A Multi-resolution Low-rank Tensor Decomposition

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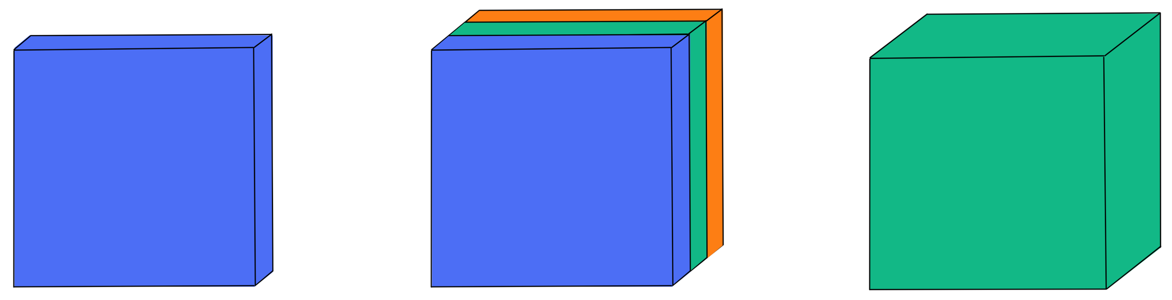
Paper ID: 2301

Multi-dimensional data

- Multi-dimensional arrays, or tensors, model highly-dimensional data
- Tensors generalize the concept of vectors and matrices to highly-dimensional domains
  - ⇒ Relevant in chemometrics, communications, or psychometrics [1, 2]
- Tensor decomposition is key to leverage parsimony
- Tensor decomposition generalizes matrix-decomposition [3, 4]



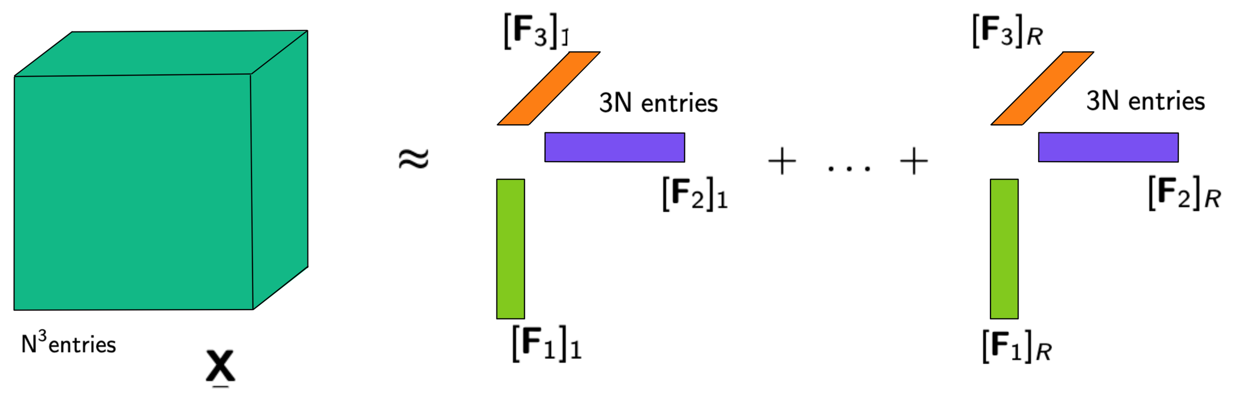
Stacking matrices to create a 3D tensor

Preliminaries of tensor decomposition

- The PARAFAC decomposition generalizes the SVD $\mathbf{X} = \sum_{r=1}^{R} \sigma_r \mathbf{u}_r \mathbf{v}_r^T$
- Consider the Ith order tensor $\underline{\mathbf{X}}$
- Consider the matrices $\mathbf{F}_i \in \mathbb{R}^{N_i \times R}$ for $i = 1, ..., I$
- Then, $\underline{\mathbf{X}}$ is said to have rank R if it can be written as

$$\underline{\mathbf{X}} = \sum_{r=1}^{R} [\mathbf{F}_1]_r \circledcirc [\mathbf{F}_2]_r \circledcirc ... \circledcirc [\mathbf{F}_I]_r \qquad (1)$$

