SAM-1.4

Robust Adaptive Beamforming Maximizing the Worst-Case SINR over Distributional Uncertainty Sets for Random INC Matrix and Signal Steering Vector

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Signal Model and Problem Formulation

- The array output signal is given by $\boldsymbol{x}(k) = \boldsymbol{w}^H(s(k)\boldsymbol{a} + \boldsymbol{i}(k) + \boldsymbol{n}(k))$, where $\boldsymbol{w}$ is the beamvector to be optimized, $s(k)$ is the desired signal waveform, $\boldsymbol{a}$ is the steering vector associated with the signal, and $\boldsymbol{i}(k)$ and $\boldsymbol{n}(k)$ are interference and sensor noise, respectively.
- The array output SINR is equal to $\frac{\boldsymbol{w}^H\boldsymbol{a}\boldsymbol{a}^H\boldsymbol{w}}{\boldsymbol{w}^H\boldsymbol{R}_{i+n}\boldsymbol{w}}$, where $\boldsymbol{R}_{i+n}$ is the interference-plus-noise covariance (INC) matrix.
- The SINR maximization problem is equivalent to the minimization problem: $\underset{\boldsymbol{w}}{\text{minimize}}\ \boldsymbol{w}^H\boldsymbol{R}_{i+n}\boldsymbol{w}$ subject to $\boldsymbol{w}^H\boldsymbol{a}\boldsymbol{a}^H\boldsymbol{w} \geq 1$.
- The distributionally robust optimization (DRO)-based robust adaptive beamforming (RAB) problem maximizing the worst-case SINR is formulated into

$$\begin{aligned} &\underset{\boldsymbol{w}}{\text{minimize}} \ \max_{G_1\in\mathcal{D}_1} \mathsf{E}_{G_1}\{\boldsymbol{w}^H\boldsymbol{R}_{i+n}\boldsymbol{w}\} \\ &\text{subject to} \ \min_{G_2\in\mathcal{D}_2} \mathsf{E}_{G_2}\{\boldsymbol{w}^H\boldsymbol{a}\boldsymbol{a}^H\boldsymbol{w}\} \geq 1, \end{aligned} \tag{1}$$

- Here, $\mathcal{D}_1$ and $\mathcal{D}_2$ are distributional sets for the INC matrix $\boldsymbol{R}_{i+n}$ and the steering vector $\boldsymbol{a}$, respectively; they are defined as
  - $\mathcal{D}_1 = \{G_1 \in \mathcal{M}_1 \mid \mathsf{Prob}_{G_1}\{\boldsymbol{R}_{i+n} \in \mathcal{Z}_1\} = 1,\ \mathsf{E}_{G_1}\{\boldsymbol{R}_{i+n}\} \succeq \boldsymbol{0},\ \|\mathsf{E}_{G_1}\{\boldsymbol{R}_{i+n}\} - \boldsymbol{S}_0\|_F \leq \rho_1\}$, where $\mathcal{Z}_1$ is a support set and $\boldsymbol{S}_0$ is the empirical mean of $\boldsymbol{R}_{i+n}$;
  - $\mathcal{D}_2 = \{G_2 \in \mathcal{M}_2 \mid \mathsf{Prob}_{G_2}\{\boldsymbol{a} \in \mathcal{Z}_2\} = 1,\ \mathsf{E}_{G_2}\{\boldsymbol{a}\} = \boldsymbol{a}_0,\ \mathsf{E}_{G_2}\{\boldsymbol{a}\boldsymbol{a}^H\} = \boldsymbol{\Sigma} + \boldsymbol{a}_0\boldsymbol{a}_0^H\}$, where $\mathcal{Z}_2$ is a support set, and $\boldsymbol{a}_0$ and $\boldsymbol{\Sigma}$ are the mean and the covariance matrix for $\boldsymbol{a}$, respectively, under the true distribution.

