# Super Nested Arrays: Sparse Arrays with Less Mutual Coupling Than Nested Arrays

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# Outline


1 Introduction (DOA, Sensor Arrays, ...)

2 Mutual Coupling in Sensor Arrays

3 Super Nested Arrays

4 Numerical Examples

5 Concluding Remarks

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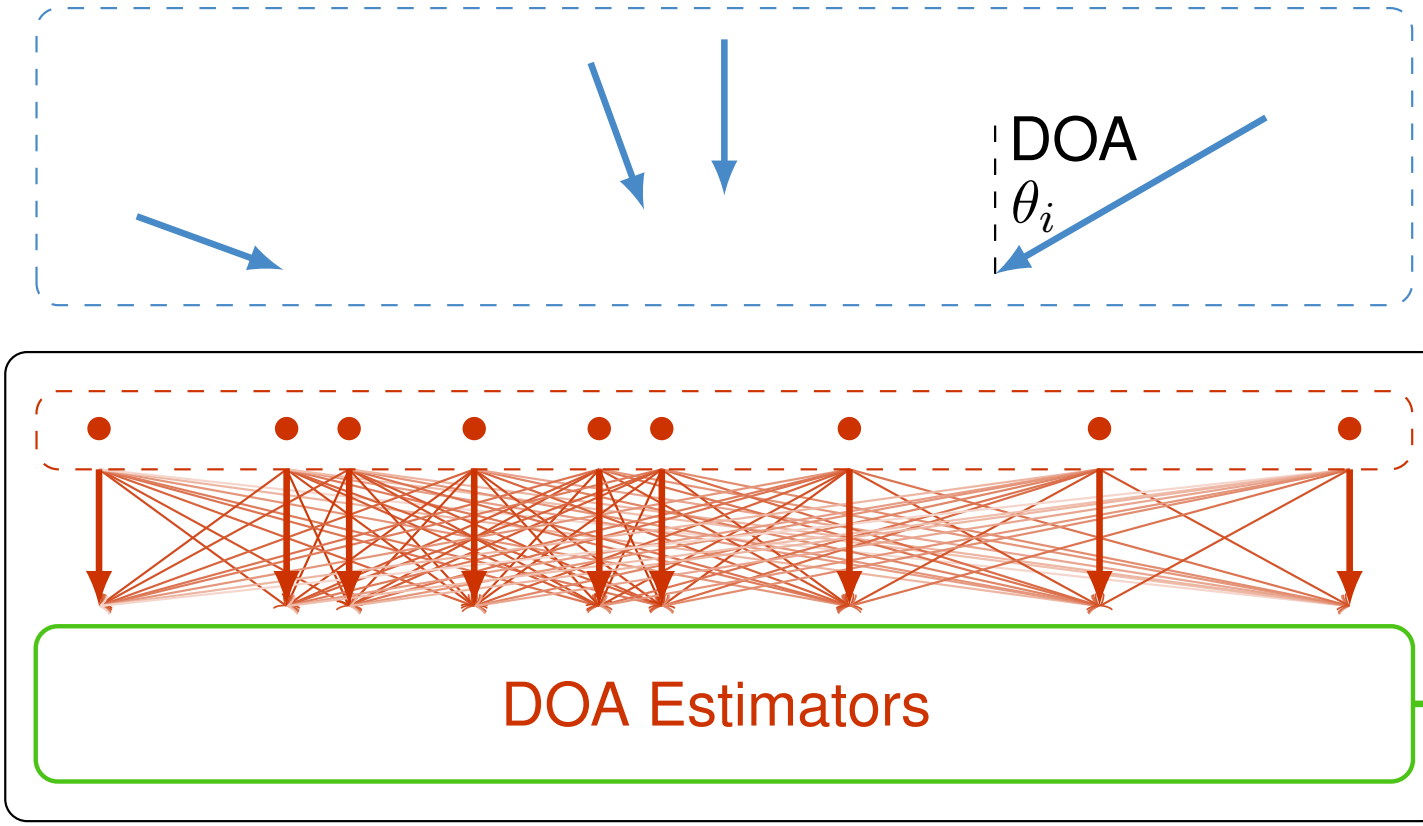
# DOA estimation in the presence of mutual coupling[1]




We will develop new sparse arrays with less mutual coupling.

