Digraph Signal Processing with Generalized Boundary Conditions

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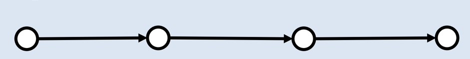
Goal

Concept	Undirected Graphs	Directed Graphs
Shift/Variation operator	✓ Symmetric	✓ Not symmetric
Convolution	✓	✓
Fourier Basis/Transform	✓	✗ **May not exist**
Orthogonality	✓	✗ (In general no)

Digraph Example:



$$A=\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{bmatrix},\quad L=D-A=\begin{bmatrix}1&-1&0&0\\0&1&-1&0\\0&0&1&-1\\0&0&0&1\end{bmatrix}$$

No eigendecomposition/Fourier basis
Only one eigenvalue (Jordan Block)

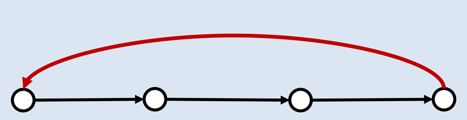
Key Idea: Boundary Conditions

Classical Signal Processing

Finite, discrete time

$$A=\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{bmatrix}$$

Add periodic extension

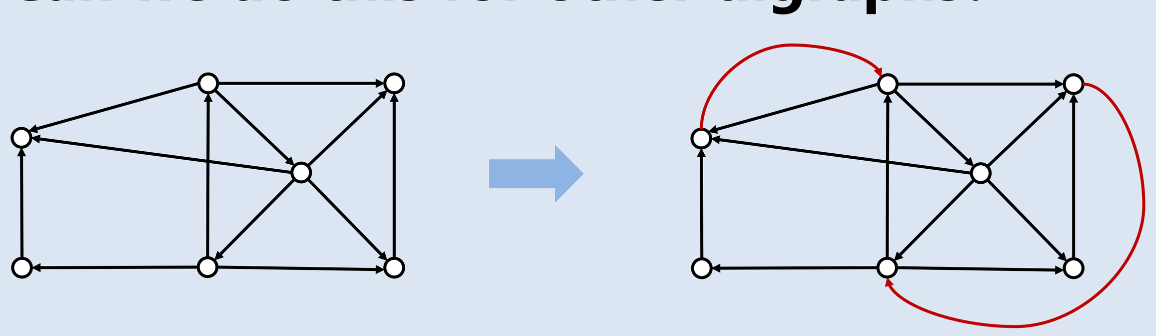


$$A+B=\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{bmatrix}$$

$$\mathrm{DFT}_4(A+B)\,\mathrm{DFT}_4^{-1}=\begin{bmatrix}1&\cdot&\cdot&\cdot\\\cdot&\mathrm{i}&\cdot&\cdot\\\cdot&\cdot&-1&\cdot\\\cdot&\cdot&\cdot&-\mathrm{i}\end{bmatrix}$$

This is done even when **the signal is not periodic**!

Can we do this for other digraphs?



$$\begin{bmatrix}0&1&&&&&\\0&0&&&&&\\&&0&1&&&\\&&0&0&&&\\&&&&1&&\\&&&&&\omega_3^2&\\&&&&&&\omega_3\end{bmatrix}\rightarrow\begin{bmatrix}\lambda_1&&&&&&\\&\lambda_2&&&&&\\&&\lambda_3&&&&\\&&&\lambda_4&&&\\&&&&\lambda_5&&\\&&&&&\lambda_6&\\&&&&&&\lambda_7\end{bmatrix}$$

Goal: Add small number of edges to obtain Fourier basis of eigenvectors

Edges to Destroy Jordan Blocks

Tool: Perturbation Theory

On the Change in the Spectral Properties of a Matrix under Perturbations of Sufficiently Low Rank

S. V. Savchenko

Theorem 1. *Let A be an arbitrary square matrix, and let $B=\sum_{i=1}^r(\cdot,\xi_i)\eta_i$ be an operator of rank r. Consider any eigenvalue λ of A. We arrange the sizes $n_1\geqslant\cdots\geqslant n_k$ of the corresponding Jordan blocks in nonascending order. Suppose that $k\geqslant r$ and*

$$n_{A+B}(\lambda)=n_A(\lambda)-n_1-\cdots-n_r. \tag{4}$$

Then $n_{r+1},\ldots,n_k$ are the sizes of Jordan blocks of the matrix $A+B$ associated with λ.

