Super-Resolution of Complex Exponentials from Modulations with Unknown Waveforms

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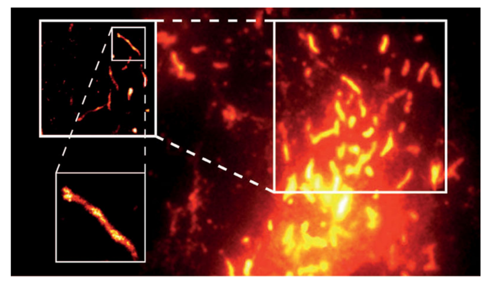
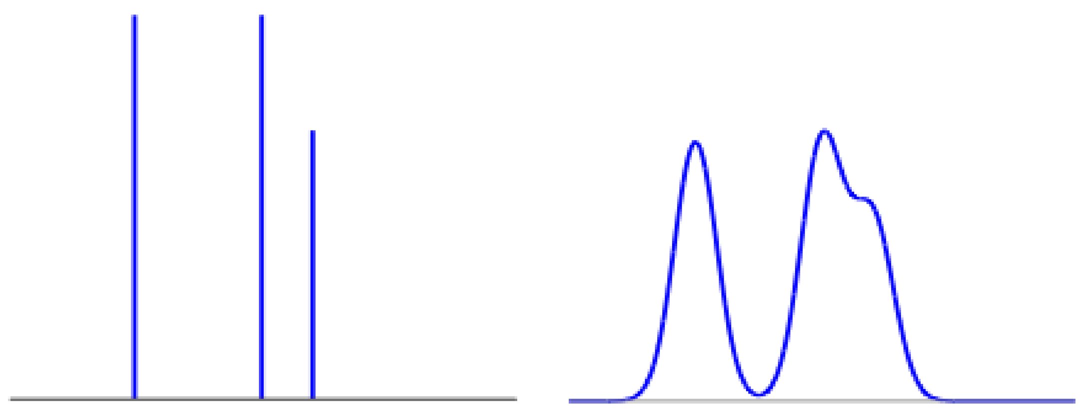
Super-Resolution

Enhancing the resolution limit of sensing systems



- single-molecule microscopy
- medical imaging
- radar imaging
- astronomy



Motivation

1. Super-resolution with unknown point spread functions

$$y(t)=\sum_{j=1}^{J} c_j \delta(t-\tau_j) * g_{\tau_j}(t)$$

- 3D single-molecule microscopy
- non-stationary blind deconvolution of seismic data

2. Parameter estimation in radar imaging

$$y(t)=\sum_{j=1}^{J} c_j e^{i2\pi\nu_j t} x(t-\mu_j)$$

New Model

Consider the observation model:

$$\boldsymbol{y}(n)=\sum_{j=1}^{J} c_j e^{-i2\pi n\tau_j}\boldsymbol{g}_j(n),\ n=-2M,\ldots,2M.$$

Given samples $\{\boldsymbol{y}(n)\}$, the goal is to

- super-resolve $\{\tau_j\}$
- recover $\{c_j\}$
- recover samples of the unknown waveforms $\{\boldsymbol{g}_j(n)\}$

This problem is severely ill-posed

- number of samples $N := 4M+1$
- number of unknowns $JN+2J$

Subspace Model and Atomic Norm Minimization

- A subspace model for $\boldsymbol{g}_j$

$$\boldsymbol{g}_j=\boldsymbol{B}\boldsymbol{h}_j,\quad \boldsymbol{B}=\left[\boldsymbol{b}_{-2M},\cdots,\boldsymbol{b}_{2M}\right]^H,\boldsymbol{b}_n\in\mathbb{C}^{K\times 1}$$

- Rewrite the observation

$$\begin{aligned}\boldsymbol{y}(n)&=\sum_{j=1}^{J} c_j \boldsymbol{a}(\tau_j)^H \boldsymbol{e}_n \boldsymbol{b}_n^H \boldsymbol{h}_j\\ &=\left\langle \sum_{j=1}^{J} c_j \boldsymbol{h}_j \boldsymbol{a}(\tau_j)^H, \boldsymbol{b}_n\boldsymbol{e}_n^H\right\rangle\\ &=:\langle \boldsymbol{X}_o, \boldsymbol{b}_n\boldsymbol{e}_n^H\rangle\end{aligned}$$

where $\boldsymbol{a}(\tau)=\left[e^{i2\pi(-2M)\tau}\cdots 1\cdots e^{i2\pi(2M)\tau}\right]^T$.

