EPFL

Spline density functions

Bona-fide Riesz projections for density estimation

Find the code at: github.com/poldap/rpde

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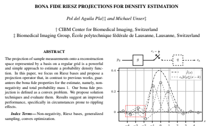
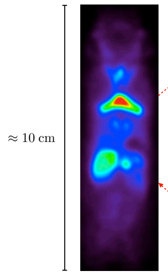


Pointillist rendering of the ICASSP 2022 screen backdrop

Joint work with Prof. M. Unser

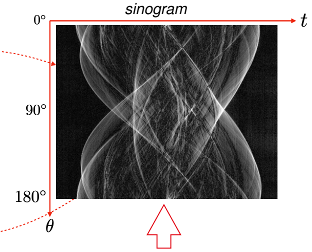
Recorded in April 2022

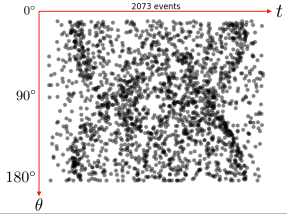
Motivation in biomedical imaging: Positron Emission Tomography

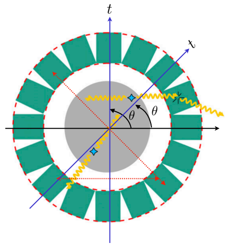


Representative PET image of a mouse injected with [18F]-F-DPA.

J. Yang et al., "Synthesis-free PET imaging of brown adipose tissue and TSPO via combination of disulfiram and ^{64}CuCl$_2$," Scientific Reports 7 (2017), 8298







Standard configuration of rectangular block detectors arranged into a cylindrical system.

C. S. Levin and H. Zaidi, "Current Trends in Preclinical PET System Design," PET Clin 2 (2007), 125–160

Generalised sampling / Approximation in shift-invariant spaces

$$f \in \mathrm{L}_2(\mathbb{R}) \longmapsto \tilde{f} \in V_{\mathrm{s}}$$

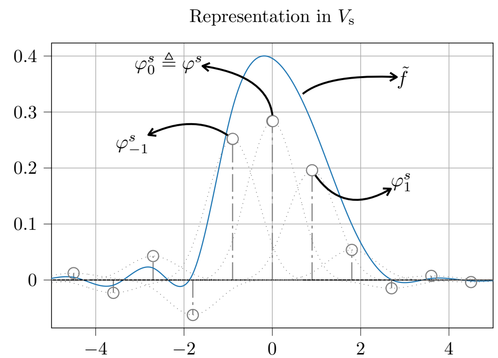
M. Unser, "Sampling—50 Years After Shannon," Proc. of the IEEE (2000), 569–587

Riesz basis condition

- Requirements for V_{s}
  - Discrete (finite/countable)
  - Good approximation properties
  - Some sort of shift invariance
- Shift invariant spaces

$$V_{\mathrm{s}} = \left\{ \tilde{f} = \sum_{k\in\mathbb{Z}} c_{\mathrm{s}}[k]\varphi^{\mathrm{s}}_k : c_{\mathrm{s}} \in \ell_2 \right\} \subset \mathrm{L}_2(\mathbb{R})$$

- with $\varphi^{\mathrm{s}}_k = \varphi^{\mathrm{s}}(\cdot - k)$
- Error order controlled by φ^{s}



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- with $\varphi^{\mathrm{s}}_k = \varphi^{\mathrm{s}}(\cdot - k)$
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$$\min_{\breve{f}\in V_{\mathrm{s}}} \left\{ \|f - \breve{f}\|_{\mathrm{L}_2(\mathbb{R})} \right\}$$

$$\Rightarrow \tilde{f} = \sum_{k\in\mathbb{Z}} \overbrace{\langle \mathring{\varphi}^{\mathrm{s}}_k, f\rangle}^{c_{\mathrm{s}}[k]} \varphi^{\mathrm{s}}_k$$

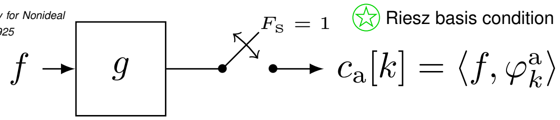
$$\exists! \mathring{\varphi}^{\mathrm{s}} \in V_{\mathrm{s}} \mid \langle \mathring{\varphi}^{\mathrm{s}}_k, \varphi^{\mathrm{s}}_l \rangle = \delta[k-l], \forall k, l \in \mathbb{Z}$$

- but we do not need to find it!

