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RECOVERY OF GRAPH SIGNALS FROM SIGN MEASUREMENTS

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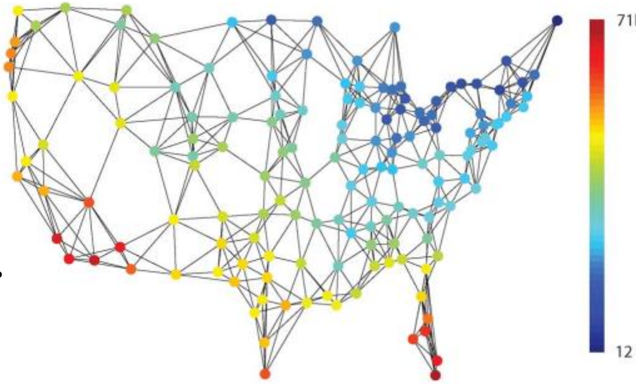
Content

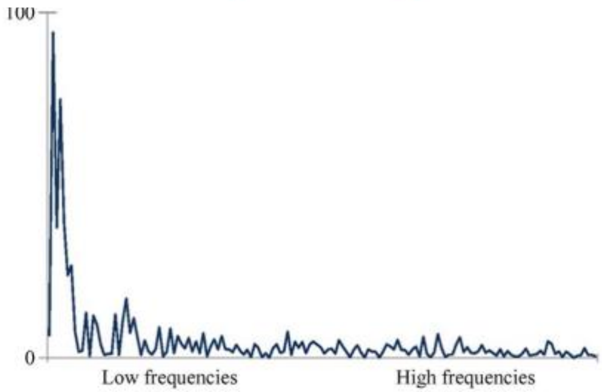
- Introduction
- Model
- Reconstruction Algorithm
- Design of Sampling Set
- Experiments

Introduction

- Graph Signals: data on an irregular topology
  - Social networks, sensor networks, …



- Graph Signal Processing (GSP)
  - Semantic segmentation, traffic prediction, …
  - Graph Fourier Transform (GFT)
    - Vertex domain → Frequency domain
  - Graph signals in life tend to be smooth
    - Signal values on adjacent vertices are similar
    - ⇔ Bandlimited in frequency domain



    - E.g. Temperature measured by 150 weather stations across the United States on February 1, 2003. [1]

Introduction

- Observing a graph signal may face
  - Not all the vertices can be observed
    - Enormous data scale, limited sampling budget, …
  - Only some simple quantized values are available
    - E.g. Rating system: no specific scores, but simple evaluations ("like","dislike", "indifference")

Sign information: $\{1, -1, 0\}$

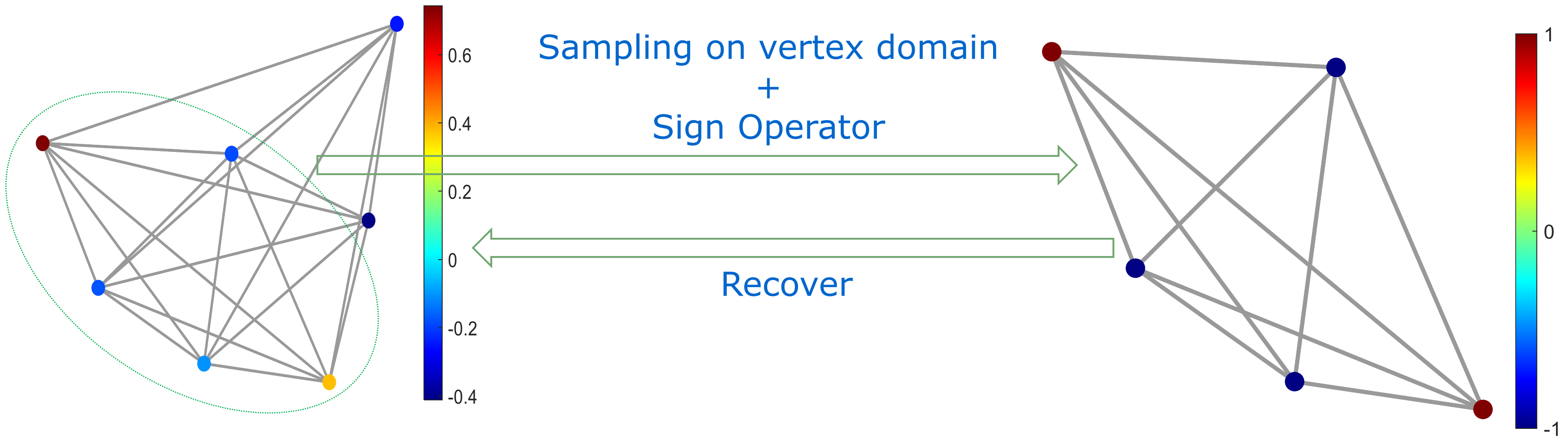
The signal value exceeds a threshold or not