- Lift the non-convex problem into a convex program

Define the atomic norm associated with the set of atoms $\mathcal{A}=\left\{\boldsymbol{h}\boldsymbol{a}(\tau)^H:\ \tau\in[0,1),\|\boldsymbol{h}\|_2=1,\boldsymbol{h}\in\mathbb{C}^{K\times 1}\right\}$ as

$$\begin{aligned}\|\boldsymbol{X}\|_{\mathcal{A}}&=\inf\{t>0:\ \boldsymbol{X}\in t\mathbf{conv}(\mathcal{A})\}\\ &=\inf_{c_k,\tau_k,\|\boldsymbol{h}_k\|_2=1}\left\{\sum_k |c_k|:\ \boldsymbol{X}=\sum_k c_k\boldsymbol{h}_k\boldsymbol{a}(\tau_k)^H\right\}.\end{aligned}$$

We solve

$$\begin{aligned}&\text{minimize} && \|\boldsymbol{X}\|_{\mathcal{A}}\\ &\text{subject to} && \boldsymbol{y}(n)=\langle\boldsymbol{X},\boldsymbol{b}_n\boldsymbol{e}_n^H\rangle,\ n=-2M,\cdots,2M.\end{aligned}$$

Denoting $\boldsymbol{q}(\tau)=\sum_{n=-2M}^{2M}\boldsymbol{\lambda}(n)e^{i2\pi n\tau}\boldsymbol{b}_n$ as the dual polynomial with $\boldsymbol{\lambda}$ being the dual optimizer, $\{\tau_j\}$ are localized by selecting out the corresponding values of τ such that $\|\boldsymbol{q}(\tau)\|_2=1$.

Main Result

If the following conditions

1. $\Delta_\tau=\min_{k\neq j}|\tau_k-\tau_j|\geq\frac{1}{M}, M\geq 64,$
2. $\boldsymbol{b}_n$ are i.i.d. samples from a distribution $\mathcal{F}$ satisfying i) $\mathbb{E}\left[\boldsymbol{b}\boldsymbol{b}^H\right]=\boldsymbol{I}_K$; ii) $\max_{1\leq p\leq K}|\boldsymbol{b}(p)|^2\leq\mu$ for $\boldsymbol{b}\in\mathcal{F}$,
3. $\boldsymbol{h}_j$ drawn i.i.d. from the uniform distribution on the complex unit sphere $\mathbb{CS}^{K-1}$,

