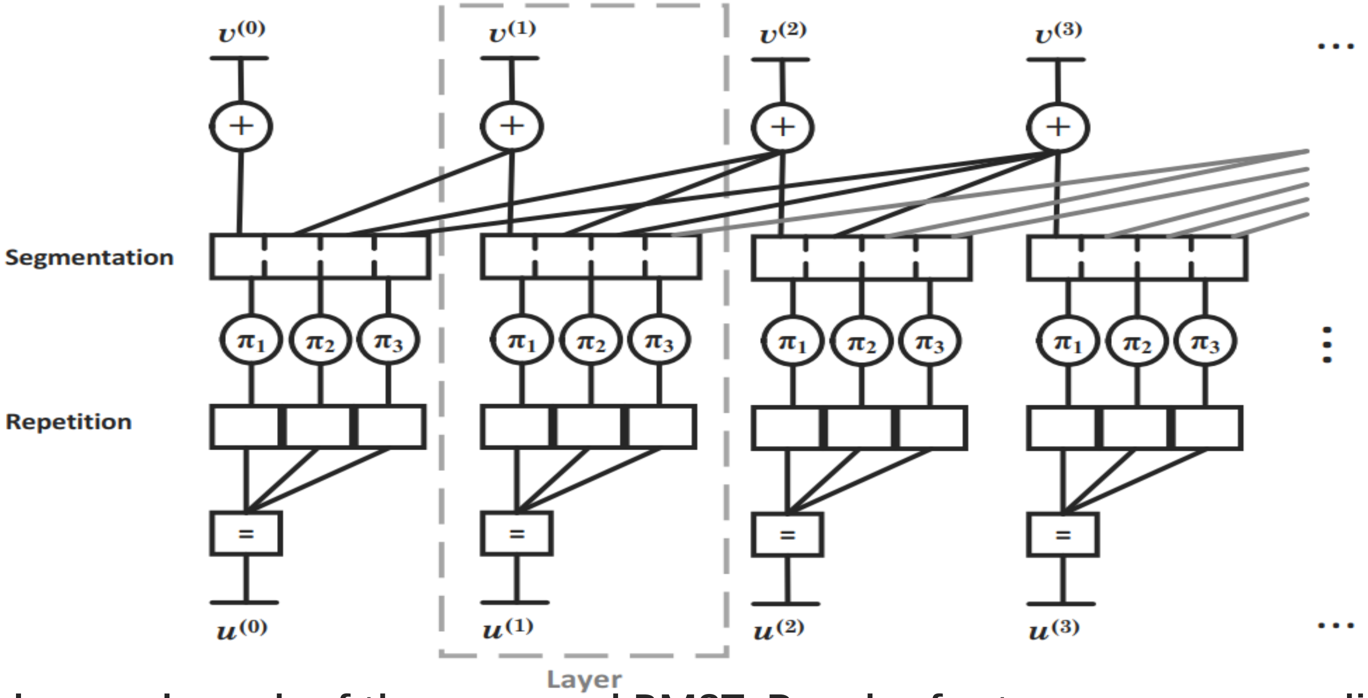

Figure: The partial normal graph of the proposed BMST-R codes for ternary source coding with an approximate rate R = n/k = 3/4, repetition degree γ = 3 and encoding memory m = 3.

Numerical Results

Example 1: Performance of the BMST-R Codes for Near-lossless Compression of Ternary Source.

- **Encoding settings**:
  - Encoding repetition degree γ = 6
  - Source block length k = 10000
  - Total number of simulated symbols of each source is 5 * 10^9.

- **BMST-R codes are universal for near-lossless compression of ternary sources.**