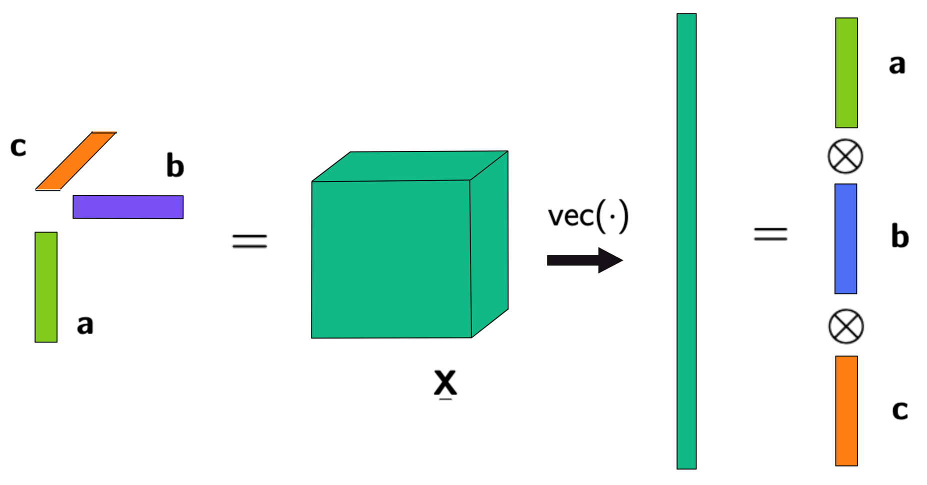
- The matrices $\mathbf{F}_i$ are obtained via ALS schemes



Tensor decomposition

Vectorization and matrization

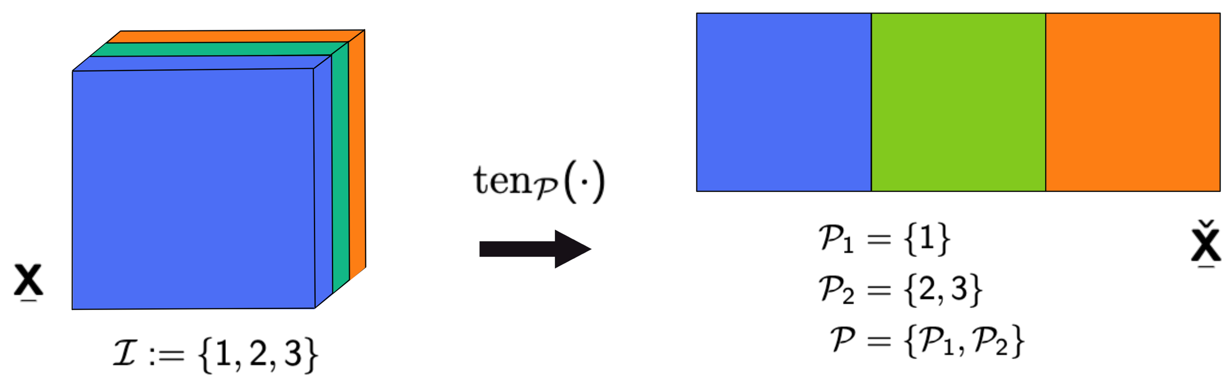
- Tensors can be unfolded into lower order structures
  - ⇒ Tensors can be unfolded into vectors (vectorization)
  - ⇒ Tensor to matrix unfolding (matrization) key in tensor decomposition
  - ⇒ Similarly, tensors can be unfolded into vectors (vectorization)
- Parsimony can be leveraged via tensor decomposition
  - ⇒ The Kronecker product factorizes a vectorization
  - ⇒ Low-rank tensor with rank R implies matrix with rank R