Equivalent Reformulation for the DRO-based RAB Problem

- The maximization problem in the objective function with $\mathcal{Z}_1 = \{\boldsymbol{R} \in \mathcal{H}^N \mid \|\boldsymbol{R}\|_F \leq \rho_2\}$ (dropping the subscript of $\boldsymbol{R}_{i+n}$), and the minimization problem in the constraint function with $\mathcal{Z}_2 = \mathbb{C}^N$, are listed, respectively, as:

$$\begin{aligned} &\underset{G_1\in\mathcal{M}_1}{\text{maximize}} \ \int_{\mathcal{Z}_1} \boldsymbol{w}^H\boldsymbol{R}\boldsymbol{w}\, dG_1(\boldsymbol{R}) \\ &\text{subject to} \ \int_{\mathcal{Z}_1} dG_1(\boldsymbol{R}) = 1 \\ &\qquad\qquad \int_{\mathcal{Z}_1} \boldsymbol{R}\, dG_1(\boldsymbol{R}) \succeq \boldsymbol{0} \\ &\qquad\qquad \left\| \int_{\mathcal{Z}_1} \boldsymbol{R}\, dG_1(\boldsymbol{R}) - \boldsymbol{S}_0 \right\|_F \leq \rho_1. \end{aligned} \tag{2}$$

$$\begin{aligned} &\underset{G_2\in\mathcal{M}_2}{\text{minimize}} \ \int_{\mathcal{Z}_2} \boldsymbol{a}^H\boldsymbol{w}\boldsymbol{w}^H\boldsymbol{a}\, dG_2(\boldsymbol{a}) \\ &\text{subject to} \ \int_{\mathcal{Z}_2} dG_2(\boldsymbol{a}) = 1 \\ &\qquad\qquad \int_{\mathcal{Z}_2} \boldsymbol{a}\, dG_2(\boldsymbol{a}) = \boldsymbol{a}_0 \\ &\qquad\qquad \int_{\mathcal{Z}_2} \boldsymbol{a}\boldsymbol{a}^H\, dG_2(\boldsymbol{a}) = \boldsymbol{\Sigma} + \boldsymbol{a}_0\boldsymbol{a}_0^H. \end{aligned} \tag{3}$$

- Key results:
  - **Proposition 1** *The dual problem for* (2) *is cast as*

$$\begin{aligned} &\underset{\boldsymbol{X},\boldsymbol{Y}}{\text{minimize}} \ \rho_1\|\boldsymbol{X}\|_F + \rho_2\|\boldsymbol{w}\boldsymbol{w}^H + \boldsymbol{X} + \boldsymbol{Y}\|_F - \text{tr}\,(\boldsymbol{S}_0\boldsymbol{X}) \\ &\text{subject to} \ \boldsymbol{X} \in \mathcal{H}^N,\ \boldsymbol{Y} \succeq \boldsymbol{0}. \end{aligned} \tag{4}$$

  Further, the strong duality between (2) *and* (4) *holds.*
  - **Proposition 2** *The dual problem for* (3) *is given by*

$$\begin{aligned} &\underset{\boldsymbol{Z},\boldsymbol{x},x}{\text{maximize}} \ x + \Re(\boldsymbol{a}_0^H\boldsymbol{x}) + \text{tr}\,(\boldsymbol{Z}(\boldsymbol{\Sigma} + \boldsymbol{a}_0\boldsymbol{a}_0^H)) \\ &\text{subject to} \ \begin{bmatrix} \boldsymbol{w}\boldsymbol{w}^H - \boldsymbol{Z} & -\frac{\boldsymbol{x}}{2} \\ -\frac{\boldsymbol{x}^H}{2} & -x \end{bmatrix} \succeq \boldsymbol{0} \\ &\qquad\qquad \boldsymbol{Z} \in \mathcal{H}^N,\ \boldsymbol{x} \in \mathbb{C}^N,\ x \in \mathbb{R}. \end{aligned} \tag{5}$$