LOW RANK PERTURBATION OF JORDAN STRUCTURE*

JULIO MORO† AND FROILÁN M. DOPICO†

CONCLUDING THEOREM. *Let A be a complex $n\times n$ matrix and λ_0 an eigenvalue of A with geometric multiplicity g. Let B be a complex $n\times n$ matrix with $\mathrm{rank}(B)\le g$ and C_0 be as in the statement of Theorem 2.1. Then the Jordan blocks of $A+B$ with eigenvalue λ_0 are just the $g-\mathrm{rank}(B)$ smallest Jordan blocks of A with eigenvalue λ_0 if and only if $C_0\neq 0$.*

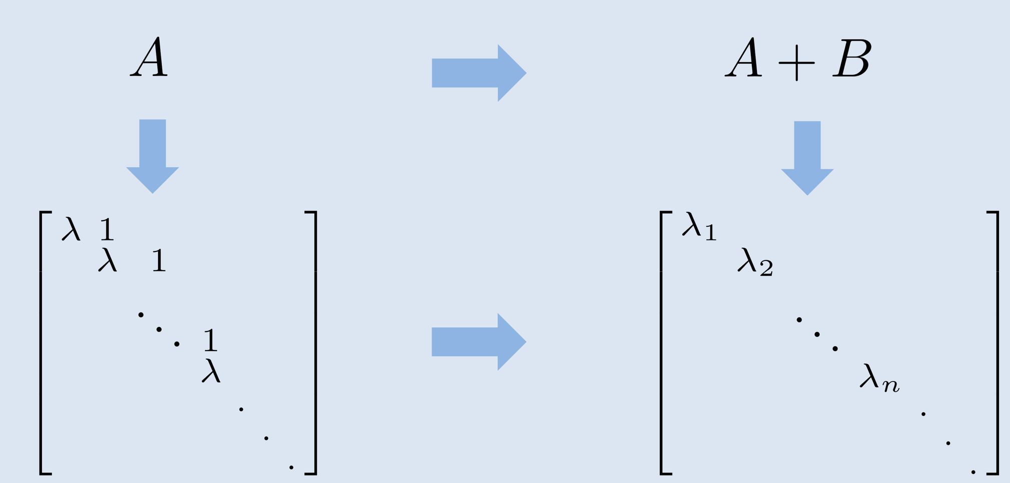
Our work: Specialize to Adjacency/Laplacian matrices

Corollay: Adding an edge is enough to **destroy** the **largest Jordan block** to a choosen eigenvalue.

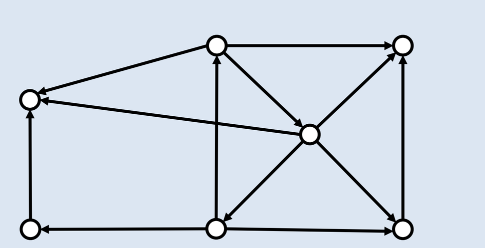
Theorem: Let $u_1,\ldots,u_r,v_1,\ldots,v_r$ be left/right eigenvectors of Jordan blocks to the eigenvalue λ and B the matrix containig only the new edge, then if

$$\sum_{k=1}^{r}u_k^T B v_k\neq 0$$

the largest Jordan block of λ gets destroyed in $A+B$.