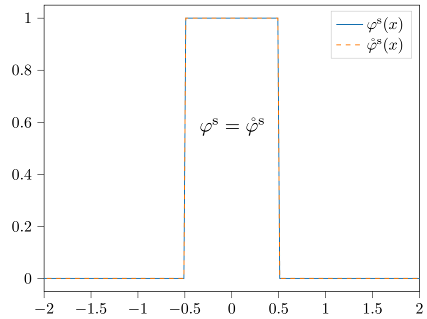
Generalised sampling: Signal processing perspective

M. Unser and A. Aldroubi, "A General Sampling Theory for Nonideal Acquisition Devices," IEEE TSP 42 (11) (1994), 2915–2925

- Real-life sampling



- $c_a[k] = (g * f)(k) = \langle f, \varphi_k^a \rangle$ where $\varphi^a(x) = g(-x)$
- The ideal case: $\varphi^a = \mathring{\varphi}^s \Longrightarrow c_s[k] = c_a[k], \forall k \in \mathbb{Z}$
- What we hope for:
  $V_a = \{\tilde{f} = \sum_{k\in\mathbb{Z}} c_a[k]\varphi_k^a : c_a \in \ell_2\} = V_s$



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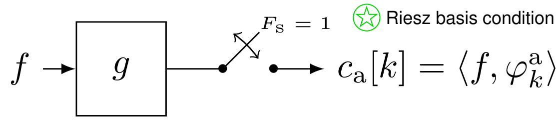
Generalised sampling: Signal processing perspective

- Real-life sampling



- What we hope for: *Orthogonal projection (min L2)*

  $V_a = \{\tilde{f} = \sum_{k\in\mathbb{Z}} c_a[k]\varphi_k^a : c_a \in \ell_2\} = V_s$

  $\Longrightarrow \mathring{\varphi}^s = \sum_{k\in\mathbb{Z}} q[k]\varphi_k^a$

  $\Longrightarrow c_s[k] = (q * c_a)[k], \forall k \in \mathbb{Z}$

  where $\hat{q}(\omega) = 1/\hat{r}_{a,s}(\omega)$, with $r_{a,s}[k] = \langle \varphi_k^a, \varphi^s \rangle$.

Consistency principle

a.k.a. measurement-reconstruction invariance

$$\begin{aligned}\langle \tilde{f}, \varphi_l^a \rangle &= \sum_{k\in\mathbb{Z}} (q * c_a)[k] \langle \varphi_k^s, \varphi_l^a \rangle \\ &= \sum_{k\in\mathbb{Z}} (q * c_a)[k] r_{a,s}[k-l] \\ &= (r_{a,s} * q * c_a)[l] = c_a[l] \\ &= \langle f, \varphi_l^a \rangle .\end{aligned}$$

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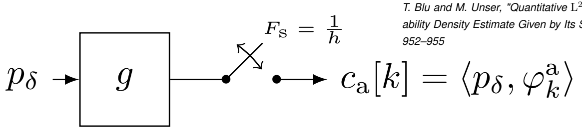
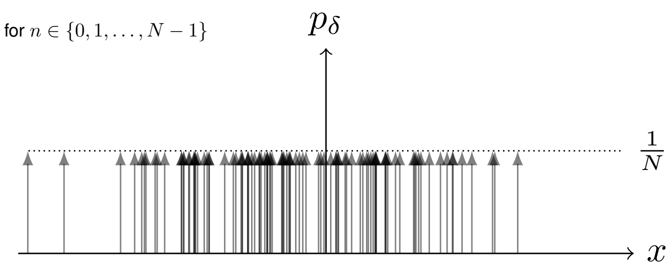
Generalised sampling for density estimation

T. Blu and M. Unser, "Quantitative L^2 Approximation Error of a Probability Density Estimate Given by Its Samples," IEEE ICASSP (2004), 952–955



For $x_n \sim \mathcal{X}$ i.i.d. for $n \in \{0, 1, \ldots, N-1\}$

$$p_\delta = \sum_{n=0}^{N-1} \delta_{x_n}$$



Empirical estimate for 100 samples of a normal

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Generalised sampling for density estimation

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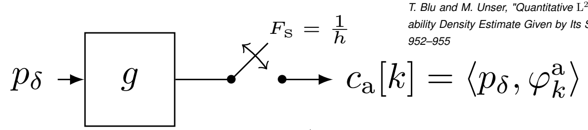