  Besides, the strong duality between (3) *and* (5) *holds.*
- An equivalent reformulation of the DRO-based RAB problem:

$$\begin{aligned} &\text{minimize} \ \rho_1\|\boldsymbol{X}\|_F + \rho_2\|\boldsymbol{w}\boldsymbol{w}^H + \boldsymbol{X} + \boldsymbol{Y}\|_F - \text{tr}\,(\boldsymbol{S}_0\boldsymbol{X}) \\ &\text{subject to} \ x + \Re(\boldsymbol{a}_0^H\boldsymbol{x}) + \text{tr}\,(\boldsymbol{Z}(\boldsymbol{\Sigma} + \boldsymbol{a}_0\boldsymbol{a}_0^H)) \geq 1 \\ &\qquad\qquad \begin{bmatrix} \boldsymbol{w}\boldsymbol{w}^H - \boldsymbol{Z} & -\frac{\boldsymbol{x}}{2} \\ -\frac{\boldsymbol{x}^H}{2} & -x \end{bmatrix} \succeq \boldsymbol{0} \\ &\qquad\qquad \boldsymbol{w}, \boldsymbol{x} \in \mathbb{C}^N,\ \boldsymbol{X}, \boldsymbol{Z} \in \mathcal{H}^N,\ \boldsymbol{Y} \succeq \boldsymbol{0}, x \in \mathbb{R}. \end{aligned} \tag{6}$$

- This is a nonconvex quadratic matrix inequality (QMI) problem.

Rank-One Solution Procedure for the LMI Relaxation Problem for (6)

- Fact: If $\text{tr}\,(\boldsymbol{W}) = \|\boldsymbol{W}\|_F (= \text{tr}\,(\boldsymbol{W}\boldsymbol{W})/\|\boldsymbol{W}\|_F)$ with $\boldsymbol{W} \succeq \boldsymbol{0} (\neq \boldsymbol{0})$, then $\boldsymbol{W}$ is of rank one.
- An iterative procedure to solve (6): At iteration k, the following LMI problem with a penalty term on the rank-one condition is solved:

$$\begin{aligned} &\text{minimize} \ \rho_1\|\boldsymbol{X}\|_F + \rho_2\|\boldsymbol{W} + \boldsymbol{X} + \boldsymbol{Y}\|_F - \text{tr}\,(\boldsymbol{S}_0\boldsymbol{X}) + \alpha\left(\text{tr}\,\boldsymbol{W} - \frac{\text{tr}\,(\boldsymbol{W}\boldsymbol{W}_k)}{\|\boldsymbol{W}_k\|_F}\right) \\ &\text{subject to} \ x + \Re(\boldsymbol{a}_0^H\boldsymbol{x}) + \text{tr}\,(\boldsymbol{Z}(\boldsymbol{\Sigma} + \boldsymbol{a}_0\boldsymbol{a}_0^H)) \geq 1 \\ &\qquad\qquad \begin{bmatrix} \boldsymbol{W} - \boldsymbol{Z} & -\frac{\boldsymbol{x}}{2} \\ -\frac{\boldsymbol{x}^H}{2} & -x \end{bmatrix} \succeq \boldsymbol{0} \\ &\qquad\qquad \boldsymbol{x} \in \mathbb{C}^N,\ \boldsymbol{X}, \boldsymbol{Z} \in \mathcal{H}^N,\ \boldsymbol{W} \succeq \boldsymbol{0},\ \boldsymbol{Y} \succeq \boldsymbol{0}, x \in \mathbb{R}, \end{aligned} \tag{7}$$

- The procedure terminates when $\left|\text{tr}\,\boldsymbol{W}_k - \frac{\text{tr}\,(\boldsymbol{W}_k\boldsymbol{W}_{k-1})}{\|\boldsymbol{W}_{k-1}\|_F}\right| \leq 10^{-6}$, and it can be shown that the sequence of the optimal values for (7) is descent.

