Fast learning of fast transforms, with guarantees

IEEE ICASSP 2022, Singapore

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May, 2022

Approximating a matrix by a product of sparse factors

Given a matrix $\mathbf{Z}$ and $J \geq 2$, find sparse factors $\mathbf{X}^{(J)}, \dots, \mathbf{X}^{(1)}$ such that

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Problem formulation

$$\min_{\mathbf{X}^{(1)},\ldots,\mathbf{X}^{(J)}} \left\| \mathbf{Z} - \mathbf{X}^{(J)}\mathbf{X}^{(J-1)} \ldots \mathbf{X}^{(1)} \right\|_F^2, \quad \text{such that } \{\mathbf{X}^{(\ell)}\}_\ell \text{ are sparse.}$$

Choices for sparsity constraint:

1. **Classical sparsity patterns**: k-sparsity by column and/or by row
2. **Fixed-support constraint**: $\mathrm{supp}(\mathbf{X}^{(\ell)}) \subseteq \mathbf{S}^{(\ell)}$ for $\ell = 1, \ldots, J$.

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A difficult problem

- Sparse coding is NP-hard [Foucart et al. 2013].
- Fixed-support setting is NP-hard for $J = 2$ factors [Le et al. 2021].
- Gradient-based methods [Le Magoarou et al. 2016] lack guarantees.

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When is the problem well-posed and tractable? (case with $J = 2$)

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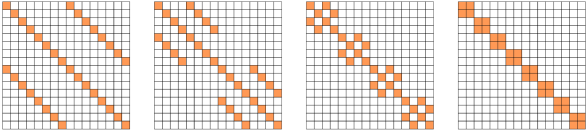
→ We study a fixed-support constraint ($J \geq 2$) satisfying such conditions.



(a) $\mathbf{S}_{\mathrm{bf}}^{(4)}$ (b) $\mathbf{S}_{\mathrm{bf}}^{(3)}$ (c) $\mathbf{S}_{\mathrm{bf}}^{(2)}$ (d) $\mathbf{S}_{\mathrm{bf}}^{(1)}$

Figure: Butterfly structure: $\mathrm{supp}(\mathbf{X}^{(\ell)}) \subseteq \mathbf{S}_{\mathrm{bf}}^{(\ell)} := \mathbf{I}_{N/2^{\ell}} \otimes \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right] \otimes \mathbf{I}_{2^{\ell-1}}$.

The butterfly structure is common to many fast transforms (e.g. DFT).

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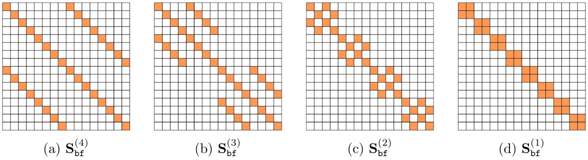

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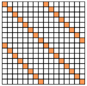
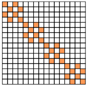
Main contribution

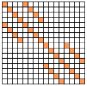
An efficient **hierarchical algorithm** to approximate **any** matrix by a product of butterfly factors.

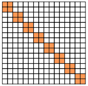
Hierarchical factorization algorithm

Let $\mathbf{Z} := \mathbf{X}^{(4)}\mathbf{X}^{(3)}\mathbf{X}^{(2)}\mathbf{X}^{(1)}$ such that:

$\mathrm{supp}(\mathbf{X}^{(4)}) \subseteq$ 

$\mathrm{supp}(\mathbf{X}^{(3)}) \subseteq$ 

$\mathrm{supp}(\mathbf{X}^{(2)}) \subseteq$ 

$\mathrm{supp}(\mathbf{X}^{(1)}) \subseteq$ 

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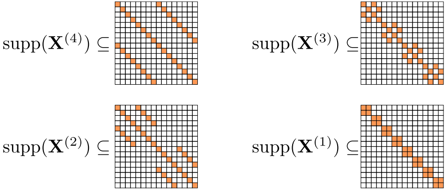