[1] Van Trees, *Optimum Array Processing: Part IV of Detection, Estimation, and Modulation Theory*, 2002.

# Known arrays and super nested arrays: An example

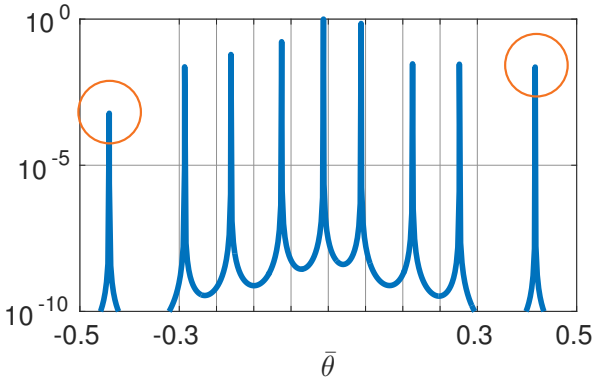
Uniform Linear Array

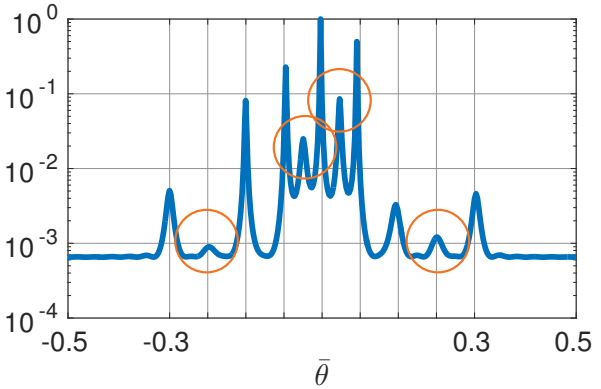
Nested Array

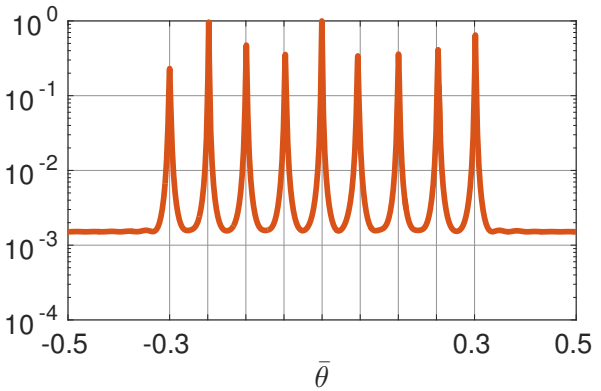
Super Nested Array



MUSIC spectra in the presence of mutual coupling $9$ sources



# ULA and sparse arrays

## ULA (not sparse)

- Identify at most $N-1$ uncorrelated sources, given $N$ sensors.[1]
- Can only find fewer sources than sensors.

## Sparse arrays

1. Minimum redundancy arrays (MRA)[2]
2. Nested arrays[3]
3. Coprime arrays[4]

- Identify $O(N^2)$ uncorrelated sources with $O(N)$ physical sensors.
- More sources than sensors!

[1] Van Trees, *Optimum Array Processing: Part IV of Detection, Estimation, and Modulation Theory*, 2002.

[2] Moffet, *IEEE Trans. Antennas Propag.*, 1968.

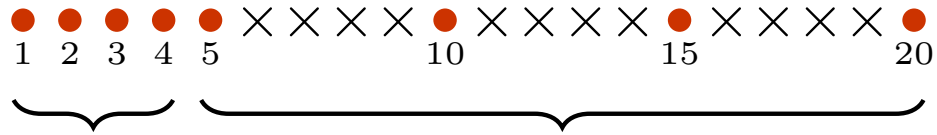
[3] Pal and Vaidyanathan, *IEEE Trans. Signal Proc.*, 2010.

[4] Vaidyanathan and Pal, *IEEE Trans. Signal Proc.*, 2011.