Our focus!

Introduction

Our research

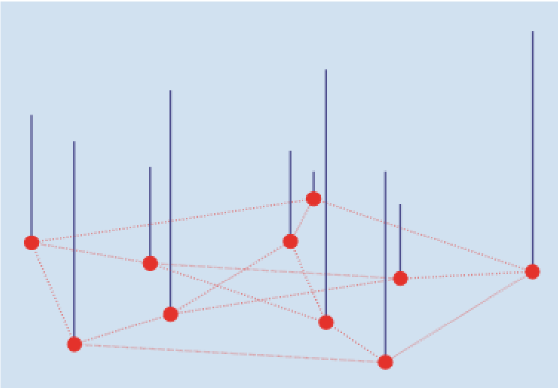
- Recover the original bandlimited signal from the sign information of partial samples



- Reconstruction algorithm + Sampling scheme

Model

- Undirected Connected Graph: $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \boldsymbol{W}\}$
  - $\mathcal{V}$: set of vertices, $|\mathcal{V}| = N$
  - $\mathcal{E}$: set of edges
  - $\boldsymbol{W}$: weighted adjacency matrix
    - $W_{ij} = W_{ji} > 0$ if $(i, j) \in \mathcal{E}$, $W_{ij} = 0$ otherwise



- Graph Signal $\boldsymbol{x} : \mathcal{V} \longmapsto \mathbb{R}$
- Graph Laplacian: $\boldsymbol{L} = \boldsymbol{D} - \boldsymbol{W}$
  - Real symmetric, positive semidefinite, $\boldsymbol{L} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^T$
    - Eigenvalues: $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$, $\boldsymbol{\Lambda} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N)$
    - Eigenvectors: $\boldsymbol{U} = [\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots \boldsymbol{u}_N]$

Model

- **Bandlimited in frequency domain**
  - GFT: $\hat{\boldsymbol{x}} = \boldsymbol{U}^T \boldsymbol{x}$ ($\mathbb{R}^N \longmapsto \mathbb{R}^N$)
  - Bandlimited: $\hat{\boldsymbol{x}}_k \neq 0$ for $f_L \leq k \leq f_U$, and $\hat{\boldsymbol{x}}_k = 0$ otherwise.

- **Sign Measurement**
  - $\boldsymbol{s}_x = \mathrm{sign}(\boldsymbol{\psi}_v \boldsymbol{x})$
    - $\mathbb{R}^N \longmapsto \{-1,1,0\}^M (M \leq N)$
    - sampling matrix $\boldsymbol{\psi}_v \in \mathbb{R}^{M \times N} \Rightarrow$ signal values on $\mathcal{V}'$ (a subset of $\mathcal{V}$)
    - $\mathrm{sign}(x) = \begin{cases} -1 & x < 0 \\ 1 & x > 0 \\ 0 & \text{otherwise} \end{cases}$
    - we assume $\|\boldsymbol{x}\| = 1$