Numerical Examples

Simulation setups: (i) the number of array sensors $N = 10$, the angular sector of interest $\Theta = [0°, 10°]$, the presumed direction of the desired signal $\theta_0 = 1°$, the actual direction $\theta = 5°$ (i.e. $4°$ look direction mismatch), and two interferers from directions $\theta_1 = -5°$ and $\theta_2 = 15°$ with INR=30 dB; (ii) waveform distortion in inhomogeneous medium is considered, the signal steering vector is distorted by wave propagation effects, i.e., independent-increment phase distortions are accumulated by the components of the steering vector, and the phase increments are independent Gaussian variables each with zero mean and standard deviation 0.02; (iii) in (2) $\boldsymbol{S}_0$ is the sampling covariance matrix (it is different in each run), and ρ_1 and ρ_2 are set to $0.001\|\boldsymbol{S}_0\|_F$ and 10^5, respectively, while in (7), α is set to 10^5; (iv) in (3), $\boldsymbol{a}_0 = \frac{1}{L}\sum_{l=1}^{L}\boldsymbol{d}(\theta_l)$ and $\boldsymbol{\Sigma} = \frac{1}{L}\sum_{l=1}^{L}(\boldsymbol{d}(\theta_l) - \boldsymbol{a}_0)(\boldsymbol{d}(\theta_l) - \boldsymbol{a}_0)^H$, where $\boldsymbol{d}(\theta_l)$ is the steering vector associated with θ_l that has the structure defined by the sensor array geometry, and $\{\theta_l\} \subseteq \Theta$ are picked up randomly following the uniform distribution; (v) all results are averaged over 200 simulation runs.

Three beamformers are compared and they are the proposed beamformer, the LRST beamformer in Ref. 13, and the ZLGL beamformer in Ref. 10.

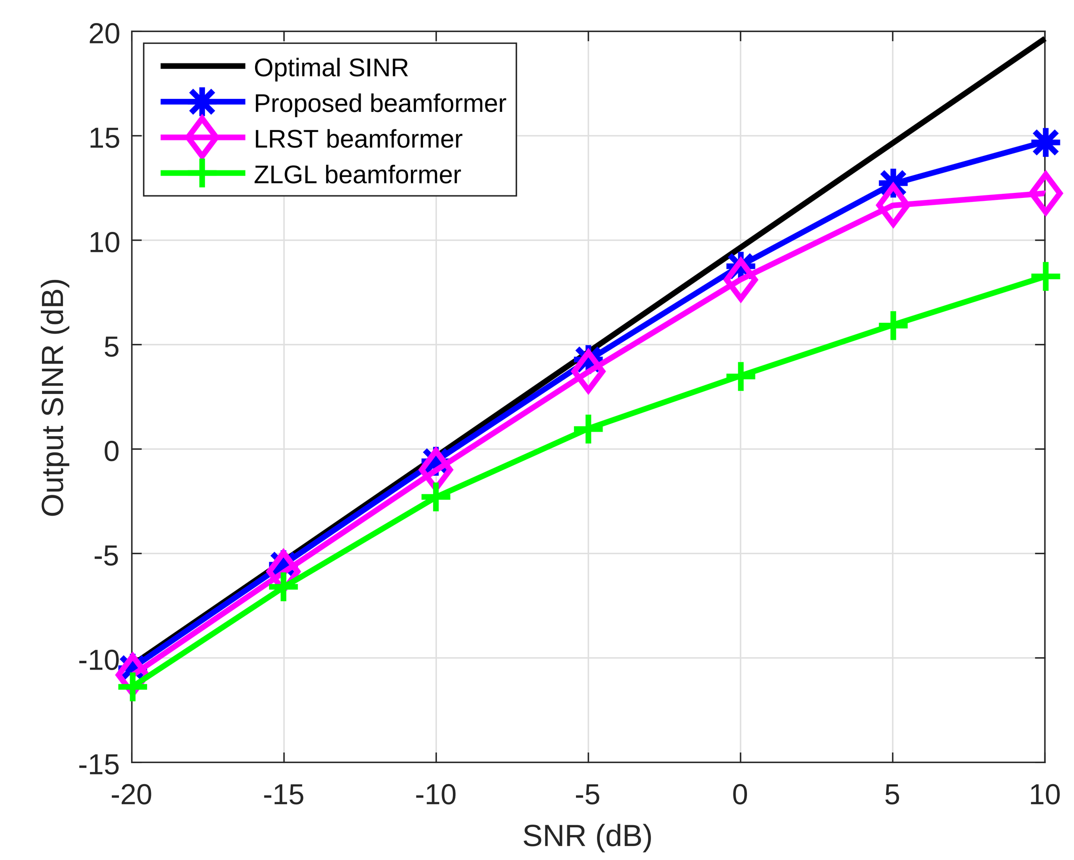

Figure 1: Average array output SINR versus SNR with the number of snapshots $T = 100$