# Nested arrays[1]

The nested array
$N_1 = 4,$
$N_2 = 4.$

$$|\mathbb{S}| = N_1 + N_2 = 8$$




Dense ULA
$N_1$ sensors
spacing 1

Sparse ULA
$N_2$ sensors
spacing $N_1 + 1$

Difference coarray

$$\mathbb{D} = \{n_1 - n_2 \mid n_1, n_2 \in \mathbb{S}\} \qquad |\mathbb{D}| = O(N_1 N_2)$$


For sufficient number of snapshots,
$(|\mathbb{U}| - 1)/2 = O(N_1 N_2)$ uncorrelated sources can be identified.
($\mathbb{U}$ = Central ULA part of $\mathbb{D}$)

[1] Pal and Vaidyanathan, *IEEE Trans. Signal Proc.*, 2010.

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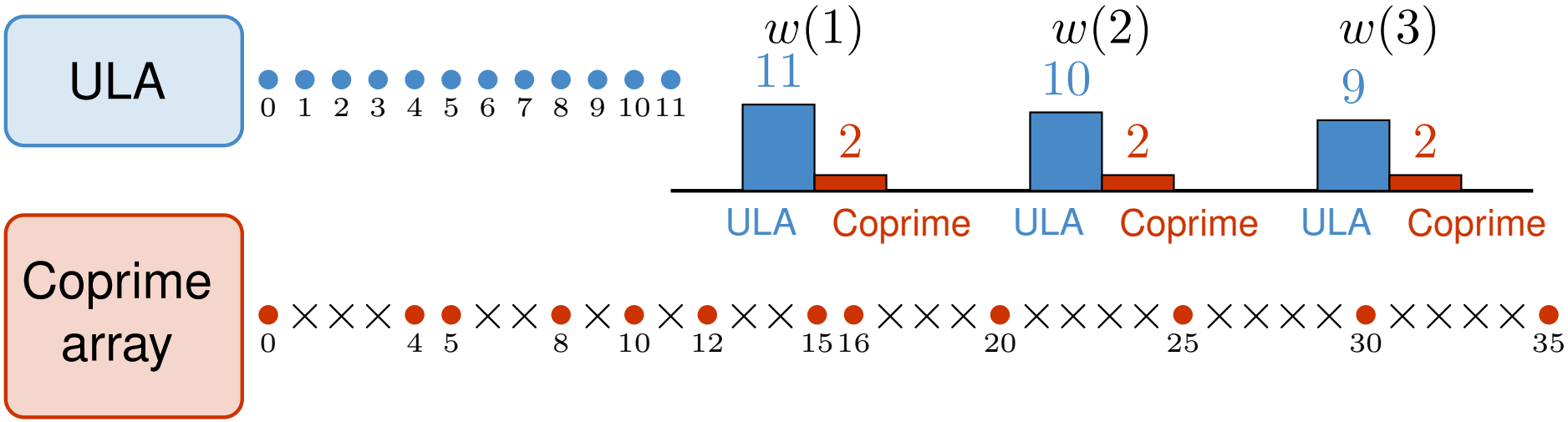
# How to characterize mutual coupling in arrays?[1]

## The weight function $w(m)$

The number of sensor pairs with separation $m$.




1. Coprime arrays are more sparse.
2. Coprime arrays enjoy **smaller** $w(1)$**,** $w(2)$ **and** $w(3)$.
3. Coprime arrays have less mutual coupling than ULA.
4. However, coprime arrays have holes in the difference coarray.

[1] Liu and Vaidyanathan, to appear in *IEEE Trans. Signal Proc.*, 2016.

# Pros and cons (✓ = Preferred, ✗ = Undesirable)

| | ULA | MRA | Nested arrays | Coprime arrays | Super nested arrays |
|---|---|---|---|---|---|
| Closed forms [Sensor locations] | Yes ✓ | No ✗ | Yes ✓ | Yes ✓ | Yes ✓ |
| Number of resolvable sources | $O(N)$ ✗ | $O(N^2)$ ✓ | $O(N^2)$ ✓ | $O(N^2)$ ✓ | $O(N^2)$ ✓ |
| Difference coarray | Hole-free ✓ | Hole-free ✓ | Hole-free ✓ | w/ holes ✗ | Hole-free ✓ |
| $w(1)$ | $N-1$ ✗ | Unknown ✗ | $\approx N/2$ ✗ | $2$ ✓ | $1$ or $2$ ✓ |

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# Goal: Desired properties of super nested arrays

