Point-Mass Filter with Decomposition of Transient Density

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Abstract

- **A novel functional decomposition of the transient density describing the system dynamics is proposed.**
- **The decomposition separates the density into functions of the future and current states.**
- **The performance of the proposed algorithm is illustrated in a terrain-aided navigation scenario.**

Problem Formulation

Consider a nonlinear stochastic dynamic system with additive noises

$$\mathbf{x}_{k+1} = \mathbf{f}_k(\mathbf{x}_k, \mathbf{u}_k) + \mathbf{w}_k, \qquad k = 0, 1, 2, \ldots, T, \qquad (1)$$

$$\mathbf{z}_k = \mathbf{h}_k(\mathbf{x}_k) + \mathbf{v}_k, \qquad k = 0, 1, 2, \ldots, T, \qquad (2)$$

where $\mathbf{x}_k \in \mathcal{R}^{n_x}$, $\mathbf{u}_k \in \mathcal{R}^{n_u}$, and $\mathbf{z}_k \in \mathcal{R}^{n_z}$ represent the *unknown* state of the system and the *known* input and measurement at time instant k, respectively.

The general solution to the state estimation is given by the BRRs for the conditional PDFs computation [4]

$$p(\mathbf{x}_k|\mathbf{z}^k) = \frac{p(\mathbf{x}_k|\mathbf{z}^{k-1})p(\mathbf{z}_k|\mathbf{x}_k)}{p(\mathbf{z}_k|\mathbf{z}^{k-1})}, \tag{3}$$

$$p(\mathbf{x}_{k+1}|\mathbf{z}^k) = \int p(\mathbf{x}_{k+1}|\mathbf{x}_k)p(\mathbf{x}_k|\mathbf{z}^k)d\mathbf{x}_k, \tag{4}$$

$p(\mathbf{x}_{k+1}|\mathbf{z}^k)$ is the one-step predictive PDF
$p(\mathbf{x}_k|\mathbf{z}^k)$ is the filtering PDF

Point Mass Density Approximation

The conditional PDF $p(x_k|\mathbf{z}^m)$ is approximated by a *piece-wise constant* point-mass density $\Xi_{k|m}$ defined at the set of the discrete grid points $\xi_k = \{\xi_k^{(i)}\}_{i=1}^N, \xi_k^{(i)} \in \mathcal{R}$, as follows

$$\Xi_{k|m} \triangleq \sum_{i=1}^{N} P_{k|m}^{(i)} S\{x_k; \xi_k^{(i)}, \Delta_k\}, \qquad (5)$$

$P_{k|m}^{(i)}$ is the conditional PDF $p(x_k|\mathbf{z}^m)$ at the i-th grid point $\xi_k^{(i)}$,

$S\{x_k; \xi_k^{(i)}, \Delta_k\} = \begin{cases} 1, \text{if } |x_k - \xi_k^{(i)}(j)| \leq \frac{\Delta_k}{2}, \\ 0, \text{otherwise.} \end{cases}$ is the *selection* function

Transient density decomposition

Assume that the state transient PDF can be decomposed as

$$p(\mathbf{x}_{k+1}|\mathbf{x}_k) \approx \sum_{r=1}^{R} \mathcal{F}_{1r}(\mathbf{x}_{k+1})\mathcal{F}_{2r}(\mathbf{x}_k) \qquad (6)$$

where $\mathcal{F}_{1r}(\cdot), \mathcal{F}_{2r}(\cdot)$, $r = 1, \ldots, R$ are suitable (non-negative) functions, known in advance, and R is the order of the approximation called *rank*. Then, the Chapman-Kolmogorov equation (4) can be written as

$$p(\mathbf{x}_{k+1}|\mathbf{z}^k) \approx \sum_{r=1}^{R} \mathcal{F}_{1r}(\mathbf{x}_{k+1}) \int \mathcal{F}_{2r}(\mathbf{x}_k) p(\mathbf{x}_k|\mathbf{z}^k) d\mathbf{x}_k \ . \qquad (7)$$