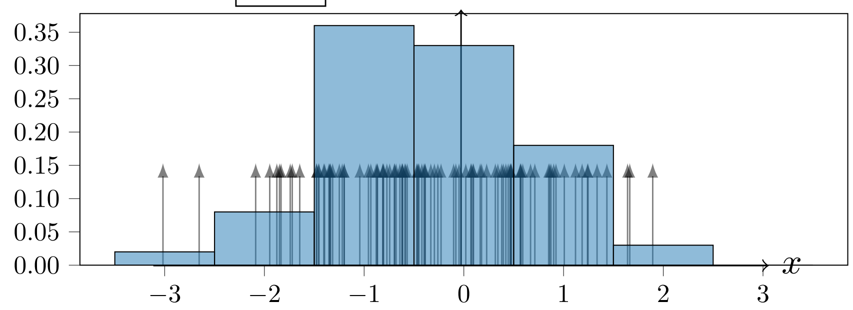

Histogram: $\varphi^a = \mathring{\varphi}^s = \varphi^s = \beta_0, r_{a,s}[k] = \beta_1(k) = \delta[k]$.

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EPFL Generalised sampling for density estimation

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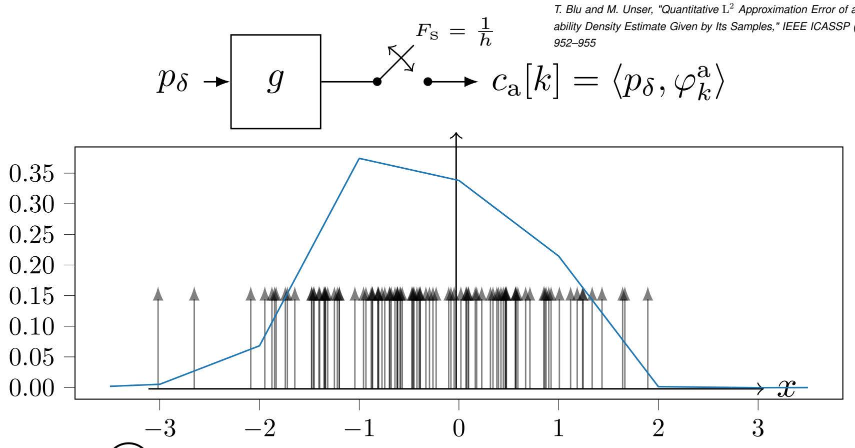

$\varphi^a = \varphi^s = \beta_1$ $r_{a,s}[k] = \beta_3(k)$.

EPFL Generalised sampling for density estimation

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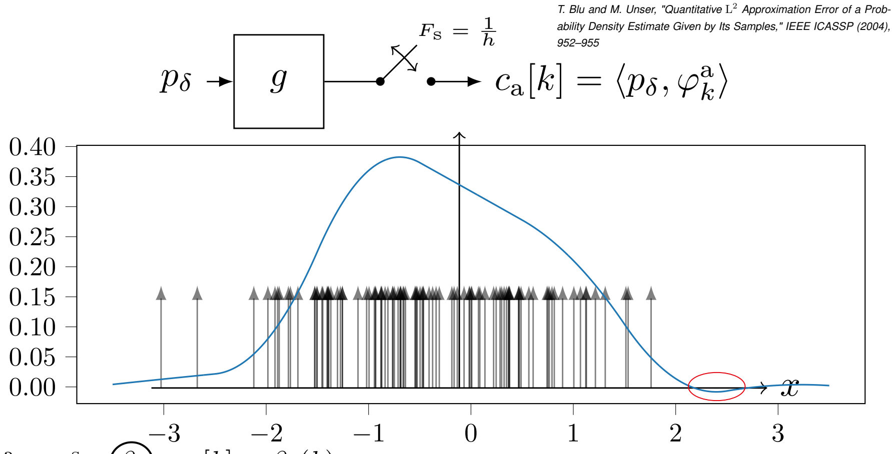
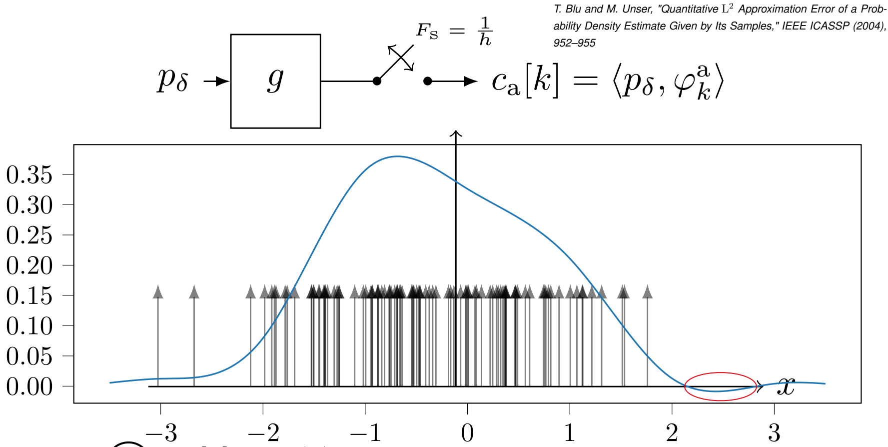