They should have the same number of sensors as nested arrays,

$$|\mathbb{S}_{\text{Super nested}}| = |\mathbb{S}_{\text{Nested}}|.$$

They should have the same difference coarray as nested arrays,

$$\mathbb{D}_{\text{Super nested}} = \mathbb{D}_{\text{Nested}}.$$

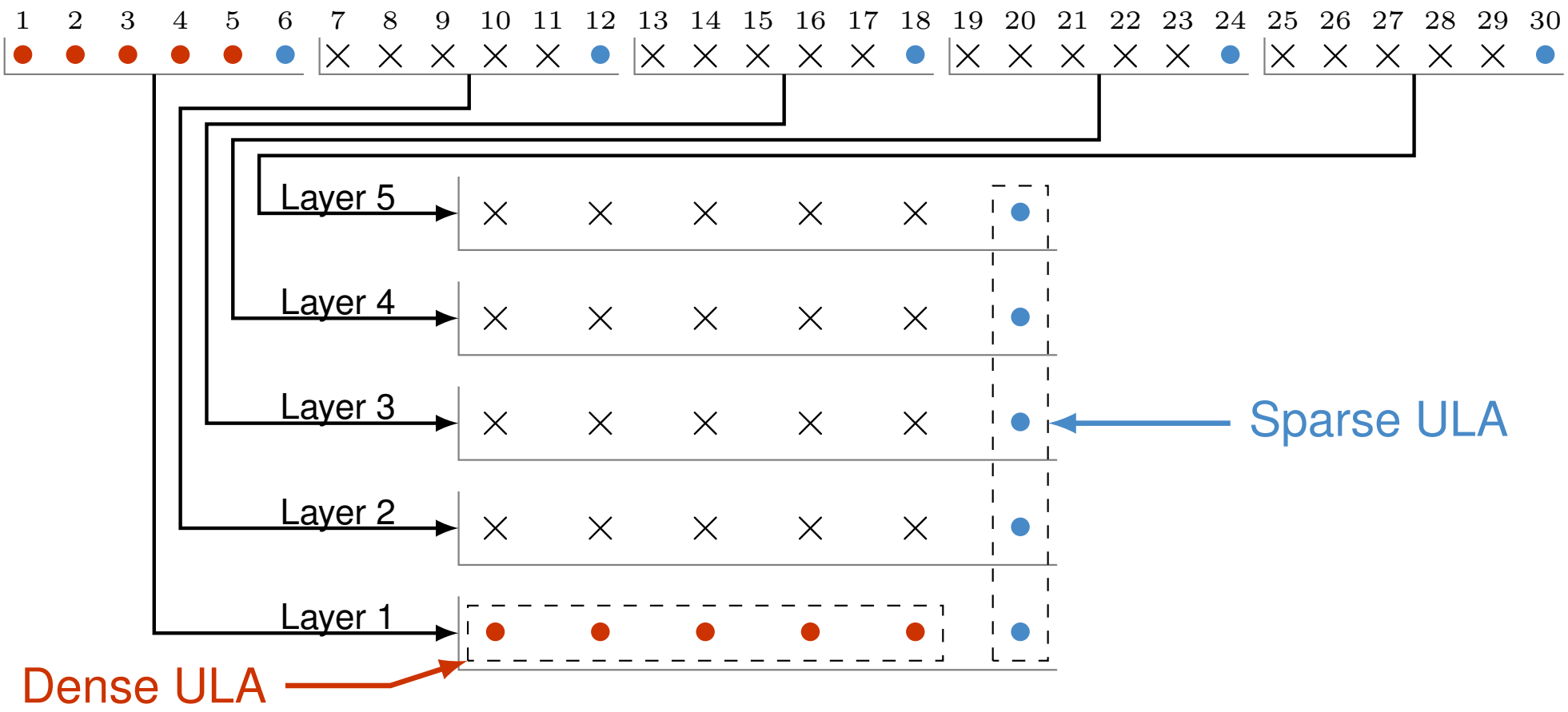
(In particular, no holes)

They should be more sparse than nested arrays,

$$w_{\text{Super nested}}(1) \leq w_{\text{Nested}}(1),$$
$$w_{\text{Super nested}}(2) \leq w_{\text{Nested}}(2),$$
$$w_{\text{Super nested}}(3) \leq w_{\text{Nested}}(3).$$