Reconstruction Algorithm

Suppose our recovery signal is $\boldsymbol{x}^*$, the original signal $\boldsymbol{x}$ has passband $f_L, \dots, f_U$ and bandwidth $B = f_U - f_L + 1$

- Consistence: $\text{sign}(\boldsymbol{\psi}_v \boldsymbol{x}^*) = \boldsymbol{s}_x$
  - For $i = 1,2, .., M$, $(\boldsymbol{\psi}_v)_i \boldsymbol{x}^* > 0 (< 0, = 0)$ according to $(\boldsymbol{s}_x)_i$
  - Constraint space:

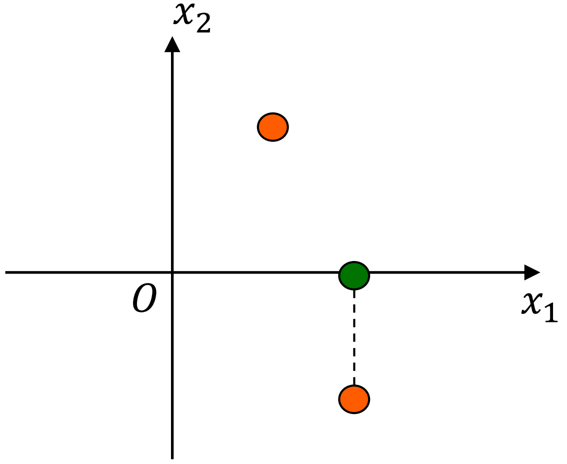
$$C_s = \bigcap_{i=1}^{M} \{\boldsymbol{\omega} \in \mathbb{R}^N | (\boldsymbol{\psi}_v)_i \boldsymbol{\omega} > 0 (< 0, = 0)\}$$

- Bandlimited: $\boldsymbol{x}^*$ has the same passband as $\boldsymbol{x}$
  - For $i = f_L, \dots, f_U$, $\boldsymbol{u}_i^T \boldsymbol{x}^* = 0$
  - Constraint space:

$$C_b = \left\{\boldsymbol{\omega} \in \mathbb{R}^N | \boldsymbol{u}_i^T \boldsymbol{\omega} = 0, i = f_L, \dots, f_U\right\}$$

Reconstruction Algorithm

- Projection operators
  - Projecting on C_v (the close convex hull of C_s)
    - Relaxation: $C_v = \cap_{i=1}^{M}\{\boldsymbol{\omega} \in \mathbb{R}^N \mid (\boldsymbol{\psi}_v)_i \boldsymbol{\omega} \geq 0 (\leq 0, = 0)\}$
    - $(\boldsymbol{P}_v\boldsymbol{\omega})_j = \begin{cases} 0 & j \in \mathcal{V}', \mathrm{sign}(\omega_j) \neq (\boldsymbol{s}_x)_i \\ \omega_j & \text{otherwise} \end{cases}$
    - A 2D example, sign information: $x_2 > 0$



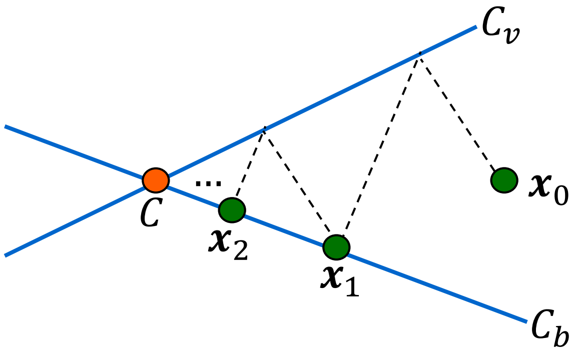
  - Projecting on C_b
    - $\boldsymbol{P}_b = \boldsymbol{U\Gamma U}^T$: A bandpass filter

