Robust Adaptive Beamforming Maximizing the Worst-Case SINR over Distributional Uncertainty Sets for Random INC Matrix and Signal Steering Vector

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Outline

- Signal model and problem formulation
- Equivalent quadratic matrix inequality problem reformulation
- Iterative procedure of finding a rank-one solution
- Numerical examples
- Summary

Signal Models

- A narrowband and point-source signal is observed by an N-antenna ULA:

$$\boldsymbol{y}(t) = s(t)\boldsymbol{a} + \boldsymbol{i}(t) + \boldsymbol{n}(t),$$

  where

  - $s(t)$ is the desired signal waveform and $\boldsymbol{a}$ is the steering vector;
  - $\boldsymbol{i}(t)$ denotes the interference;
  - $\boldsymbol{n}(t)$ represents the array noise;
  - $s(t)\boldsymbol{a}$ and $\boldsymbol{i}(t) + \boldsymbol{n}(t)$ are statistically independent.

- The array outputs the weighted signal

$$x(t) = \boldsymbol{w}^H \boldsymbol{y}(t),$$

- $\boldsymbol{w}$ is an $N \times 1$ weight vector (to be optimized in subsequent designs).

Maximization of the Array Output SINR

- Output SINR of the array

$$\text{SINR} = \frac{\sigma_s^2 |\boldsymbol{w}^H \boldsymbol{a}|^2}{\boldsymbol{w}^H \boldsymbol{R}_{i+n} \boldsymbol{w}},$$

- The SINR maximization problem is equivalent to the following problem:

$$\underset{\boldsymbol{w}}{\text{minimize}} \ \boldsymbol{w}^H \boldsymbol{R}_{i+n} \boldsymbol{w} \ \ \text{subject to} \ |\boldsymbol{w}^H \boldsymbol{a}| \geq 1.$$

- A closed-form solution (MVDR beamformer): $\boldsymbol{w}^\star = \frac{1}{\boldsymbol{a}^H \boldsymbol{R}_{i+n}^{-1} \boldsymbol{a}} \boldsymbol{R}_{i+n}^{-1} \boldsymbol{a}$.

- The interference-plus-noise covariance (INC) matrix $\boldsymbol{R}_{i+n}$ is often unavailable.

- The true steering vector $\boldsymbol{a}$ cannot be predefined accurately.

- The beamformer performance is degraded significantly even if there is a small mismatch between $\boldsymbol{R}_{i+n}$ and its estimate, or/and between $\boldsymbol{a}$ and its estimate.

DRO-Based Robust Adaptive Beamforming Maximizing the Worst-Case SINR

- Assume that both $\boldsymbol{R}_{i+n} \in \mathcal{H}^N$ and $\boldsymbol{a} \in \mathbb{C}^N$ are random variables.

- The DRO-based RAB problem maximizing the worst-case SINR is formulated:

$$\begin{aligned} \underset{\boldsymbol{w}}{\text{minimize}} \quad & \max_{G_1 \in \mathcal{D}_1} \mathsf{E}_{G_1}\{\boldsymbol{w}^H \boldsymbol{R}_{i+n} \boldsymbol{w}\} \\ \text{subject to} \quad & \min_{G_2 \in \mathcal{D}_2} \mathsf{E}_{G_2}\{\boldsymbol{w}^H \boldsymbol{a}\boldsymbol{a}^H \boldsymbol{w}\} \geq 1. \end{aligned} \tag{1}$$

  - Here, the set $\mathcal{D}_1$ of probability distributions is defined as

$$\mathcal{D}_1 = \left\{ G_1 \in \mathcal{M}_1 \;\middle|\; \begin{array}{l} \mathsf{Prob}_{G_1}\{\boldsymbol{R}_{i+n} \in \mathcal{Z}_1\} = 1 \\ \mathsf{E}_{G_1}\{\boldsymbol{R}_{i+n}\} \succeq \boldsymbol{0} \\ \|\mathsf{E}_{G_1}\{\boldsymbol{R}_{i+n}\} - \boldsymbol{S}_0\|_F \leq \rho_1 \end{array} \right\}, \tag{2}$$