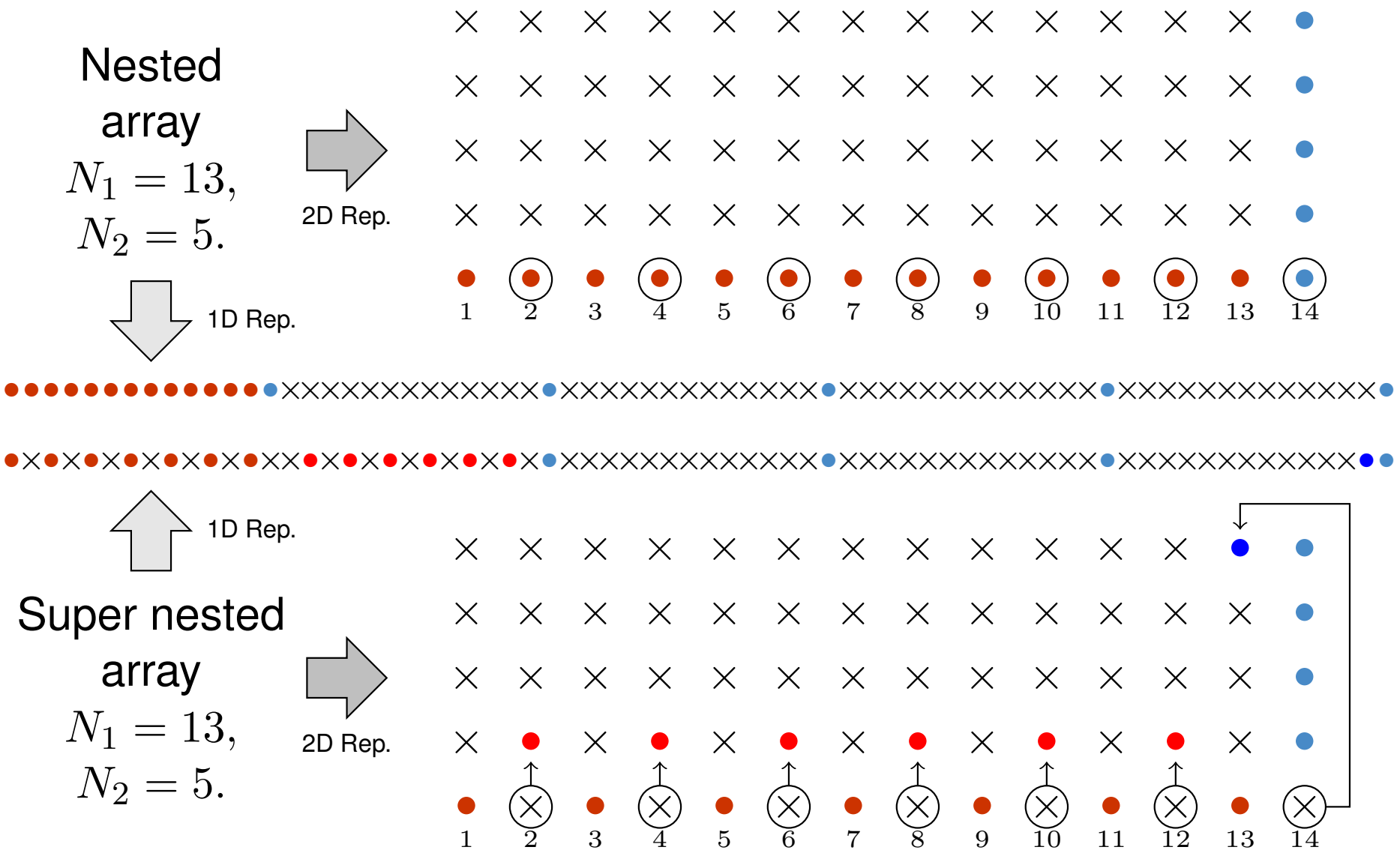
# 2D representations for 1D nested arrays[1]

The nested array with $N_1 = N_2 = 5$




[1] Liu and Vaidyanathan, to appear in *IEEE Trans. Signal Proc.*, 2016.