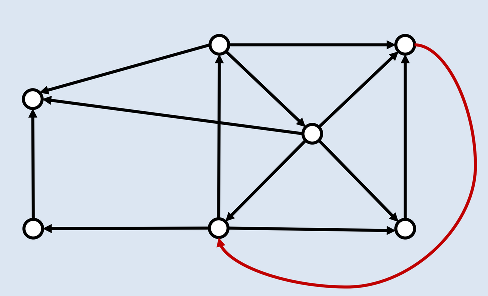


Algorithm



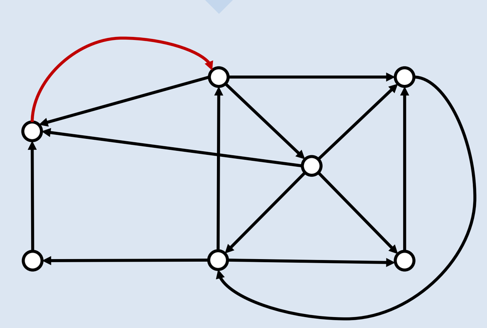
$$\begin{bmatrix}0&1&0&1&1&0&0\\0&0&1&0&0&1&1\\1&0&0&0&1&1&1\\0&0&0&0&0&0&1\\0&0&0&0&0&1&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{bmatrix}\qquad\begin{bmatrix}0&1&&&&&\\0&0&&&&&\\&&0&1&&&\\&&0&0&&&\\&&&&1&&\\&&&&&\omega_3&\\&&&&&&\omega_3^2\end{bmatrix}$$

$$-b_{6,1}-b_{6,2}+b_{6,5}-b_{7,2}+b_{7,4}\neq 0$$



$$\begin{bmatrix}0&1&0&1&1&0&0\\0&0&1&0&0&1&1\\1&0&0&0&1&1&1\\0&0&0&0&0&0&1\\0&0&0&0&0&1&0\\1&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{bmatrix}\qquad\begin{bmatrix}\mu_1&&&&&&\\&\mu_2&&&&&\\&&0&1&&&\\&&0&0&&&\\&&&&\mu_5&&\\&&&&&\mu_6&\\&&&&&&\mu_7\end{bmatrix}$$

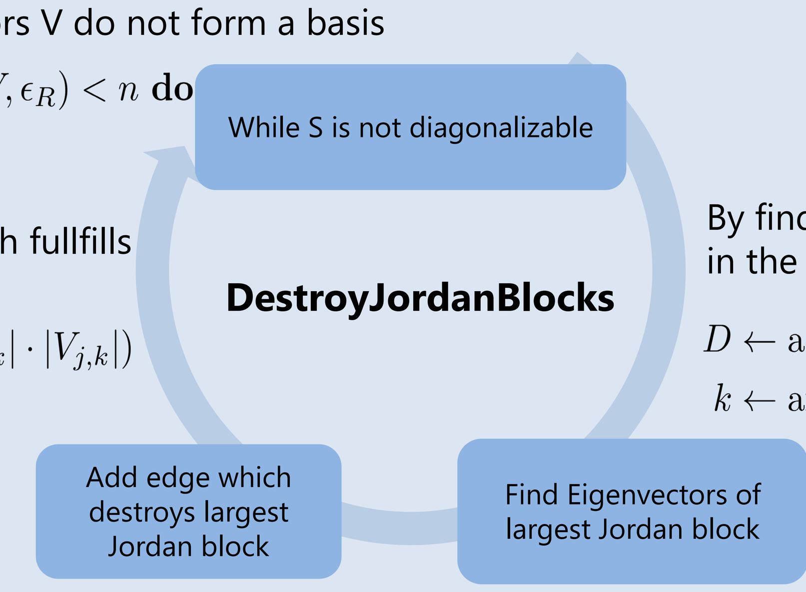
$$-b_{7,2}+b_{7,4}\neq 0$$



$$\begin{bmatrix}0&1&0&1&1&0&0\\0&0&1&0&0&1&1\\1&0&0&0&1&1&1\\0&0&0&0&0&0&1\\0&0&0&0&0&1&0\\1&0&0&0&0&0&0\\0&1&0&0&0&0&0\end{bmatrix}\qquad\begin{bmatrix}\lambda_1&&&&&&\\&\lambda_2&&&&&\\&&\lambda_3&&&&\\&&&\lambda_4&&&\\&&&&\lambda_5&&\\&&&&&\lambda_6&\\&&&&&&\lambda_7\end{bmatrix}$$

If eigenvectors V do not form a basis

while $\mathrm{rank}(V,\epsilon_R)<n$ **do**



By finding eigenvectors in the same subspace

$$D\leftarrow \mathrm{acos}(|V^T\cdot V|)$$
$$k\leftarrow \mathrm{argmax}_k(\#(D_{k,i}<\epsilon_D))$$