Reconstruction Algorithm

- Continuously projecting onto the convex sets (POCS) [2]

  Denote C as the feasible region, i.e. $C = C_b \cap C_v$

  - Iterative process : $\boldsymbol{x}_{n+1} = \boldsymbol{P}_b \boldsymbol{P}_v \boldsymbol{x}_n$



- Convergence Analysis [3]

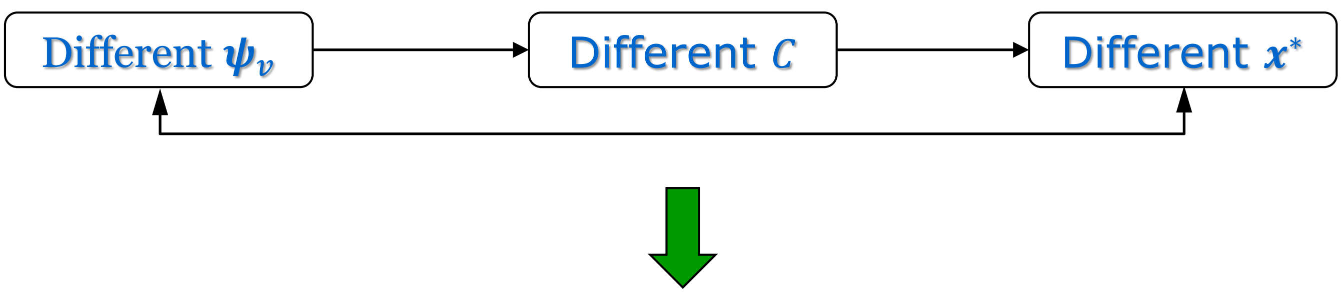
  1. The iterative sequence $\{\boldsymbol{x}_n\}$ converges to some point $\boldsymbol{x}^*$ in C
  2. The convergence rate is independent of the selection of the initial point $\boldsymbol{x}_0$

Design of Sampling Set

- Goal: Find a sampling set so that $\boldsymbol{x}^*$ is closer to $\boldsymbol{x}$

  $\Rightarrow$ Find a sampling set that makes C smaller

  - C (C_v) is decided by $\boldsymbol{\psi}_v$
  - Different C may lead to different $\boldsymbol{x}^*$



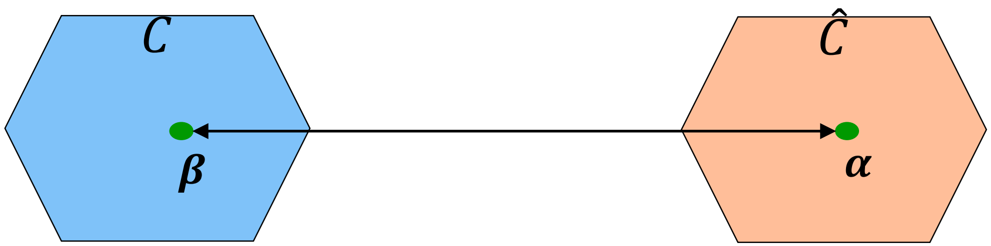
We can modify the sampling set to adjust recovery quality !

Design of Sampling Set

- Feasible Region Analysis
  - Denote $\boldsymbol{U}_B = [\boldsymbol{u}_{f_L}, \ldots \boldsymbol{u}_{f_U}]$, sampling set $S = \{S(1), \ldots S(M)\}$. Define

$$\hat{C} = \bigcap_{i=1}^{M} \{\boldsymbol{\alpha} \in \mathbb{R}^B | \ (\boldsymbol{\psi}_v)_i \boldsymbol{U}_B \boldsymbol{\alpha} \geq 0 (\leq 0, = 0)\}$$

$$= \bigcap_{i=1}^{M} \{\boldsymbol{\alpha} \in \mathbb{R}^B | (\boldsymbol{U}_B)_{S(i)} \boldsymbol{\alpha} \geq 0 (\leq 0, = 0)\}$$