Parsimony in vectorizations

Tensorization

- Matrization and vectorization can be generalized
- Tensors can be unfolded into arbitrary low-order tensors
  - ⇒ Matrization is an special case, the lower-order tensor is a matrix
  - ⇒ Vectorization is another, the lower-order tensor is a vector
- Partitions define how to unfold a tensor to a lower-order tensor:
  - ⇒ Partitions tells how to group dimensions



Tensorization

Design idea 1: Partitions

- Consider a tensor $\underline{\mathbf{X}}$ of order I:
  - ⇒ Let $\mathcal{I} := \{1, 2, ..., I\}$ denote the set containing all indexes

Partition

The ordered set $\mathcal{P} = \{\mathcal{P}_1, ..., \mathcal{P}_P\}$ is a partition of the set $\mathcal{I}$ if it holds that: $\mathcal{P}_p \neq \emptyset$ for all p, $\mathcal{P}_p \cap \mathcal{P}_{p'} = \emptyset$ for all $p' \neq p$, and $\bigcup_{p=1}^{P} \mathcal{P}_p = \mathcal{I}$.

Design idea 2: Tensor unfolding

- The tensor $\underline{\mathbf{X}}$ can be re-arranged into a lower-order tensor $\check{\underline{\mathbf{X}}}$

Tensorization

The elements of the lower-order tensor $\check{\underline{\mathbf{X}}}$ are defined as follows

$$\begin{aligned} &\check{\underline{\mathbf{X}}} = \text{ten}_{\mathcal{P}}(\underline{\mathbf{X}}) \in \mathbb{R}^{\prod_{j=1}^{|\mathcal{P}_1|} |\mathcal{P}_1(j)| \times ... \times \prod_{j=1}^{|\mathcal{P}_P|} |\mathcal{P}_P(j)|} \\ &[\check{\underline{\mathbf{X}}}]_{k_1,...,k_{|\mathcal{P}|}} = [\underline{\mathbf{X}}]_{n_1,...,n_I} \quad \text{and} \\ &k_p = n_{\mathcal{P}_p(1)} \quad \text{if} \quad |\mathcal{P}_p| = 1 \\ &k_p = n_{\mathcal{P}_p(1)} + \sum_{i=2}^{|\mathcal{P}_p|} (n_{\mathcal{P}_p(i)} - 1) \prod_{j=1}^{i-1} N_{\mathcal{P}_p(j)} \quad \text{if} \quad |\mathcal{P}_p| > 1 \end{aligned} \qquad (2)$$

Contribution: MR-LR Tensor Decomposition

- We propose a Multi-Resolution Low Rank (MRLR) decomposition:
  - ⇒ Consider a *collection* of lower-order representations of the tensor
  - ⇒ Postulate a low-rank decomposition for each representation
  - ⇒ Map each representation back to the original tensor domain
  - ⇒ Model the tensor as the sum of the low-rank representations

MRLR decomposition

Formally, consider a collection of partitions $\mathcal{P}^{(1)}, ..., \mathcal{P}^{(L)}$ with $|\mathcal{P}^{(l)}| \leq |\mathcal{P}^{(l')}|$ for $l < l'$. Given the Ith order tensor $\underline{\mathbf{X}}$, we propose

$$\underline{\mathbf{X}} = \sum_{l=1}^{L} \underline{\mathbf{Z}}_l, \quad \text{with} \quad \text{rank}(\text{ten}_{\mathcal{P}^{(l)}}(\underline{\mathbf{Z}}_l)) \leq R_l, \qquad (3)$$

Robust problem formulation

- The MRLR needs to be formalized
  - ⇒ Consider a known tensor $\underline{\mathbf{X}}$

MRLR decomposition optimization problem

The MRLR decomposition can be obtained via solving:

$$\min_{\underline{\mathbf{Z}}_1 ... \underline{\mathbf{Z}}_L} \left\| \underline{\mathbf{X}} - \sum_{l=1}^{L} \underline{\mathbf{Z}}_l \right\|_F \qquad (4)$$
$$\text{s. t.} \quad \text{rank}(\text{ten}_{\mathcal{P}^{(l)}}(\underline{\mathbf{Z}}_l)) \leq R_l.$$