By finding index which fullfills the condition best

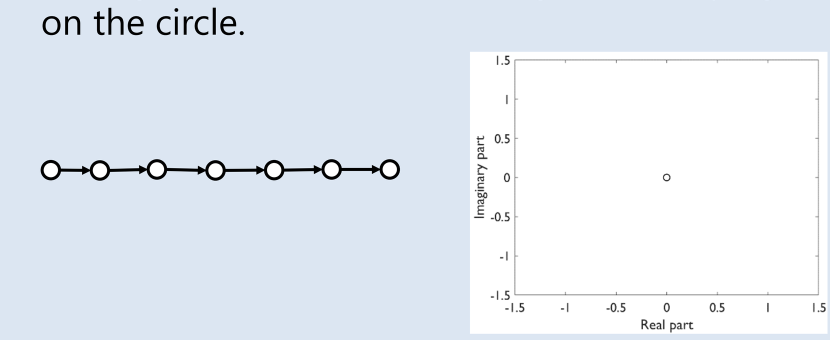
$$(i,j)\leftarrow \mathrm{argmax}_{i,j}(|U_{i,k}|\cdot|V_{j,k}|)\quad \text{s.t. } A_{i,j}=0$$
$$A_{i,j}\leftarrow 1$$

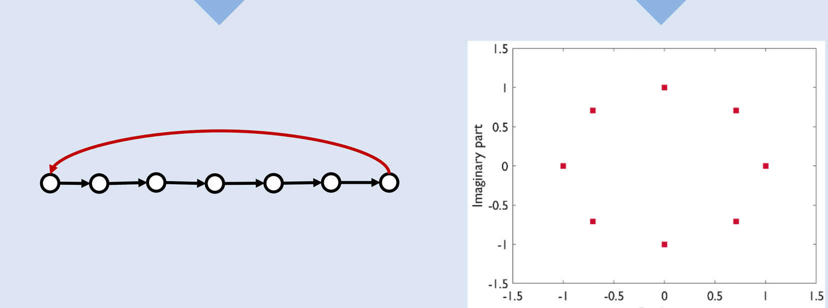
Implementation: https://github.com/bseifert-HSA/digraphSP-generalized-boundaries

Spectrum

Finite, discrete time

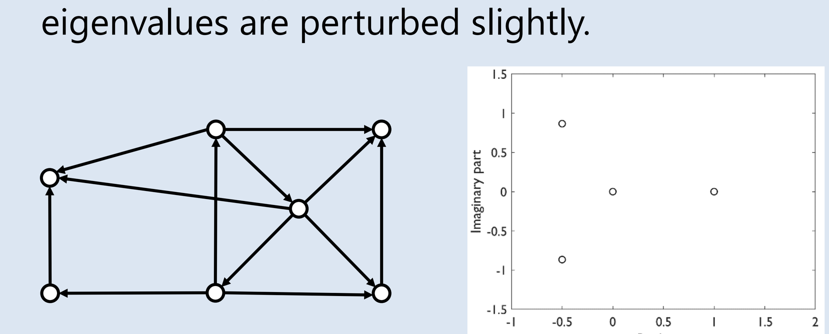
Adding the periodic boundary condition splits the eigenvalue 0 into simple eigenvalues lying on the circle.

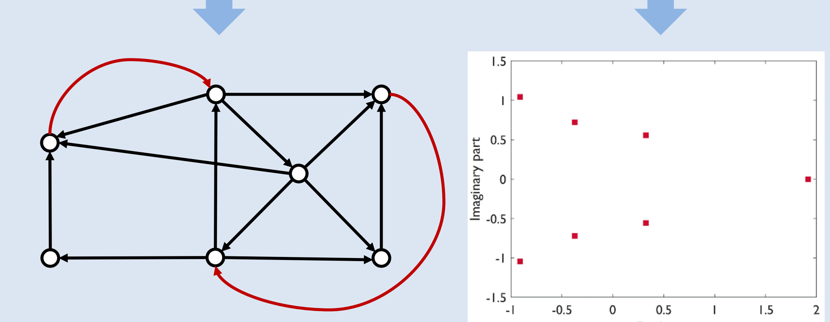




General digraphs

By adding an edge a Jordan block gets split into simple eigenvalues, but also the other eigenvalues are perturbed slightly.