# From the nested array to the super nested array



# The formal definition of super nested arrays[1]

Assume $N_1$ and $N_2$ are integers satisfying $N_1 \geq 4$ and $N_2 \geq 3$. Super nested arrays are specified by the integer set $\mathbb{S}^{(2)}$, defined by

$$\mathbb{S}^{(2)} = \mathbb{X}_1^{(2)} \cup \mathbb{Y}_1^{(2)} \cup \mathbb{X}_2^{(2)} \cup \mathbb{Y}_2^{(2)} \cup \mathbb{Z}_1^{(2)} \cup \mathbb{Z}_2^{(2)},$$

- $\mathbb{X}_1^{(2)} = \{1 + 2\ell \mid 0 \leq \ell \leq A_1\}$,
- $\mathbb{Y}_1^{(2)} = \{(N_1 + 1) - (1 + 2\ell) \mid 0 \leq \ell \leq B_1\}$,
- $\mathbb{X}_2^{(2)} = \{(N_1 + 1) + (2 + 2\ell) \mid 0 \leq \ell \leq A_2\}$,
- $\mathbb{Y}_2^{(2)} = \{2(N_1 + 1) - (2 + 2\ell) \mid 0 \leq \ell \leq B_2\}$,
- $\mathbb{Z}_1^{(2)} = \{\ell(N_1 + 1) \mid 2 \leq \ell \leq N_2\}$,
- $\mathbb{Z}_2^{(2)} = \{N_2(N_1 + 1) - 1\}$.

} six ULA

- $$(A_1, B_1, A_2, B_2) = \begin{cases} (r, r-1, r-1, r-2), & \text{if } N_1 = 4r, \\ (r, r-1, r-1, r-1), & \text{if } N_1 = 4r+1, \\ (r+1, r-1, r, r-2), & \text{if } N_1 = 4r+2, \\ (r, r, r, r-1), & \text{if } N_1 = 4r+3, \end{cases}$$

  where $r$ is an integer.

[1] MATLAB routines are available at `http://systems.caltech.edu/dsp/students/clliu/SuperNested.html`

# Main properties of super nested arrays: 1) Difference coarray[1]

Difference coarray of the nested array with $N_1$ and $N_2$

$\mathbb{D}_{\text{Nested}}$

$\mathbb{D}_{\text{Super nested}}$

Difference coarray of the super nested array with $N_1$ and $N_2$

- Based on the formal definition, it can be proved that $\mathbb{D}_{\text{Super nested}} = \mathbb{D}_{\text{Nested}}$.
- Properties of $\mathbb{D}_{\text{Super nested}}$:
  - Contiguous integers.
  - Hole-free.

[1] Liu and Vaidyanathan, *IEEE ICASSP*, 2016; Liu and Vaidyanathan, to appear in *IEEE Trans. Signal Proc.*, 2016.

# Main properties of super nested arrays: 2) Weight functions[1]

- The formal definition for super nested arrays allows us to prove closed-form expressions for $w(1)$, $w(2)$, and $w(3)$.

| | Nested arrays | Super nested arrays |
|---|---|---|
| $w(1)$ | ✗ $N_1$ | ✓ $\begin{cases} 2, & \text{if } N_1 \text{ is even,} \\ 1, & \text{if } N_1 \text{ is odd.} \end{cases}$ |
| $w(2)$ | $N_1 - 1$ | $\begin{cases} N_1 - 3, & \text{if } N_1 \text{ is even,} \\ N_1 - 1, & \text{if } N_1 \text{ is odd,} \end{cases}$ |
| $w(3)$ | ✗ $N_1 - 2$ | ✓ $\begin{cases} 3, & \text{if } N_1 = 4, 6, \\ 4, & \text{if } N_1 \text{ is even, } N_1 \geq 8, \\ 1, & \text{if } N_1 \text{ is odd,} \end{cases}$ |

[1] Liu and Vaidyanathan, *IEEE ICASSP*, 2016; Liu and Vaidyanathan, to appear in *IEEE Trans. Signal Proc.*, 2016.