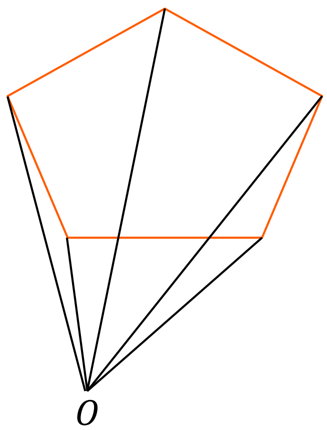
Any vector in C has a one - to - one correspondence in $\hat{C}$

  - For any signals $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2$ in C with coordinates $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2$ under $\boldsymbol{U}_B$, then $\langle \boldsymbol{\beta}_1, \boldsymbol{\beta}_2 \rangle = \langle \boldsymbol{U}_B \boldsymbol{\alpha}_1, \boldsymbol{U}_B \boldsymbol{\alpha}_2 \rangle = \langle \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2 \rangle$

Design of Sampling Set

- Feasible Region Analysis
  - $\hat{C}$ is a closed convex cone
    - Any vector in $\hat{C}$ can be represented linearly by extreme vectors (EVs) with non-negative coefficients [4] .

$$\hat{C} = \left\{ \sum_{i=1}^{r} k_i \boldsymbol{\varphi}_i \,\middle|\, k_i \geq 0 \ (i, 1,2, \dots r) \right\}$$

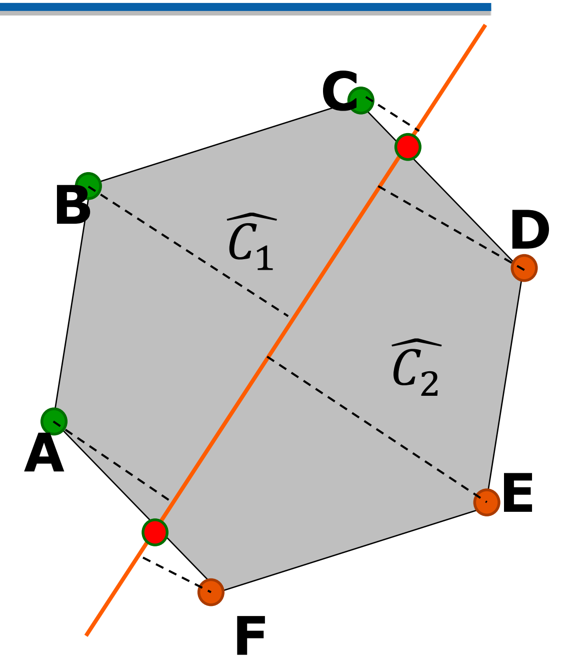


    - $\mathcal{Z} = \{\boldsymbol{\varphi}_i\}_{i=1}^{r}$ (normalized) are called extreme vectors
  - The size metric of $\hat{C}$

$$\theta = \max_{\boldsymbol{\gamma}, \boldsymbol{\mu} \in \mathcal{Z}} \arccos\langle \boldsymbol{\gamma}, \boldsymbol{\mu} \rangle$$

Design of Sampling Set

- Sampling process
  - sampling ⇔ adding constraints
  - select ξ as the next sample:
    - $\hat{C} \longrightarrow \hat{C} \cap \{\boldsymbol{\alpha} \in \mathbb{R}^B | (\boldsymbol{U}_B)_\xi \boldsymbol{\alpha} \geq 0 (\leq 0, = 0)\}$
    - Divide $\hat{C}$ by hyperplane $\{\boldsymbol{\alpha} \in \mathbb{R}^B | (\boldsymbol{U}_B)_\xi \boldsymbol{\alpha} = 0\}$



- Brute Force Approach
  - calculate the EVs for every unsampled vertex
  - calculate θ and determine the smallest one