Results

Generally applicable & fast

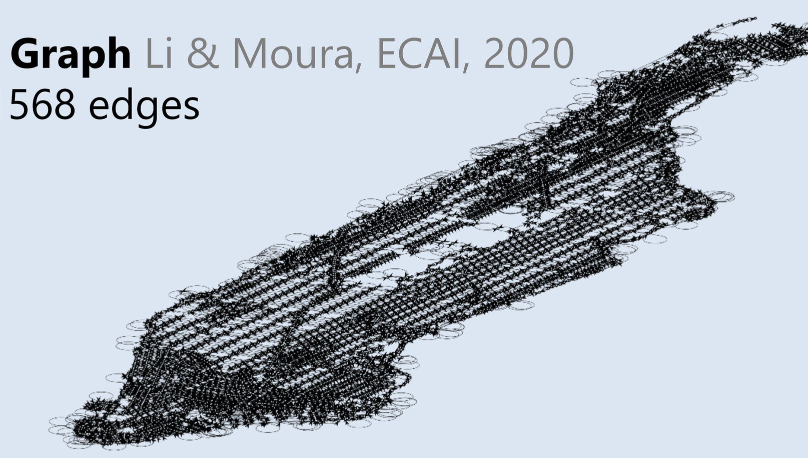
Random digraphs with different properties, **500 nodes & ~5000 edges**

	min		median		max	
	edges	time	edges	time	edges	time
Watts-Strogatz	0	0.2s	1	0.5s	3	1.3s
Barabási-Albert	36	4.4s	44	10s	55	31s
Klemm-Eguílez	10	2.2s	27	6s	47	9s

Medium number of edges added: 27
Median runtime: 6 seconds

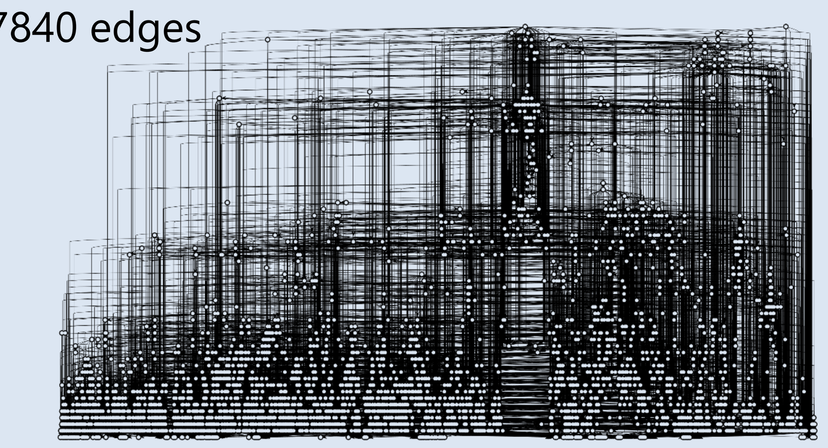
Scalable

Manhattan Taxi Graph Li & Moura, ECAI, 2020
5464 nodes & 11568 edges



Runtime: 19 hours, 243 edges added
Runtime (inexact algo): **5 min**, 772 edges added

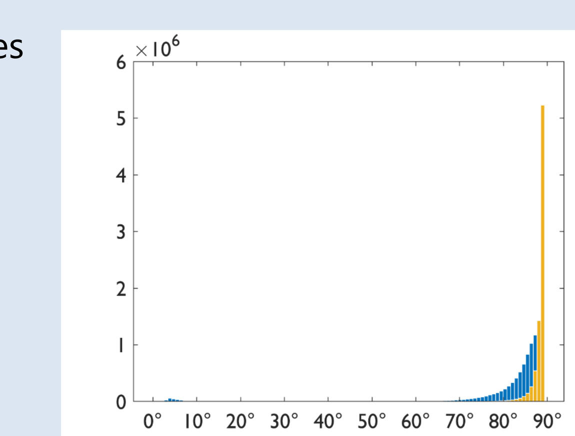
Citation Graph https://snap.stanford.edu/data/cit-HepPh.html
4989 nodes & 17840 edges



Runtime (inexact algo): 31.5 min, 1911 edges added

Fourier bases found almost orthogonal

Histograms of pairwise angles between basis vectors



Total variation almost preserved

Eigenvectors of A+B are approximate eigenvectors of A, the total variation

$$\mathrm{TV}_A(v)=\|v-\tfrac{1}{|\lambda_{\max}|}Av\|$$

barely changes.