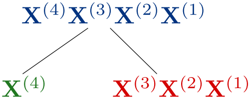

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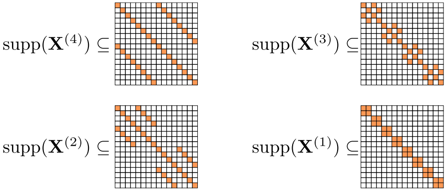

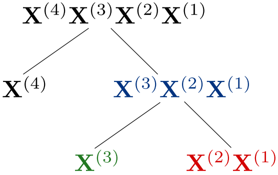

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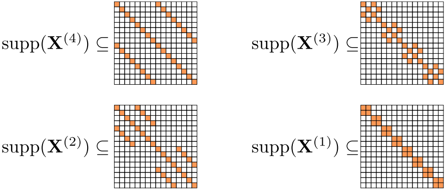

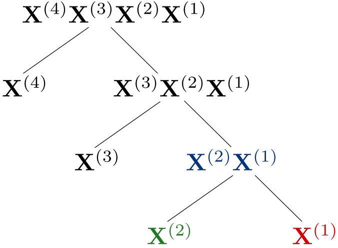

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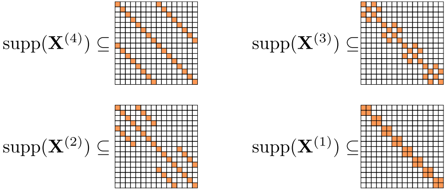

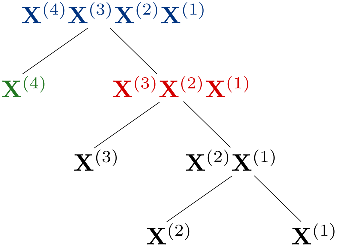

How to recover the partial products?

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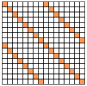
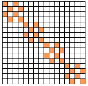
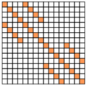
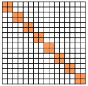
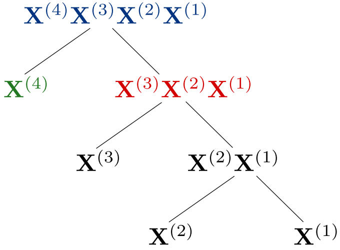
$\mathrm{supp}(\mathbf{X}^{(4)}) \subseteq$ 

$\mathrm{supp}(\mathbf{X}^{(3)}) \subseteq$ 

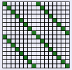
$\mathrm{supp}(\mathbf{X}^{(2)}) \subseteq$ 

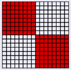
$\mathrm{supp}(\mathbf{X}^{(1)}) \subseteq$ 



How to recover the partial products? $\rightarrow$ use their known supports

Lemma (Supports of the partial products)

$\mathrm{supp}(\mathbf{X}^{(4)}) \subseteq$  $= \mathbf{S}_{\mathtt{bf}}^{(4)}$

$\mathrm{supp}(\mathbf{X}^{(3)}\mathbf{X}^{(2)}\mathbf{X}^{(1)}) \subseteq$  $= \mathbf{S}_{\mathtt{bf}}^{(3)}\mathbf{S}_{\mathtt{bf}}^{(2)}\mathbf{S}_{\mathtt{bf}}^{(1)}$

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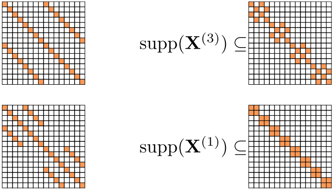