Often, the transient PDF

$$p(\mathbf{x}_{k+1}|\mathbf{x}_k) = p_{\mathbf{w}_k}\left(\mathbf{x}_{k+1} - \mathbf{f}_k(\mathbf{x}_k, \mathbf{u}_k)\right)$$

is a function of $\mathbf{x}_{k+1}$ and $\mathbf{f}_k$, where $\mathbf{f}_k = \mathbf{f}_k(\mathbf{x}_k, \mathbf{u}_k)$. Then,

$$p(\mathbf{x}_{k+1}|\mathbf{x}_k) = p_{\mathbf{w}_k}\left(\mathbf{x}_{k+1} - \mathbf{f}_k\right) \ .$$

Subsequently, the decomposition (6) in the form

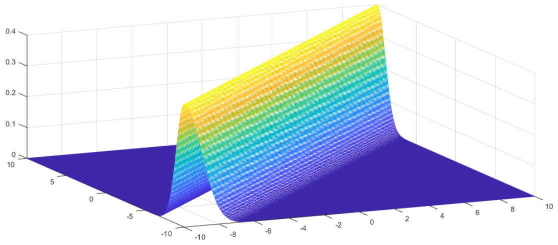
$$p_{\mathbf{w}_k}(\mathbf{x}_{k+1} - \mathbf{f}_k) \approx \sum_{r=1}^{R} \mathcal{F}_{1r}(\mathbf{x}_{k+1})\mathcal{F}_{2r}(\mathbf{f}_k) \tag{8}$$

needs to be computed only over a region of differences $\mathbf{x}_{k+1} - \mathbf{f}_k$. If the function $p_{\mathbf{w}_k}(\mathbf{x}_{k+1} - \mathbf{f}_k)$ is symmetric (invariant) with respect to permutation of its arguments $\mathbf{x}_{k+1}$ and $\mathbf{f}_k$, we may assume that the decomposition (8) is symmetric as well, i.e., $\mathcal{F}_{1r} = \mathcal{F}_{2r}$ for $r = 1, \ldots, R$.

Gaussian Transient PDF

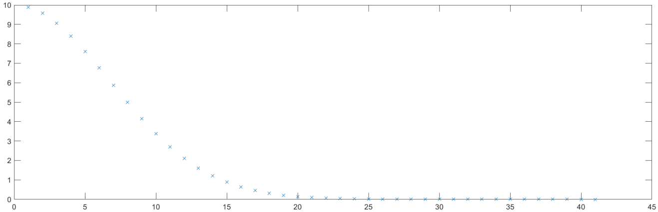
Consider first the scalar case, i.e, $n_x = 1$, and process noise variance $\mathrm{var}[w_k] = \sigma^2 = 1$

$$p(x_{k+1}|x_k) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x_{k+1}-f_k)^2} \quad . \tag{9}$$



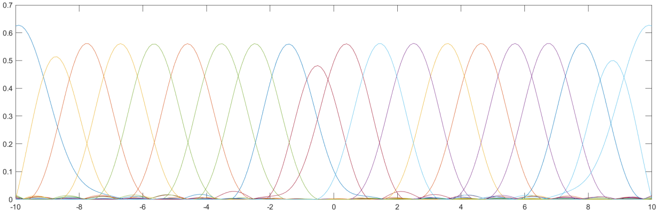
Let the transient PDF be evaluated in the grid $x_{k+1} \in [-L, L]$, $f_k \in [-L, L]$, $L = 10$, with granularity $1/10$ to obtain a $D \times D$ matrix $\mathbf{M}$, $D = 201$.

Then, $\mathbf{M}$ is subject to a symmetric nonnegative matrix factorization $\mathbf{M} = \mathbf{W}\mathbf{W}^T$, where $\mathbf{W} \in \mathcal{R}^{D \times R}$. What rank should one to use ?