  - $\boldsymbol{S}_0$ is the empirical mean of $\boldsymbol{R}_{i+n}$, and the sampling covariance matrix $\hat{\boldsymbol{R}} = \frac{1}{T}\sum_{t=1}^{T} \boldsymbol{y}(t)\boldsymbol{y}^H(t)$ is regarded as an alterative of $\boldsymbol{S}_0$;

– the set $\mathcal{D}_2$ of probability distributions is given by

$$\mathcal{D}_2 = \left\{ G_2 \in \mathcal{M}_2 \;\middle|\; \begin{array}{l} \text{Prob}_{G_2}\{\boldsymbol{a} \in \mathcal{Z}_2\} = 1 \\ \text{E}_{G_2}\{\boldsymbol{a}\} = \boldsymbol{a}_0 \\ \text{E}_{G_2}\{\boldsymbol{a}\boldsymbol{a}^H\} = \boldsymbol{\Sigma} + \boldsymbol{a}_0\boldsymbol{a}_0^H \end{array} \right\}, \tag{3}$$

– $\boldsymbol{a}_0 \in \mathbb{C}^N$ is the mean, and $\boldsymbol{\Sigma} \succ \mathbf{0}$ is the covariance matrix of random vector $\boldsymbol{a}$ under the true distribution $\bar{G}_2$.
– Both of $\boldsymbol{a}_0$ and $\boldsymbol{\Sigma}$ are known.

The Maximization Problem in the Objective

- The inner maximization problem in the objective of the DRO-based RAB problem is written as

$$\begin{array}{ll} \underset{G_1 \in \mathcal{M}_1}{\text{maximize}} & \int_{\mathcal{Z}_1} \boldsymbol{w}^H \boldsymbol{R} \boldsymbol{w} \, dG_1(\boldsymbol{R}) \\ \text{subject to} & \int_{\mathcal{Z}_1} dG_1(\boldsymbol{R}) = 1 \\ & \int_{\mathcal{Z}_1} \boldsymbol{R} \, dG_1(\boldsymbol{R}) \succeq \boldsymbol{0} \\ & \left\| \int_{\mathcal{Z}_1} \boldsymbol{R} \, dG_1(\boldsymbol{R}) - \boldsymbol{S}_0 \right\|_F \leq \rho_1. \end{array} \quad (4)$$

- Here, the subscript of $\boldsymbol{R}_{i+n}$ is dropped for notational simplicity.

- The dual problem is cast as

$$\begin{array}{ll} \text{minimize} & \rho_1 \|\boldsymbol{X}\|_F + \delta_{\mathcal{Z}_1}(\boldsymbol{w}\boldsymbol{w}^H + \boldsymbol{X} + \boldsymbol{Y}) - \operatorname{tr}(\boldsymbol{S}_0 \boldsymbol{X}) \\ \text{subject to} & \boldsymbol{X} \in \mathcal{H}^N, \, \boldsymbol{Y} \succeq \boldsymbol{0} (\in \mathcal{H}_+^N). \end{array} \quad (5)$$

- In (5), $\delta_{\mathcal{Z}_1}(\cdot)$ stands for the support function of $\mathcal{Z}_1$, and it is convex.

- Clearly, the dual problem is a finite-dimension convex optimization problem.

- It is verified that the strong duality holds between them.

- Suppose that the support set $\mathcal{Z}_1 = \{\boldsymbol{R} \in \mathcal{H}^N \mid \|\boldsymbol{R}\|_F \leq \rho_2\}$ is considered.

- Then, the support function $\delta_{\mathcal{Z}_1}(\boldsymbol{w}\boldsymbol{w}^H + \boldsymbol{X} + \boldsymbol{Y}) = \rho_2\|\boldsymbol{w}\boldsymbol{w}^H + \boldsymbol{X} + \boldsymbol{Y}\|_F$.