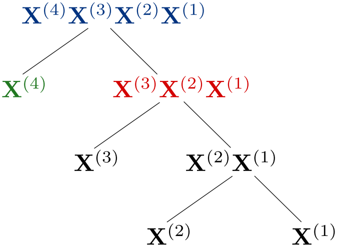

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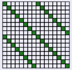
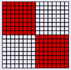
$\mathrm{supp}(\mathbf{X}^{(4)}) \subseteq$  $= \mathbf{S}_{\mathtt{bf}}^{(4)}$

$\mathrm{supp}(\mathbf{X}^{(3)}\mathbf{X}^{(2)}\mathbf{X}^{(1)}) \subseteq$  $= \mathbf{S}_{\mathtt{bf}}^{(3)}\mathbf{S}_{\mathtt{bf}}^{(2)}\mathbf{S}_{\mathtt{bf}}^{(1)}$

Two-layer fixed-support problem:

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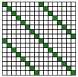
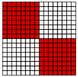
$\operatorname{supp}(\mathbf{A}) \subseteq$  $= \mathbf{S}_{\mathtt{bf}}^{(4)}$

$\operatorname{supp}(\mathbf{B}) \subseteq$  $= \mathbf{S}_{\mathtt{bf}}^{(3)}\mathbf{S}_{\mathtt{bf}}^{(2)}\mathbf{S}_{\mathtt{bf}}^{(1)}$

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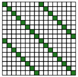
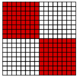
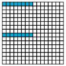
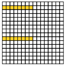
$\mathrm{supp}(\mathbf{A}) \subseteq$  $= \mathbf{S}_{\mathtt{bf}}^{(4)}$

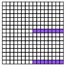
$\mathrm{supp}(\mathbf{B}) \subseteq$  $= \mathbf{S}_{\mathtt{bf}}^{(3)}\mathbf{S}_{\mathtt{bf}}^{(2)}\mathbf{S}_{\mathtt{bf}}^{(1)}$

Constraint on the rank-one matrices

$\mathrm{supp}(\mathbf{A}_{\bullet,1}\mathbf{B}_{1,\bullet}) \subseteq$  $= \mathcal{S}_1$

$\mathrm{supp}(\mathbf{A}_{\bullet,2}\mathbf{B}_{2,\bullet}) \subseteq$  $= \mathcal{S}_2$

$\vdots$

$\mathrm{supp}(\mathbf{A}_{\bullet,N}\mathbf{B}_{N,\bullet}) \subseteq$  $= \mathcal{S}_N$

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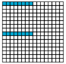
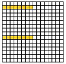
$\operatorname{supp}(\mathbf{A}_{\bullet,1}\mathbf{B}_{1,\bullet}) \subseteq$  $= \mathcal{S}_1$

$\operatorname{supp}(\mathbf{A}_{\bullet,2}\mathbf{B}_{2,\bullet}) \subseteq$  $= \mathcal{S}_2$

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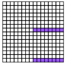
$\operatorname{supp}(\mathbf{A}_{\bullet,N}\mathbf{B}_{N,\bullet}) \subseteq$  $= \mathcal{S}_N$

Theorem ([Le et al. 2021; Zheng et al. 2022])

The rank-one matrices have **pairwise disjoint supports**. *Consequently, (1) is polynomially solvable and admits an essentially unique solution.*

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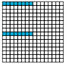
$\operatorname{supp}(\mathbf{A}_{\bullet,1}\mathbf{B}_{1,\bullet}) \subseteq$  $= \mathcal{S}_1$

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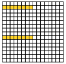
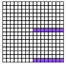
$\operatorname{supp}(\mathbf{A}_{\bullet,2}\mathbf{B}_{2,\bullet}) \subseteq$  $= \mathcal{S}_2$

$\operatorname{supp}(\mathbf{A}_{\bullet,N}\mathbf{B}_{N,\bullet}) \subseteq$  $= \mathcal{S}_N$

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Algorithm to solve (1):

1. Extract the submatrices $\mathbf{Z}_{|\mathcal{S}_i}$, $i = 1, \ldots, N$
2. Perform best rank-one approximation for each submatrix