Eigenvalues of $\mathbf{M}$

Columns of **W** for $R = 20$ look like

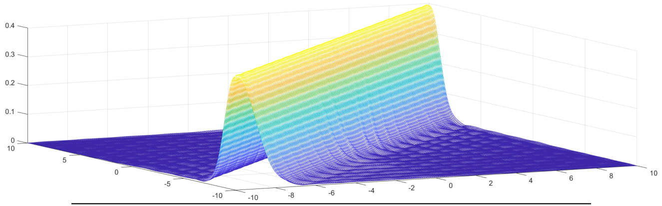


The columns of **W** will be modeled by the functions $\mathcal{F}_{1r} = \mathcal{F}_{2r} = \mathcal{F}_r$ as

$$\mathcal{F}_r(x) = h \cdot e^{-\frac{1}{2}(x - m_r)^2 / w^2} \tag{10}$$

parameterized by the peak position m_r, width w, and height h.

The resultant approximation of $p(x_{k+1}|f_k)$ for $R = 20$



R	d	w	h	E
10	2	0.8701	0.6411	1.72e-2
15	1.33	0.7326	0.6266	4.70e-3
20	1.05	0.7088	0.5768	9.6077e-4
25	0.82	0.7071	0.5109	5.5380e-5
30	0.68	0.7071	0.4653	3.8459e-6
35	0.58	0.7071	0.4297	3.9440e-6
40	0.52	0.7071	0.4068	4.2067e-6

Gaussian Transient PDF in higher dimension

Let

$$p(\mathbf{x}_{k+1}|\mathbf{x}_k) = \frac{1}{\sqrt{(2\pi)^{n_x}|\mathbf{Q}|}} e^{-\frac{1}{2}(\mathbf{x}_{k+1}-\mathbf{f}_k)^T\mathbf{Q}^{-1}(\mathbf{x}_{k+1}-\mathbf{f}_k)},$$

where $\mathbf{f}_k = \mathbf{f}_k(\mathbf{x}_k, \mathbf{u}_k)$. Now, if $\mathbf{Q}$ is a diagonal matrix with elements Q^i, $i = 1, \ldots, n_x$ on its diagonal, then

$$p(\mathbf{x}_{k+1}|\mathbf{x}_k) = \prod_{i=1}^{n_x} \frac{1}{\sqrt{(2\pi)Q^i}} e^{-\frac{1}{2}(\mathbf{x}^i_{k+1}-\mathbf{f}^i_k)^2/Q^i}, \qquad (11)$$

where $\mathbf{x}^i_{k+1}$ and $\mathbf{f}^i_k$ are i-th elements of $\mathbf{x}_{k+1}$ and $\mathbf{f}_k$, respectively. The decomposition of the transient PDF as a product of the two scalar decompositions applied to each element $\mathbf{x}^i_{k+1}$ and $\mathbf{f}^i_k$, $i = 1, \ldots, n_x$.

Terrain-aided navigation

Let a state-space transition be linear, $\mathbf{f}_k(\mathbf{x}_k, \mathbf{u}_k) = \mathbf{x}_k + \mathbf{u}_k$, $\mathbf{x}_k$ is a *two*-dimensional state vector describing the vehicle position, $\mathbf{u}_k = [300, 300]^T$ is an available shift vector provided, e.g., by the inertial navigation system or odometer.

The measurement z_k is nonlinear function $h_k(\cdot)$ realized through a terrain map. We compare performance of three point-mass filter algorithms:

- PMF_{TRUE} with a high number of grid points $N = 150^2$
- PMF_{ST} with the traditional computation with $N = 20^2$ and $N = 50^2$
- PMF_{D} with the *proposed* transient PDF decomposition with $N = 20^2$ and $N = 50^2$ with three different ranks.

Results

N		PMF_{STD}	PMF_D		
			$R = 10$	$R = 20$	$R = 25$
20^2	IE	30e-3	130e-3	31e-3	30e-3
	τ	96e-4	12e-4	13e-4	14e-4
50^2	IE	49e-4	1147e-4	62e-4	56e-4
	τ	229e-3	31e-3	33e-3	33e-3

Conclusions

- The paper proposed a non-negative functional decomposition of the transition density, through which the convolution in the point-mass filter (PMF) can efficiently be calculated.
- With an appropriate rank of the decomposition, significant computational costs savings can be achieved with only negligible loss of PMF estimate quality.