$\varphi^a = \varphi^s = \beta_2$ $a_{a,s}[k] = \beta_5(k)$.

EPFL Generalised sampling for density estimation

T. Blu and M. Unser, "Quantitative L^2 Approximation Error of a Probability Density Estimate Given by Its Samples," IEEE ICASSP (2004), 952–955



$\varphi^a = \varphi^s = \beta_3$ $r_{a,s}[k] = \beta_7(k)$.

EPFL Generalised sampling for density estimation: Error characteristics

- Let $f_\tau(x) \triangleq f(x-\tau)$, consider $\tilde{f} : \mathbb{R}^N \to V_s$

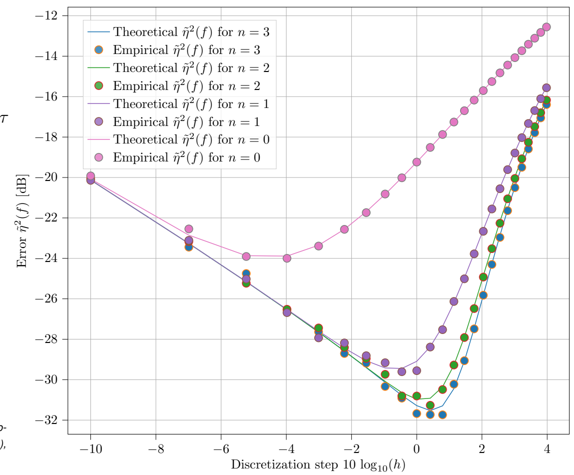
$$\tilde{\eta}^2(f) = \frac{1}{h}\int_0^h \mathrm{E}\left\{\|f_\tau - \tilde{f}(\{x_n + \tau\})\|^2_{L_2}\right\} \mathrm{d}\tau$$

- Closed form expressions for $\tilde{\eta}^2(f)$
- Compromise in setting h, statistical vs approximation error
- Estimating a standard Gaussian from 10^3 samples, h going from 0.1 to 2.5

$\varphi^a = \varphi^s = \beta_n$

$r_{a,s}[k] = \beta_{2n+1}(k)$

T. Blu and M. Unser, "Quantitative L^2 Approximation Error of a Probability Density Estimate Given by Its Samples," IEEE ICASSP (2004), 952–955



Non-negative approximation

$$f \in \mathrm{L}_2(\mathbb{R}) \longmapsto \tilde{f} \in V_{\mathrm{s}}^{+}$$

- V_{s} such that $\varphi^s = \beta_n$
  - V_{s}^{+} is a closed convex cone in V_{s} (and in $\mathrm{L}_2(\mathbb{R})$)
  - Because it is a closed convex set, the projection is unique, i.e.,

$$\exists! \tilde{f}_{+} \in V_{\mathrm{s}}^{+} \mid \|f - \tilde{f}_{+}\|_{\mathrm{L}_2} \leq \|f - f_{+}\|_{\mathrm{L}_2}, \forall f_{+} \in V_{\mathrm{s}}^{+}$$

  - How to find $\tilde{f}_{+}$: an open problem

Non-negative approximation

$$f \in \mathrm{L}_2(\mathbb{R}) \longmapsto \tilde{f} \in V_{\mathrm{s}}^{+}$$

- V_{s} such that $\varphi^s = \beta_n$
  - How to find $\tilde{f}_{+}$: an open problem
- In PDF estimation, *bona-fide* functions

$$\tilde{f}_{+}(x) \geq 0, \forall x \in \mathbb{R} \text{ and } \int \tilde{f}_{+}(x)\mathrm{d}x = 1$$

Recall: Consistency principle, *a.k.a.* measurement-reconstruction invariance

$$\langle \tilde{f}, \varphi_k^{\mathrm{a}} \rangle = \langle f, \varphi_k^{\mathrm{a}} \rangle, \forall l \in \mathbb{Z}.$$