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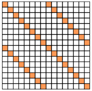
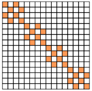
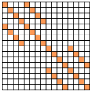
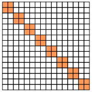
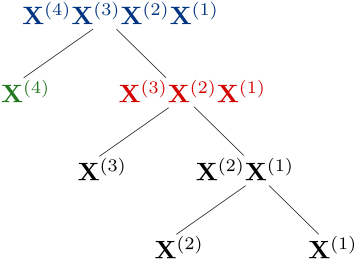
$\mathrm{supp}(\mathbf{X}^{(4)}) \subseteq$ 

$\mathrm{supp}(\mathbf{X}^{(3)}) \subseteq$ 

$\mathrm{supp}(\mathbf{X}^{(2)}) \subseteq$ 

$\mathrm{supp}(\mathbf{X}^{(1)}) \subseteq$ 



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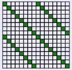
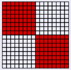
Lemma (Support of the partial products)

$\mathrm{supp}(\mathbf{X}^{(4)}) \subseteq$  $= \mathbf{S}_{\mathtt{bf}}^{(4)}$

$\mathrm{supp}(\mathbf{X}^{(3)}\mathbf{X}^{(2)}\mathbf{X}^{(1)}) \subseteq$  $= \mathbf{S}_{\mathtt{bf}}^{(3)}\mathbf{S}_{\mathtt{bf}}^{(2)}\mathbf{S}_{\mathtt{bf}}^{(1)}$

The corresponding rank-one supports are pairwise disjoint.

Hierarchical factorization algorithm

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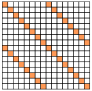
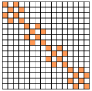
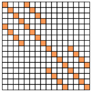
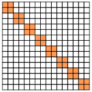
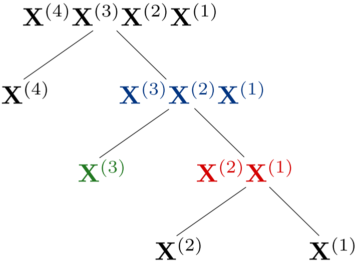
$\mathrm{supp}(\mathbf{X}^{(4)}) \subseteq$ 

$\mathrm{supp}(\mathbf{X}^{(3)}) \subseteq$ 

$\mathrm{supp}(\mathbf{X}^{(2)}) \subseteq$ 

$\mathrm{supp}(\mathbf{X}^{(1)}) \subseteq$ 



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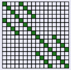
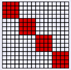
$\mathrm{supp}(\mathbf{X}^{(3)}) \subseteq$  $= \mathbf{S}_{\mathtt{bf}}^{(3)}$

$\mathrm{supp}(\mathbf{X}^{(2)}\mathbf{X}^{(1)}) \subseteq$  $= \mathbf{S}_{\mathtt{bf}}^{(2)}\mathbf{S}_{\mathtt{bf}}^{(1)}$

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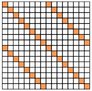
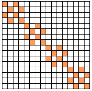
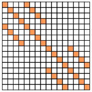
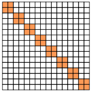
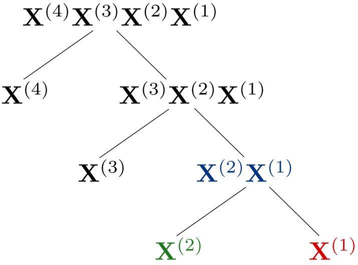
$\mathrm{supp}(\mathbf{X}^{(4)}) \subseteq$ 

$\mathrm{supp}(\mathbf{X}^{(3)}) \subseteq$ 

$\mathrm{supp}(\mathbf{X}^{(2)}) \subseteq$ 

$\mathrm{supp}(\mathbf{X}^{(1)}) \subseteq$ 



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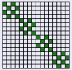
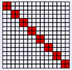
$\mathrm{supp}(\mathbf{X}^{(2)}) \subseteq$  $= \mathbf{S}_{\mathtt{bf}}^{(2)}$