- The dual problem is specified to

$$\begin{array}{ll} \text{minimize} & \rho_1\|\boldsymbol{X}\|_F + \rho_2\|\boldsymbol{w}\boldsymbol{w}^H + \boldsymbol{X} + \boldsymbol{Y}\|_F - \operatorname{tr}(\boldsymbol{S}_0\boldsymbol{X}) \\ \text{subject to} & \boldsymbol{X} \in \mathcal{H}^N,\ \boldsymbol{Y} \succeq \boldsymbol{0} (\in \mathcal{H}_+^N). \end{array} \tag{6}$$

The Minimization Problem in the Constraint

- The minimization problem in the constraint of the DRO-based RAB problem can be expressed as

$$
\begin{array}{ll}
\underset{G_2 \in \mathcal{M}_2}{\text{minimize}} & \int_{\mathcal{Z}_2} \boldsymbol{a}^H \boldsymbol{w}\boldsymbol{w}^H \boldsymbol{a} \, dG_2(\boldsymbol{a}) \\
\text{subject to} & \int_{\mathcal{Z}_2} dG_2(\boldsymbol{a}) = 1 \\
& \int_{\mathcal{Z}_2} \boldsymbol{a} \, dG_2(\boldsymbol{a}) = \boldsymbol{a}_0 \\
& \int_{\mathcal{Z}_2} \boldsymbol{a}\boldsymbol{a}^H \, dG_2(\boldsymbol{a}) = \boldsymbol{\Sigma} + \boldsymbol{a}_0 \boldsymbol{a}_0^H .
\end{array} \tag{7}
$$

- The dual problem can be derived as follows.

$$
\begin{array}{ll}
\text{maximize} & x + \Re(\boldsymbol{a}_0^H \boldsymbol{x}) + \mathrm{tr}\left(\boldsymbol{Z}(\boldsymbol{\Sigma} + \boldsymbol{a}_0 \boldsymbol{a}_0^H)\right) \\
\text{subject to} & \boldsymbol{a}^H (\boldsymbol{w}\boldsymbol{w}^H - \boldsymbol{Z})\boldsymbol{a} - \Re(\boldsymbol{a}^H \boldsymbol{x}) - x \geq 0, \ \forall \boldsymbol{a} \in \mathcal{Z}_2 \\
& \boldsymbol{Z} \in \mathcal{H}^N, \ \boldsymbol{x} \in \mathbb{C}^N, \ x \in \mathbb{R}.
\end{array} \tag{8}
$$

- The strong duality between the two problems holds.

- Suppose that $\mathcal{Z}_2 = \mathbb{C}^N$. The semi-infinite inequality constraint in the dual is equivalent to the following quadratic matrix inequality (QMI):

$$\begin{bmatrix} \boldsymbol{w}\boldsymbol{w}^H - \boldsymbol{Z} & -\frac{\boldsymbol{x}}{2} \\ -\frac{\boldsymbol{x}^H}{2} & -x \end{bmatrix} \succeq \boldsymbol{0}. \tag{9}$$

Equivalent QMI Reformulation for the DRO-based RAB Problem

- The original DRO-based RAB problem (1) can be transformed into

$$\begin{aligned} \text{minimize} \quad & \rho_1\|\boldsymbol{X}\|_F + \rho_2\|\boldsymbol{w}\boldsymbol{w}^H + \boldsymbol{X} + \boldsymbol{Y}\|_F - \operatorname{tr}(\boldsymbol{S}_0\boldsymbol{X}) \\ \text{subject to} \quad & x + \Re(\boldsymbol{a}_0^H\boldsymbol{x}) + \operatorname{tr}(\boldsymbol{Z}(\boldsymbol{\Sigma} + \boldsymbol{a}_0\boldsymbol{a}_0^H)) \geq 1 \\ & \begin{bmatrix} \boldsymbol{w}\boldsymbol{w}^H - \boldsymbol{Z} & -\frac{\boldsymbol{x}}{2} \\ -\frac{\boldsymbol{x}^H}{2} & -x \end{bmatrix} \succeq \boldsymbol{0} \\ & \boldsymbol{w}, \boldsymbol{x} \in \mathbb{C}^N, \boldsymbol{X}, \boldsymbol{Z} \in \mathcal{H}^N, \boldsymbol{Y} \succeq \boldsymbol{0}, x \in \mathbb{R}. \end{aligned} \tag{10}$$

- This is a nonconvex QMI problem (w.r.t. $\boldsymbol{w}$).