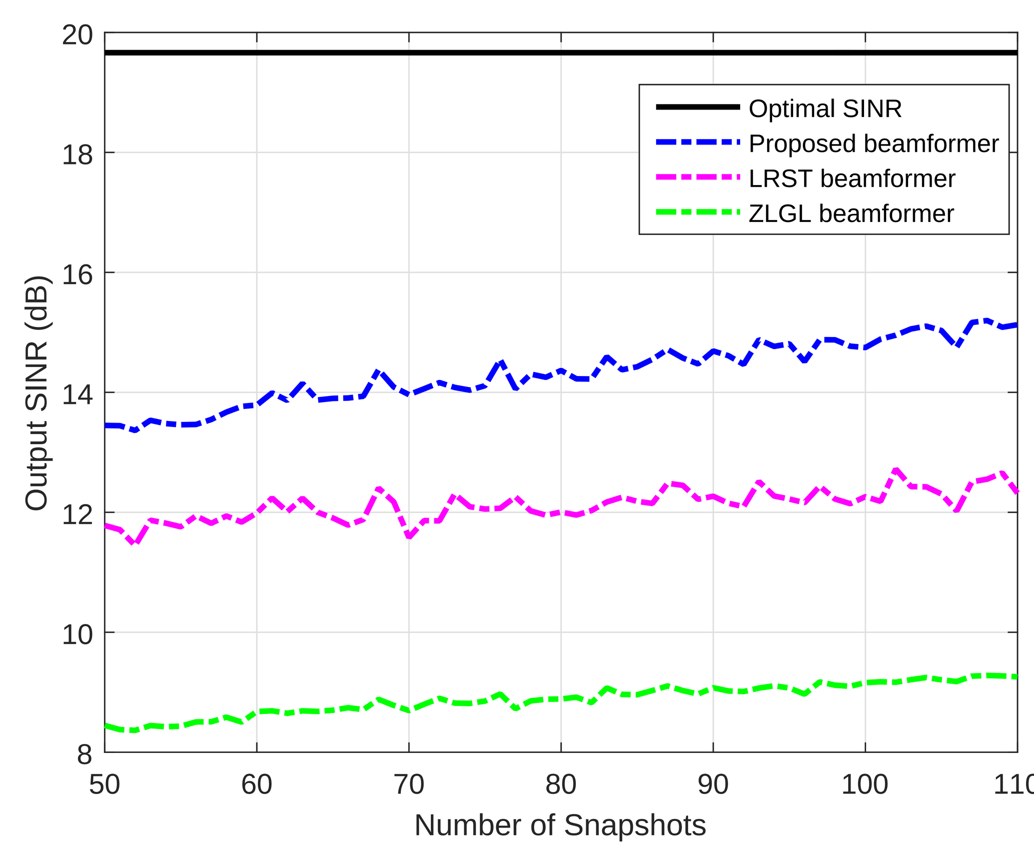

Figure 2: Average beamformer output SINR versus number of snapshots with SNR equal to 10 dB

Conclusions

- The DRO-based RAB problem of maximizing the worst-case output SINR over the distributional uncertainty sets for the INC matrix and the signal steering vector has been addressed.
- The distributional set for the INC matrix includes constraints on the support of the distribution, the positive semidefiniteness and similarity of the mean; the distributional set for the steering vector accounts for the first- and second-order moments and support of the distribution.
- The RAB problem has been transformed into a nonconvex QMI problem via the strong duality of linear conic programming.
- The QMI problem has been tackled by solving a sequence of LMI problems with a penalty term on the rank-one constraint in the objective function.
- The improved performance of the proposed DRO-based robust adaptive beamformer has been demonstrated by simulations in terms of the array output SINR with comparison to two existing beamforming techniques.