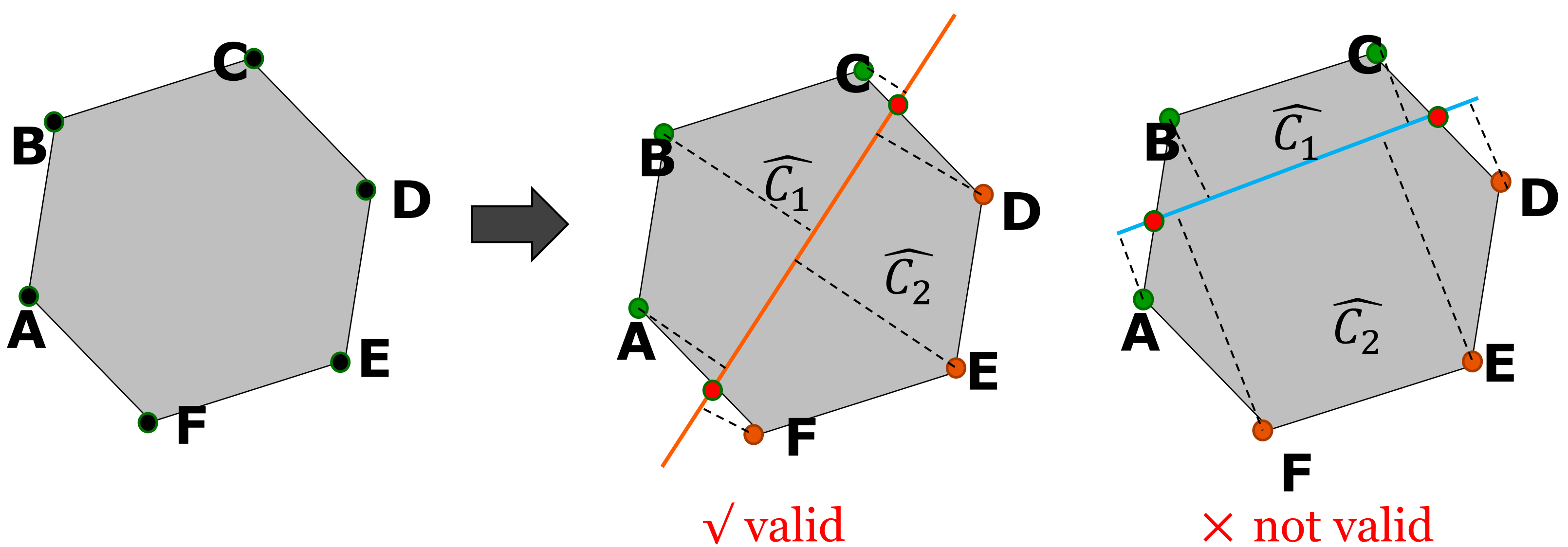
Unbearable Complexity !!

Design of Sampling Set

- Greedy Sampling

  1. Find the "valid" vertices

     - Valid: the corresponding hyperplane separates the EV pair associated with θ.
     - Example: (A,D) is the target EV pair