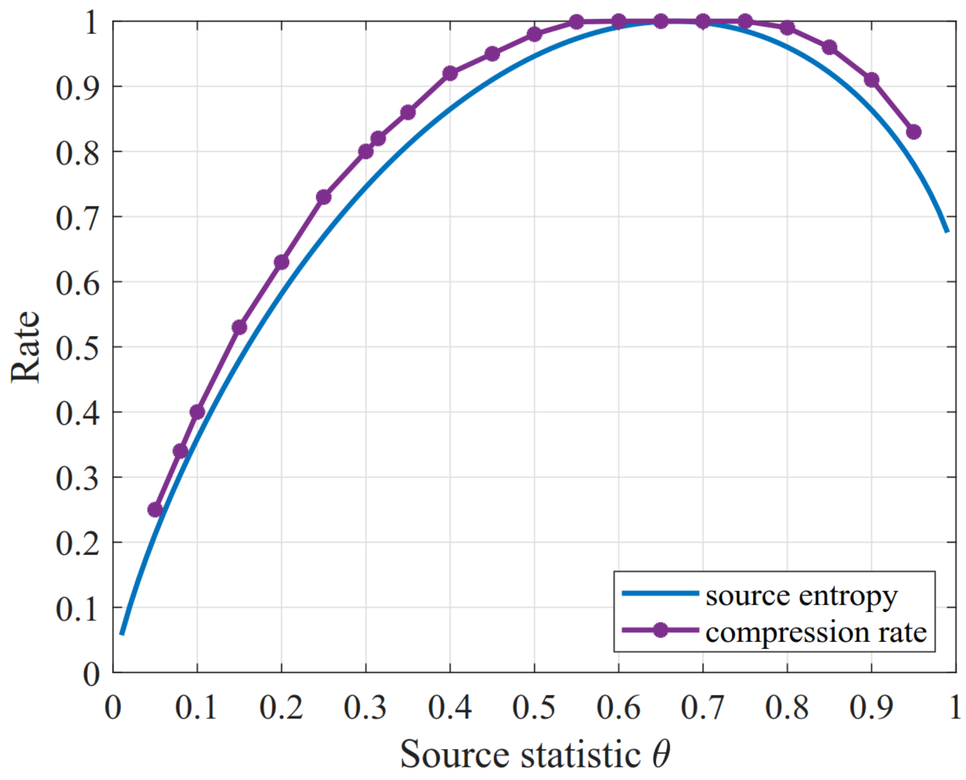

Figure: The compression rates with SER performance lower than 10^{-5}.

Numerical Results

Example 2: Performance of Dead-zone Scalar Quantization.

- As the interval Δ shrinks, the distortion decreases but the entropy increases.
- Minimizing the distortion must take into account the available bandwidth (the rate).
- The distortion is mainly caused by the quantization once the rate is higher than a certain threshold.

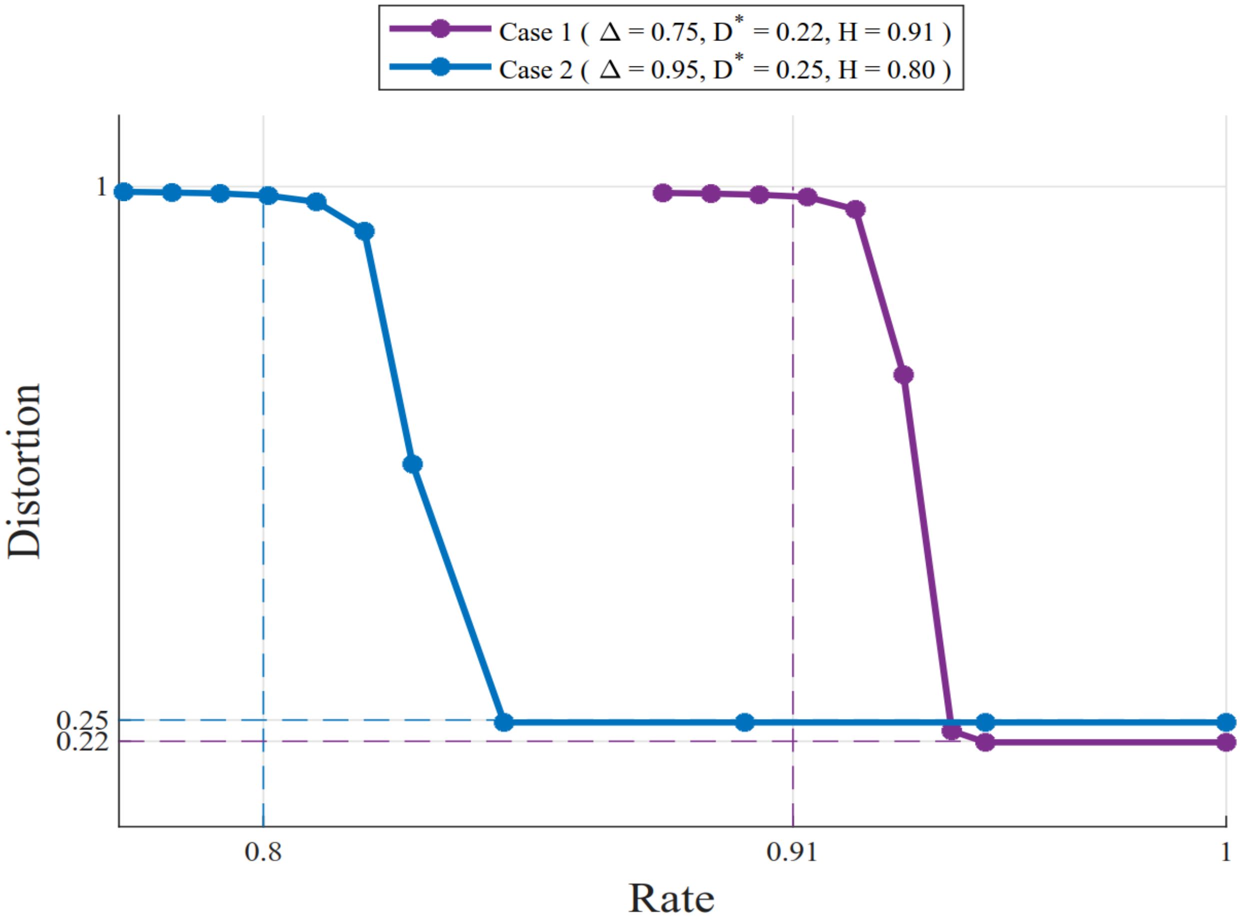

Figure: Performance of BMST-R coding at various rates for dead-zone scalar quantizers with the quantization level P = 1 but different quantization intervals Δ.