$\mathrm{supp}(\mathbf{X}^{(1)}) \subseteq$  $= \mathbf{S}_{\mathtt{bf}}^{(1)}$

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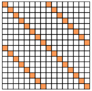
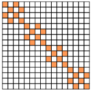
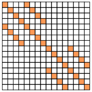
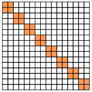
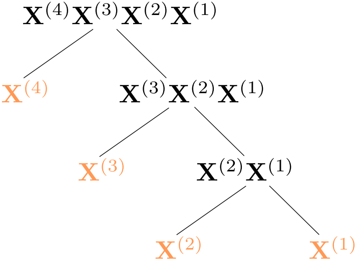
$\mathrm{supp}(\mathbf{X}^{(4)}) \subseteq$ 

$\mathrm{supp}(\mathbf{X}^{(3)}) \subseteq$ 

$\mathrm{supp}(\mathbf{X}^{(2)}) \subseteq$ 

$\mathrm{supp}(\mathbf{X}^{(1)}) \subseteq$ 



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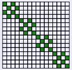
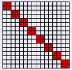
$\mathrm{supp}(\mathbf{X}^{(2)}) \subseteq$  $= \mathbf{S}_{\mathtt{bf}}^{(2)}$

$\mathrm{supp}(\mathbf{X}^{(1)}) \subseteq$  $= \mathbf{S}_{\mathtt{bf}}^{(1)}$

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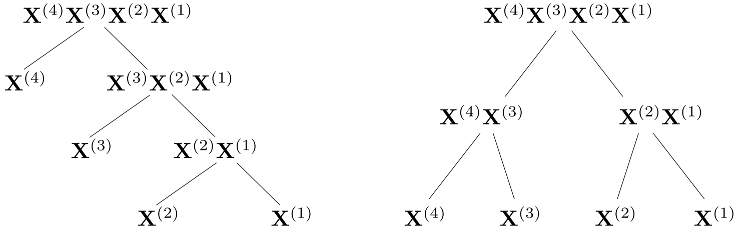
The butterfly factors $\{\mathbf{X}^{(\ell)}\}_{\ell=1}^{4}$ are recovered (up to scaling ambiguities) from the product $\mathbf{Z}$.

Theoretical guarantees

The algorithm works for any number of factors and any binary tree.



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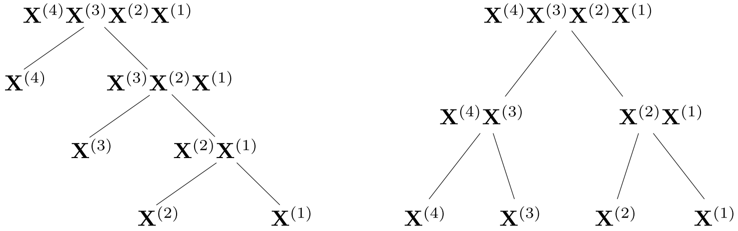

Theorem (Exact recovery guarantees [Zheng et al. 2022])

Except for trivial degeneracies, every tuple $(\mathbf{X}^{(\ell)})_{\ell=1}^{J}$ *satisfying the butterfly constraint can be reconstructed by the algorithm from* $\mathbf{Z} := \mathbf{X}^{(J)} \ldots \mathbf{X}^{(1)}$ *(up to unavoidable scaling ambiguities).*

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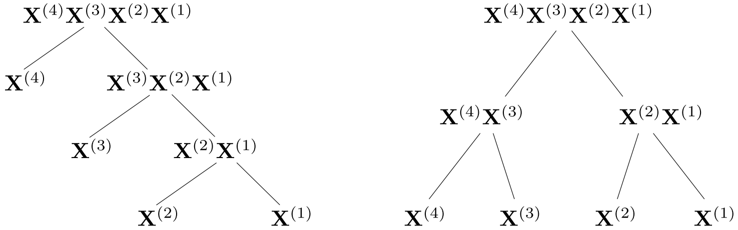