Design of Sampling Set

- Greedy Sampling

2. Among all the valid vertices, calculate the **distances** of EVs

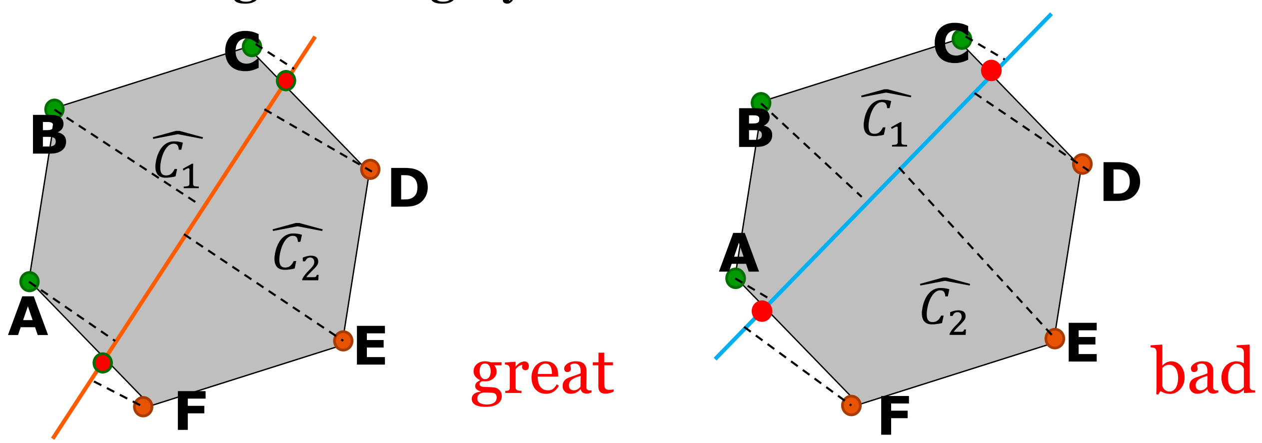
$$d_j = \sum_{\gamma \in \mathcal{Z}} \frac{(\boldsymbol{U}_B)_j \boldsymbol{\gamma}}{\|(\boldsymbol{U}_B)_j\|}$$

3. Next sample ξ:

$$\xi = \mathrm{argmin}_j \left| d_j \right|$$

To make the feasible region roughly "cut" in half.

Example :



Experiments

Performance on a sensor graph

Procedure

1. Obtain the greedy sampling set and 50 random sampling sets.
2. Arbitrarily select 50 initial points.
3. Recover the signal from the initial points on each sampling set.

Evaluation criterion

$$\delta = \frac{1}{50}\sum_{i=1}^{50} \arccos \langle \boldsymbol{x}, \boldsymbol{x}_i^* \rangle$$

- $\boldsymbol{x}_i^*$ stands for the normalized recovery signal of the ith initial signal.
- The larger δ is, the worse the recovery is.