- Proposal: Relax consistency, impose *bona-fide*

$$\min_{\breve{f} \in V_{\mathrm{s}}} \left\{ \|\langle f, \varphi_k^{\mathrm{a}} \rangle - \langle \breve{f}, \varphi_k^{\mathrm{a}} \rangle\|_2^2 \right\}$$

$$\text{such that } \breve{f}(x) \geq 0, \forall x \in \mathbb{R} \text{ and } \int \breve{f}(x)\mathrm{d}x = 1$$

Non-negative approximation

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$$\text{such that } \breve{f}(x) \geq 0, \forall x \in \{mh_c : m \in \mathbb{Z}\} \text{ and } \int \breve{f}(x)\mathrm{d}x = 1$$

$$\Updownarrow$$

$$\min_{c_{\mathrm{s}} \in \ell_2} \left\{ \|c_{\mathrm{a}}[k] - (r_{\mathrm{a,s}} * c_{\mathrm{s}})[k]\|_2^2 \right\}$$

$$\text{such that } (c_{\mathrm{s}}^{\uparrow M}[k] * \varphi^{\mathrm{s}}(k/M))[q] \geq 0, \forall q \in \mathbb{Z} \text{ and } \sum_{k \in \mathbb{Z}} c_{\mathrm{s}}[k] = 1$$

✓ convex ✓ quadratic ✓ linear constraints ⟳ finite dimensional for finite support f

Relaxed consistency, Bona-fide PDF estimation: Error characteristics

$$\tilde{\eta}^2(f) = \frac{1}{h}\int_0^h \mathrm{E}\left\{\|f_\tau - \tilde{f}(\{x_n + \tau\})\|^2_{\mathrm{L}_2}\right\} \mathrm{d}\tau$$

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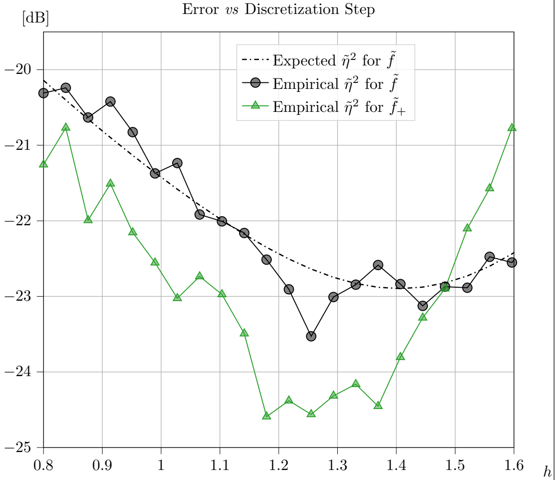
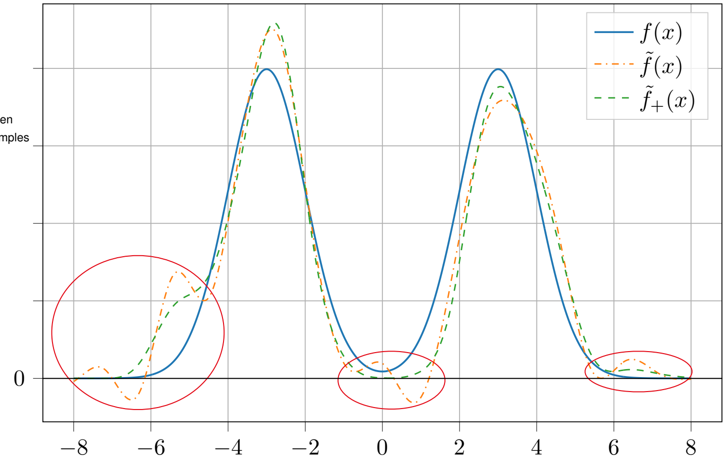


Relaxed consistency, Bona-fide PDF estimation

- Estimating an equal mixture between $\mathcal{N}(-3,1)$ and $\mathcal{N}(3,1)$ from 10^2 samples

$$\varphi^{\mathrm{a}} = \varphi^{\mathrm{s}} = \beta_3$$

$$r_{\mathrm{a,s}}[k] = \beta_7(k)$$



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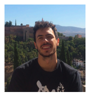

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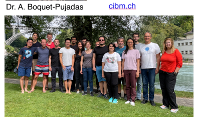

Everyone at BIG, EPFL (bigwww.epfl.ch)

Joint work with Prof. M. Unser