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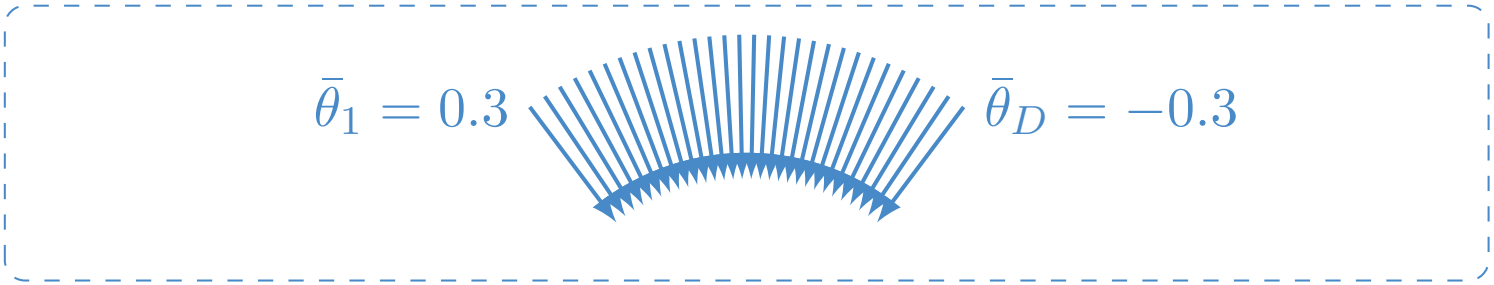
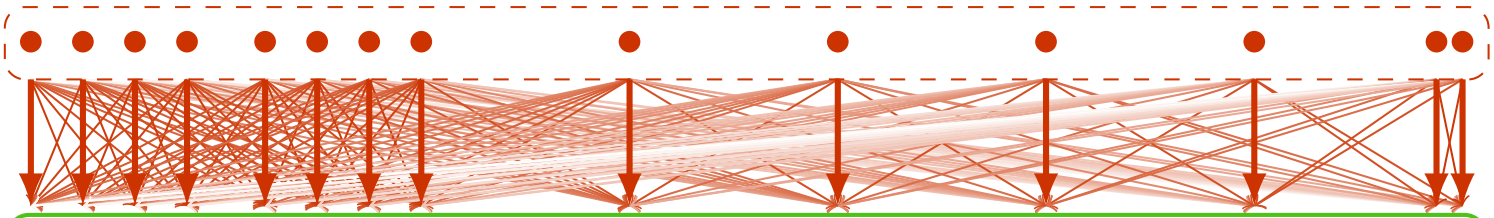
# Simulation procedure




$D = 30$ Uncorrelated Sources



14 Sensors

Spatial Smoothing MUSIC (SS MUSIC)[1]

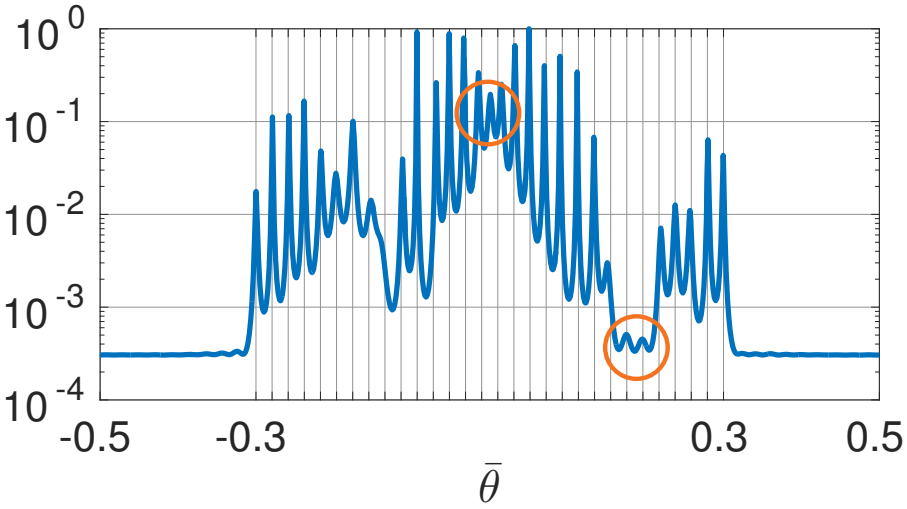
Estimated normalized DOA $\hat{\bar{\theta}}_i$

10 dB SNR, 500 snapshots, RMSE $E = (\frac{1}{D}\sum_{i=1}^{D}(\hat{\bar{\theta}}_i - \bar{\theta}_i)^2)^{1/2}$