- The conventional LMI relaxation technique can be applied, namely, the following

LMI problem is solved:

$$
\begin{aligned}
\text{minimize} \quad & \rho_1 \|\boldsymbol{X}\|_F + \rho_2 \|\boldsymbol{W} + \boldsymbol{X} + \boldsymbol{Y}\|_F - \operatorname{tr}(\boldsymbol{S}_0 \boldsymbol{X}) \\
\text{subject to} \quad & x + \Re(\boldsymbol{a}_0^H \boldsymbol{x}) + \operatorname{tr}(\boldsymbol{Z}(\boldsymbol{\Sigma} + \boldsymbol{a}_0 \boldsymbol{a}_0^H)) \geq 1 \\
& \begin{bmatrix} \boldsymbol{W} - \boldsymbol{Z} & -\frac{\boldsymbol{x}}{2} \\ -\frac{\boldsymbol{x}^H}{2} & -x \end{bmatrix} \succeq \boldsymbol{0} \\
& \boldsymbol{x} \in \mathbb{C}^N, \boldsymbol{X}, \boldsymbol{Z} \in \mathcal{H}^N, \boldsymbol{W} \succeq \boldsymbol{0}, \boldsymbol{Y} \succeq \boldsymbol{0}, x \in \mathbb{R}.
\end{aligned}
\tag{11}
$$

- If the LMI problem has a rank-one optimal solution $\boldsymbol{w}^\star \boldsymbol{w}^{\star H}$, then $\boldsymbol{w}^\star$ is optimal for the QMI problem (10).

- A rank-one solution procedure is desired when the LMI relaxation problem admits an optimal solution $\boldsymbol{W}^\star$ of rank more than one.

Rank-One Solution Procedure for the LMI Relaxation Problem

- Observe that if nonzero $\boldsymbol{W} \succeq \mathbf{0}$ is of rank one, then $\operatorname{tr}\boldsymbol{W} = \|\boldsymbol{W}\|_F$, and vice versa.

- The previous condition can also take the form: $\operatorname{tr}\boldsymbol{W} - \frac{\operatorname{tr}(\boldsymbol{W}\boldsymbol{W})}{\|\boldsymbol{W}\|_F} = 0$.

- Therefore, at iteration k of a procedure, the following LMI problem with a penalty term on the rank-one solution constraint is solved:

$$
\begin{array}{ll}
\text{minimize} & \rho_1\|\boldsymbol{X}\|_F + \rho_2\|\boldsymbol{W}+\boldsymbol{X}+\boldsymbol{Y}\|_F - \operatorname{tr}(\boldsymbol{S}_0\boldsymbol{X}) + \alpha\left(\operatorname{tr}\boldsymbol{W} - \frac{\operatorname{tr}(\boldsymbol{W}\boldsymbol{W}_k)}{\|\boldsymbol{W}_k\|_F}\right) \\
\text{subject to} & x + \Re(\boldsymbol{a}_0^H\boldsymbol{x}) + \operatorname{tr}(\boldsymbol{Z}(\boldsymbol{\Sigma} + \boldsymbol{a}_0\boldsymbol{a}_0^H)) \geq 1 \\
& \begin{bmatrix} \boldsymbol{W}-\boldsymbol{Z} & -\frac{\boldsymbol{x}}{2} \\ -\frac{\boldsymbol{x}^H}{2} & -x \end{bmatrix} \succeq \mathbf{0} \\
& \boldsymbol{x} \in \mathbb{C}^N, \boldsymbol{X}, \boldsymbol{Z} \in \mathcal{H}^N, \boldsymbol{W} \succeq \mathbf{0}, \boldsymbol{Y} \succeq \mathbf{0}, x \in \mathbb{R},
\end{array}
\tag{12}
$$

Finding a Rank-One Solution for the LMI Relaxation Problem (11)

Input: $\boldsymbol{S}_0$, $\boldsymbol{\Sigma}$, $\boldsymbol{a}_0$, ρ_1, ρ_2, α;

Output: A rank-one solution $\boldsymbol{w}^\star \boldsymbol{w}^{\star H}$;

1. set $k = 0$; let $\boldsymbol{W}_k$ be a high-rank optimal solution $\boldsymbol{W}^\star$ for (11);

2. **do**

3. solve the LMI problem (12), obtaining solution $\boldsymbol{W}_{k+1}$;

4. $k := k + 1$;

5. **until** $\left| \operatorname{tr} \boldsymbol{W}_k - \frac{\operatorname{tr}(\boldsymbol{W}_k \boldsymbol{W}_{k-1})}{\|\boldsymbol{W}_{k-1}\|_F} \right| \leq 10^{-6}$

6. output $\boldsymbol{w}^\star$ with $\boldsymbol{W}_k = \boldsymbol{w}^\star \boldsymbol{w}^{\star H}$.