Algorithmic implementation

- The expression in (4) can be worked out
  - ⇒ Sequentially solving for each $\underline{\mathbf{Z}}_l$
  - ⇒ Leveraging low rank in each $\underline{\mathbf{Z}}_l$ via the PARAFAC decomposition

MRLR decomposition optimization problem

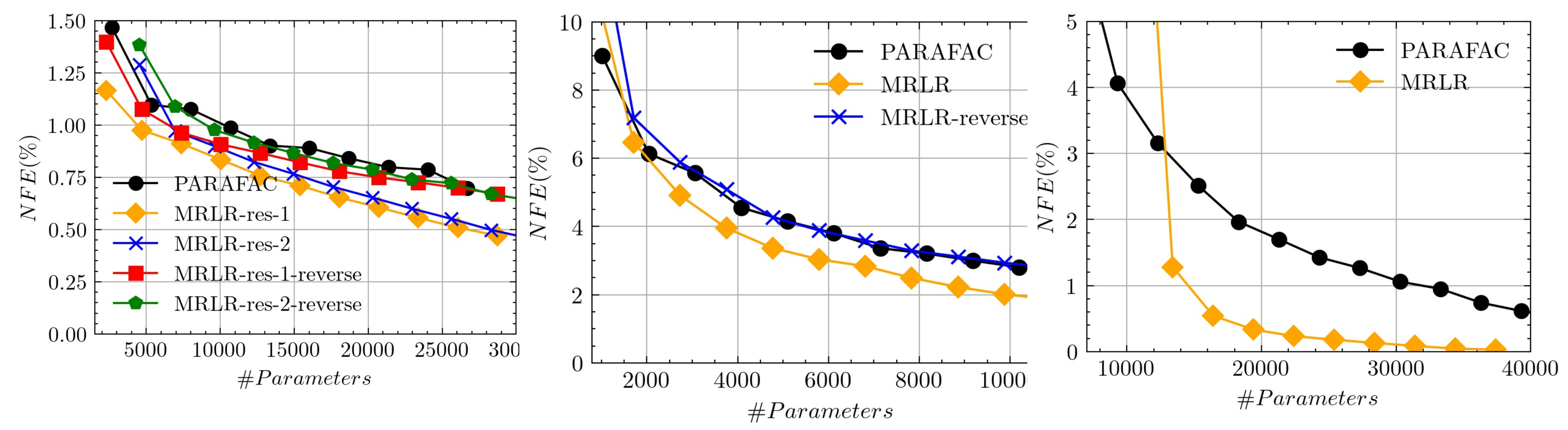
The MRLR decomposition optimization problem becomes now:

$$\min_{\mathbf{H}_1^i, ..., \mathbf{H}_{J_i}^i} \left\| \underline{\mathbf{X}} - \sum_{l \neq i}^{L} \underline{\mathbf{Z}}_l - \sum_{j=1}^{R_i} [\mathbf{H}_1^i]_j \circledcirc ... \circledcirc [\mathbf{H}_{J_i}^i]_j \right\|_F. \qquad (5)$$

Numerical results

- MRLR decomposition has been tested in three scenarios (left to right):
  - ⇒ The aminoacids dataset, a three-mode tensor ($5 \times 201 \times 61$) $\rightarrow I = 3$
  - ⇒ An RGB video signal of 173 frames ($3 \times 173 \times 1080 \times 720$) $\rightarrow I = 4$
  - ⇒ A discretized three-dimensional function $f(x_1, x_2, x_3) = \frac{x_1^2 + x_2^2}{e^{|x_2 + x_3|}} \rightarrow I = 3$
- Performance measured in terms of Normalized Frobenius Error

$$\text{NFE} = ||\underline{\mathbf{X}} - \check{\underline{\mathbf{X}}}||_F / ||\underline{\mathbf{X}}||_F. \qquad (6)$$



- For the same number of parameters, the MRLR decomposition outperforms PARAFAC
- The MRLR decomposition depends on the partitions

Bibliography

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