Numerical Results

Example 3: Performance of BMST-R Codes for Gaussian Source Compression.

- **The flexibility of the BMST-R codes plays an important role in this construction.**
- **The quantized sequence at the i-th level, $1 \leq i \leq P$, is compressed by a BMST-R code with rate $\mathrm{R}(U_i)$ targeting to approach the conditional entropy $\mathrm{H}(U_i|U_1,\cdots,U_{i-1})$.**
- **The decoding is accomplished sequentially level-by-level starting from $i = 1$.**
- **The total rate of this multi-level coding is given by $\mathrm{R}(U_1,\cdots,U_P) = \sum_{i=1}^{P} \mathrm{R}(U_i)$, which is near to the joint entropy $\mathrm{H}(U_1,\cdots,U_P)$.**

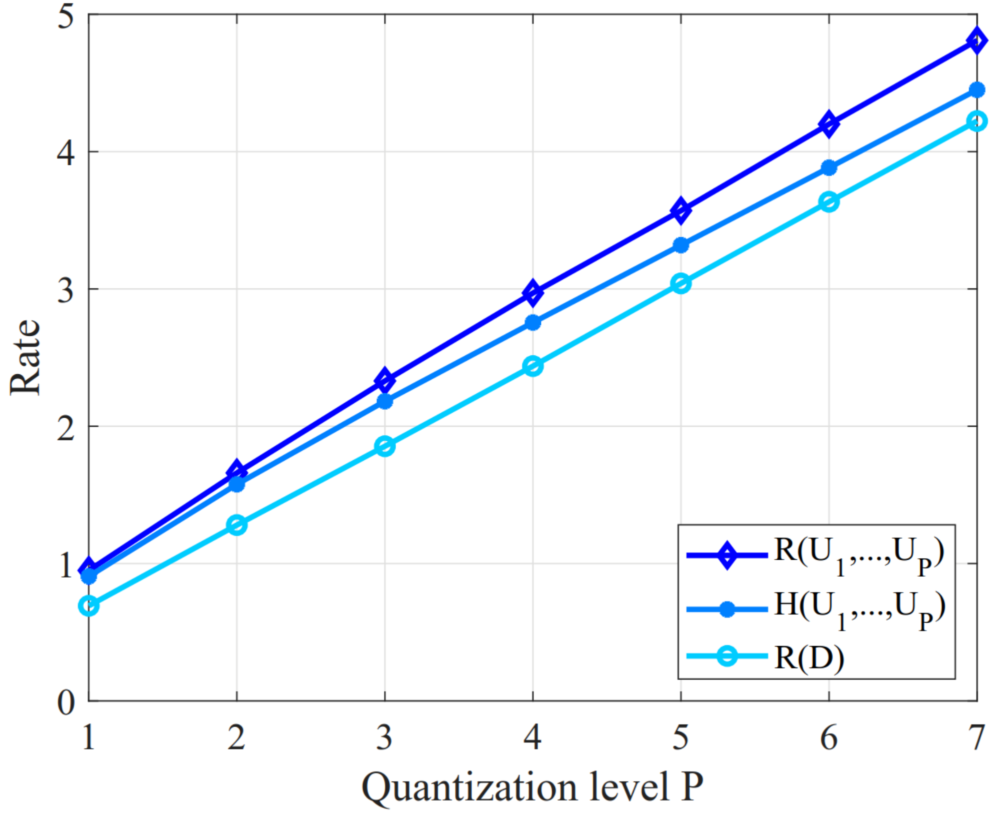

Figure: Comparison of the entropies and the rates with respect to distortion with different quantization levels.

Conclusions and Future Work

A tandem scheme for Gaussian source compression (Dead-zone scalar quantization & LDGM codes)

- ✓ **LDGM codes:** universally optimal for near-lossless compression
- ✓ **Distortion:** mainly caused by the quantization
- ✓ **Advantages:** universality and flexibility, enabling an easily configurable trade-off between bandwidth and distortion.

- ➢ Combined with modulation in 3-PAM (equivalently, 9-QAM)
- ➢ Joint source-channel coding (JSCC)

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Thank you for your attention!