- It can be shown that the sequence of the optimal values for (12) is descent, namely, $v_1 \geq v_2 \geq \cdots$, where v_k is the optimal value for (12) in iteration $k-1$.

- The terminating condition implies that the output solution $\boldsymbol{W}_k$ is a rank-one solution for (12), since $\mathrm{tr}(\boldsymbol{W}_k) \approx \frac{\mathrm{tr}(\boldsymbol{W}_k \boldsymbol{W}_{k-1})}{\|\boldsymbol{W}_{k-1}\|_F} \approx \|\boldsymbol{W}_k\|_F$.

- The computational complexity is dominated by solving the LMI problem (12) in each iteration, which is manageable since the problem has only one inequality constraint and one LMI constraint with size $N+1$.

Simulation Setup

- Scenario: A uniform linear array with $N = 10$ sensors spaced half a wavelength;
  - The angular sector of interest $\Theta = [0^\circ, 10^\circ]$;
  - The actual signal direction $\theta = 5^\circ$;
  - The presumed direction $\theta_0 = 1^\circ$;
  - Two interferers located in the directions of $\theta_1 = -5^\circ$ and $\theta_2 = 15^\circ$ with the same interference-to-noise ratio (INR) of 30 dB;
  - The array noise: a spatially and temporally white Gaussian vector with zero mean and covariance $\boldsymbol{I}$;
  - Wavefront distortion: The phase increments are independent Gaussian variables each with zero mean and standard deviation 0.02;
  - $\boldsymbol{S}_0$ is the sampling covariance matrix (it is different in each run);
  - $\boldsymbol{a}_0 = \frac{1}{L}\sum_{l=1}^{L} \boldsymbol{d}(\theta_l)$ and $\boldsymbol{\Sigma} = \frac{1}{L}\sum_{l=1}^{L}(\boldsymbol{d}(\theta_l) - \boldsymbol{a}_0)(\boldsymbol{d}(\theta_l) - \boldsymbol{a}_0)^H$;
  - Parameters $\rho_1 = 0.001\|\boldsymbol{S}_0\|_F$, $\rho_2 = 10^5$, and $\alpha = 10^5$.

- All results are averaged over 200 simulation runs.

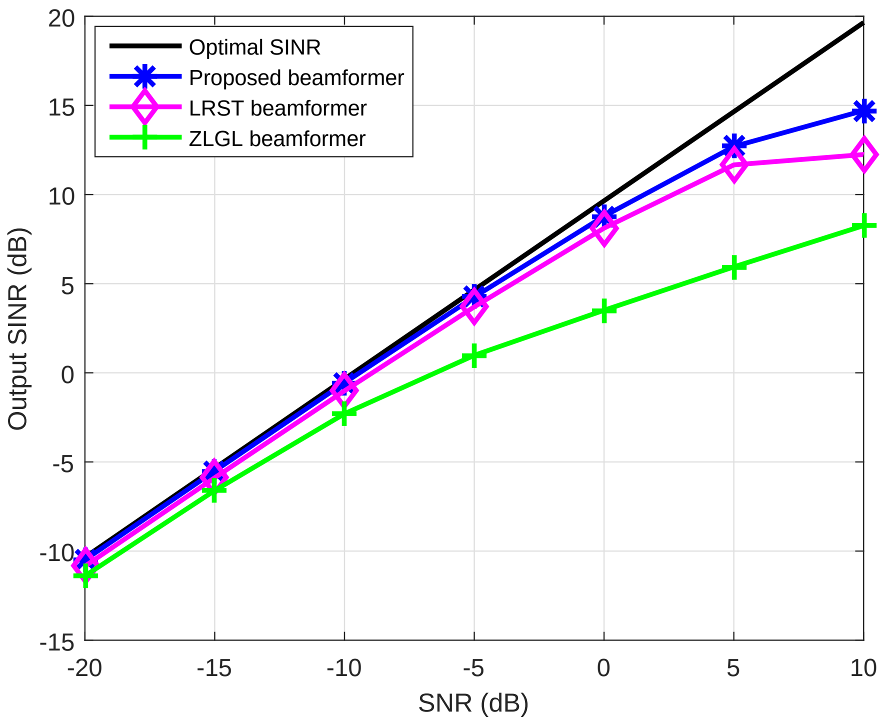
- Three DRO-based beamformers are compared: The proposed one, LRST beamformer[1], and ZLGL beamformer[2].