Theorem (Exact recovery guarantees [Zheng et al. 2022])

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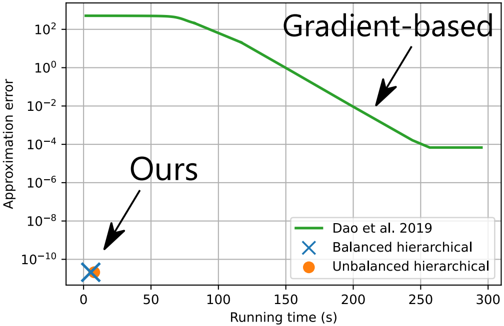
- Complexity is $\mathcal{O}(N^2)$ for both trees.
- We can use the algorithm in the non-exact setting.

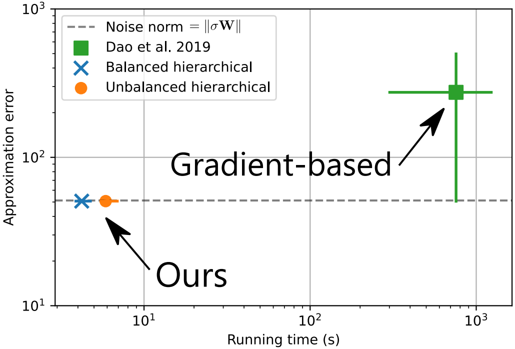
Faster and more accurate in the noiseless setting

Approximation of the DFT matrix by a product of $J = 9$ butterfly factors:



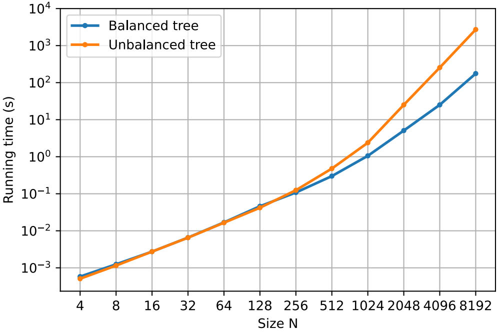
Also more robust in the noisy setting

Approximation of $\mathbf{Z} = \mathbf{DFT_N} + \sigma\mathbf{W}$ by a product of $J = 9$ butterfly factors:



Our method scales with the matrix size

Approximation of the (noisy) DFT matrix of size $N = 2^J$ by a product of J butterfly factors:



Conclusion and perspectives

Hierarchical algorithm: $\mathcal{O}(N^2)$

$\mathbf{Z} \in \mathbb{R}^{N \times N}$ (dense)

Storage: $\mathcal{O}(N^2)$

Cost for evaluation: $\mathcal{O}(N^2)$
$\mathbf{x} \mapsto \mathbf{Z}\mathbf{x}$

$\tilde{\mathbf{Z}} := \mathbf{X}^{(J)}\mathbf{X}^{(J-1)} \ldots \mathbf{X}^{(1)}$

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Implementation in the FAµST toolbox at `https://faust.inria.fr`.

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Future work

- Application in dictionary learning, sparse neural network training, ...
- Stability properties of the hierarchical algorithm

Thank you for your attention!

To know more:

Q.-T. Le, E. Riccietti, and R. Gribonval (2022)
Spurious Valleys, Spurious Minima and NP-hardness of Sparse Matrix Factorization With Fixed Support
arXiv preprint, arXiv:2112.00386.

L. Zheng, E. Riccietti, and R. Gribonval (2022)
Efficient Identification of Butterfly Sparse Matrix Factorizations
arXiv preprint, arXiv:2110.01235.