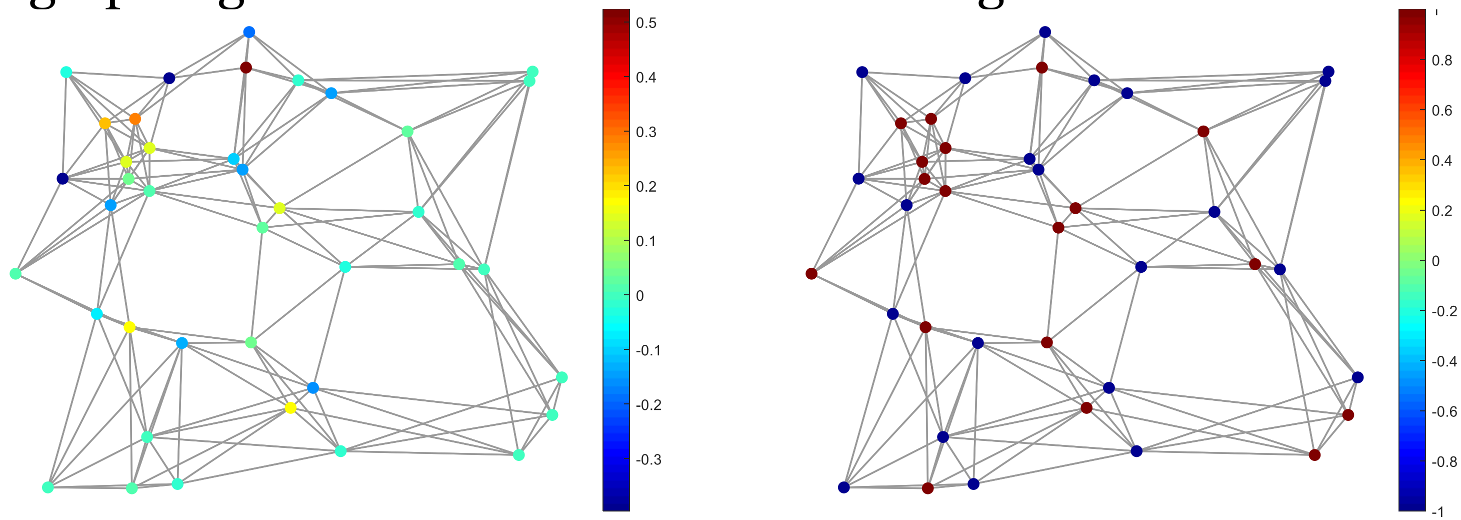
Experiments

Performance on a sensor graph

Parameters

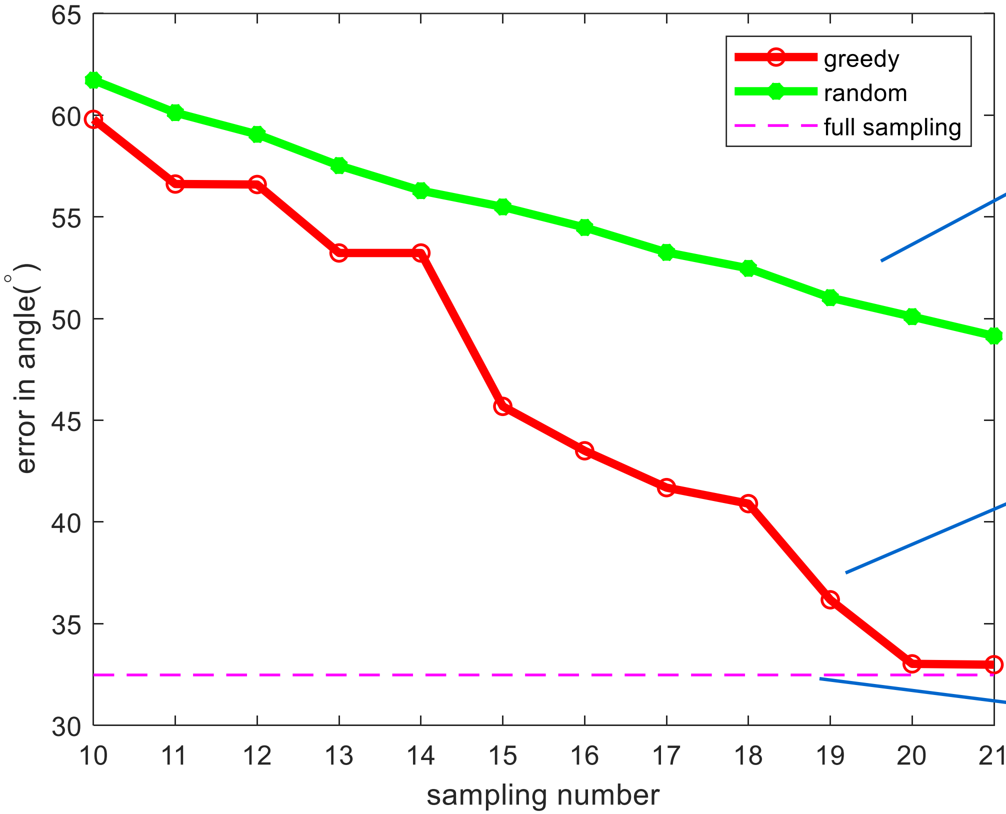
Vertices N	Edges $\lvert\mathcal{E}\rvert$	Lower bound f_L	Upper bound f_U
40	153	29	35

a graph signal with unit norm | its sign information



Experiments

- Performance on a sensor graph



Average error for 50 random sampling sets

Average error for our greedy sampling set

Average error when all the vertices are sampled

Reference

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Thank you