[1] B. Li, Y. Rong, J. Sun, and K.L. Teo, "A distributionally robust minimum variance beamformer design," *IEEE Signal Processing Letters*, vol. 25, no. 1, pp. 105–109, Jan. 2018.

[2] X. Zhang, Y. Li, N. Ge, and J. Lu, "Robust minimum variance beamforming under distributional uncertainty," in *Proc. 40th IEEE Int. Conf. Acoustics, Speech, and Signal Processing*, Brisbane, Australia, Apr. 2015, pp. 2514–2518.

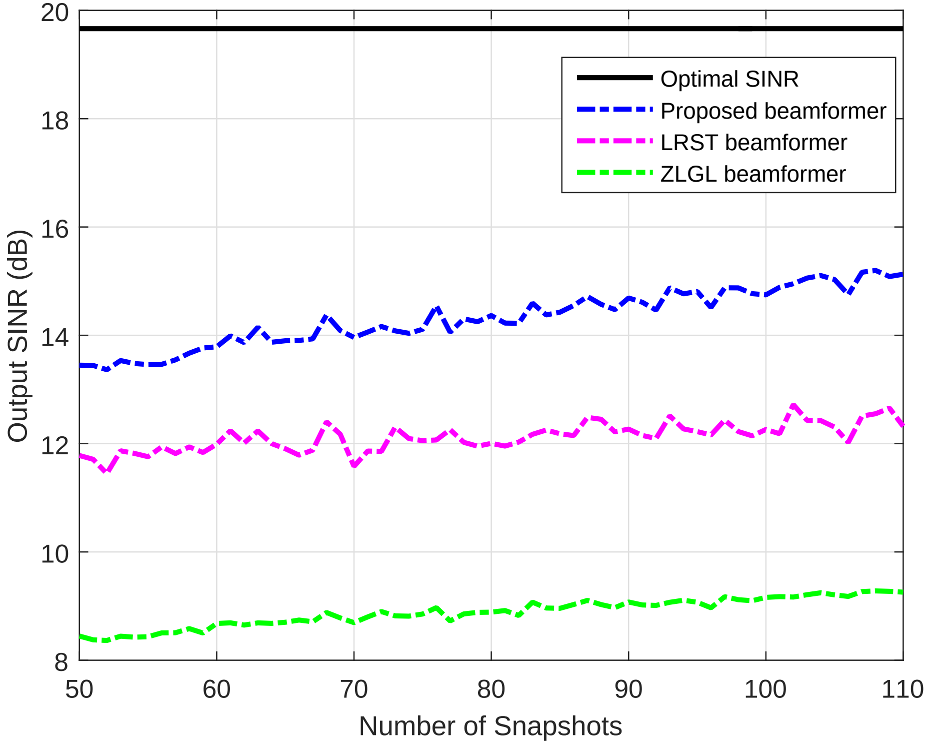
Simulation Output: Average Array Output SINR versus SNR

- Average array output SINR versus SNR with the number of snapshots $T = 100$:



Simulation Output: Average Array Output SINR versus Number of Snapshots

- Average array output SINR versus number of snapshots with SNR equal to 10 dB:



Summary

- We have studied the DRO-based RAB problem of maximizing the worst-case SINR over the distributional sets for random INC matrix and desired signal steering vector.
- The RAB problem is transformed into a nonconvex QMI problem via the strong duality theorem of linear conic programming.
- The QMI problem is tackled by iteratively solving a sequence of LMI relaxation problems with a penalty term on the rank-one constraint.
- The sequence of the optimal values for the LMI relaxation problems is descent, which means that the algorithm always outputs a rank-one solution when the penalty weight is large enough.
- Numerical results show that the proposed beamformer outperforms the other two existing beamformers in terms of the array output SINR.