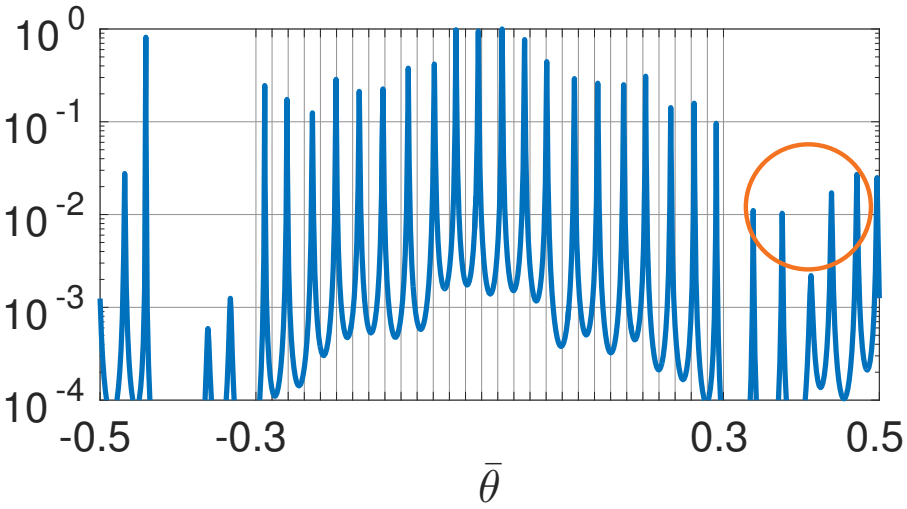
[1] Pal and Vaidyanathan, *IEEE Trans. Signal Proc.*, 2010; Liu and Vaidyanathan, *IEEE Signal Proc. Lett.*, 2015.

# MUSIC spectra

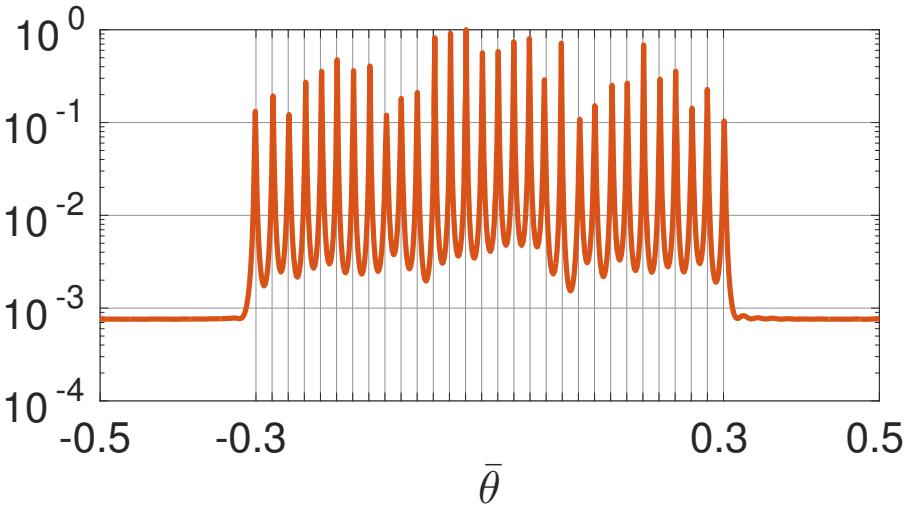
Nested array ($E = 0.012066$)




Coprime array ($E = 0.10529$)




Super nested array ($E = 0.001201$)




The super nested array is much better.

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# Concluding remarks

- Super nested arrays
  - They have the same number of sensors, the same difference coarray as nested arrays.
  - They have reduced mutual coupling than nested arrays.
  - They can be constructed from nested arrays or from the formal definition.
- In the future, decoupling algorithms will improve the performance.[1]
- For more information, please go to our project website: `http://systems.caltech.edu/dsp/students/clliu/SuperNested.html`

Thank you!

[1] Friedlander and Weiss, *IEEE Trans. Antennas Propag.*, 1991; BouDaher, Ahmad, Amin, and Hoorfar, *EUSIPCO*, 2015.