are satisfied, then there exists some some C such that

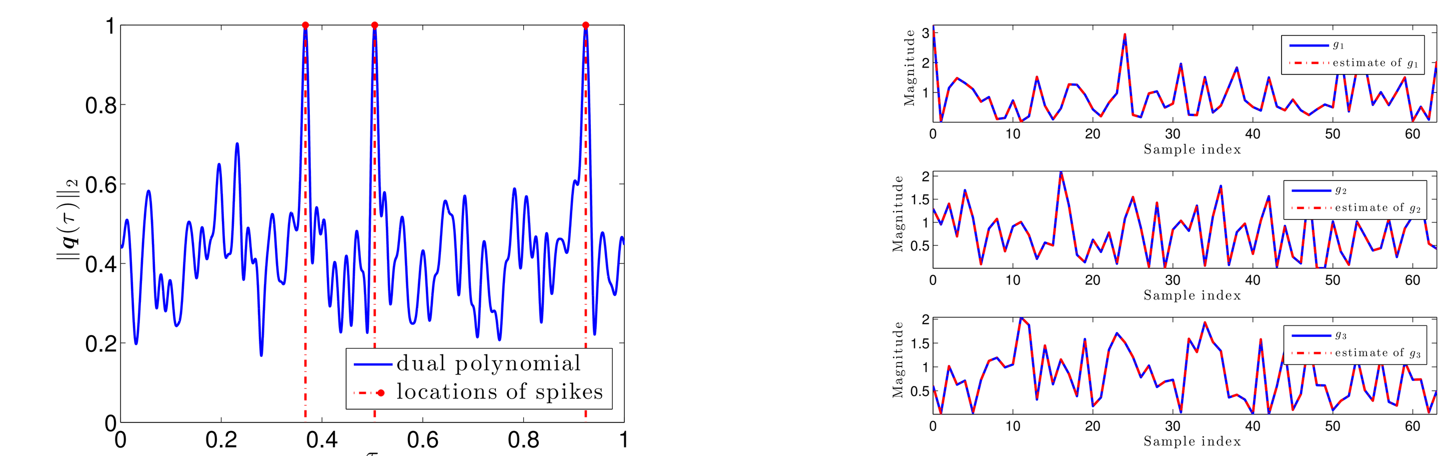
$$M\geq C\mu JK\log\left(\frac{MJK}{\delta}\right)\log^2\left(\frac{MK}{\delta}\right)$$

is sufficient to guarantee that we can recover $\boldsymbol{X}_o$ with probability at least $1-\delta$.

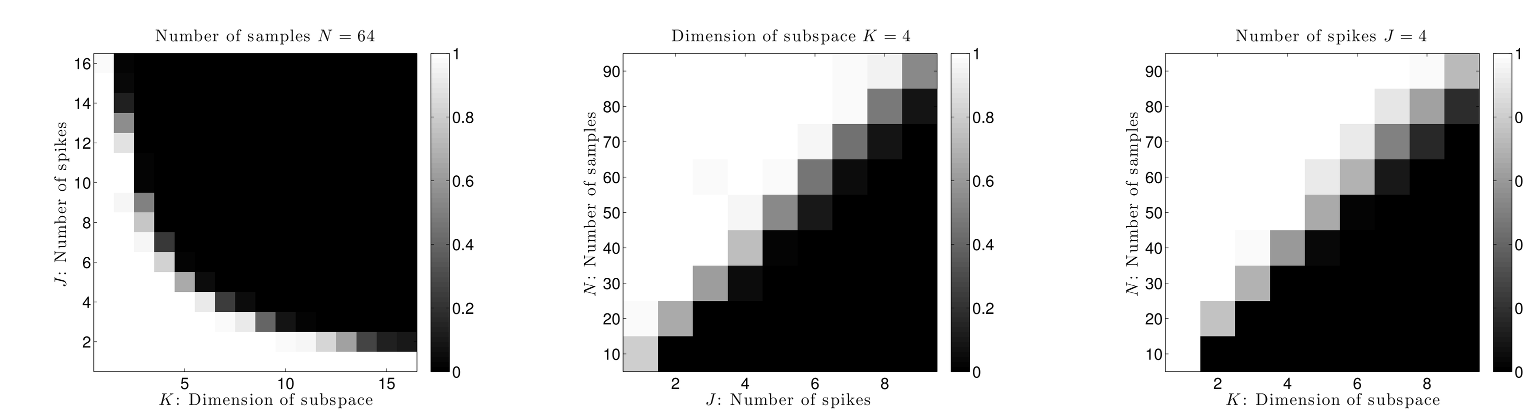
Numerical Simulations

1. A simple example

- we use CVX to solve the optimization problem (SDP)
- set $N=64$, $J=3$ and $K=4$, randomly generate the locations of J spikes on $[0,1)$ under the minimum separation condition $\Delta_\tau=\frac{1}{N}$
- build $\boldsymbol{B}$ with entries generated randomly from the standard Gaussian distribution
- $\boldsymbol{h}_j$ is also generated using i.i.d. real Gaussian random variables and is then normalized

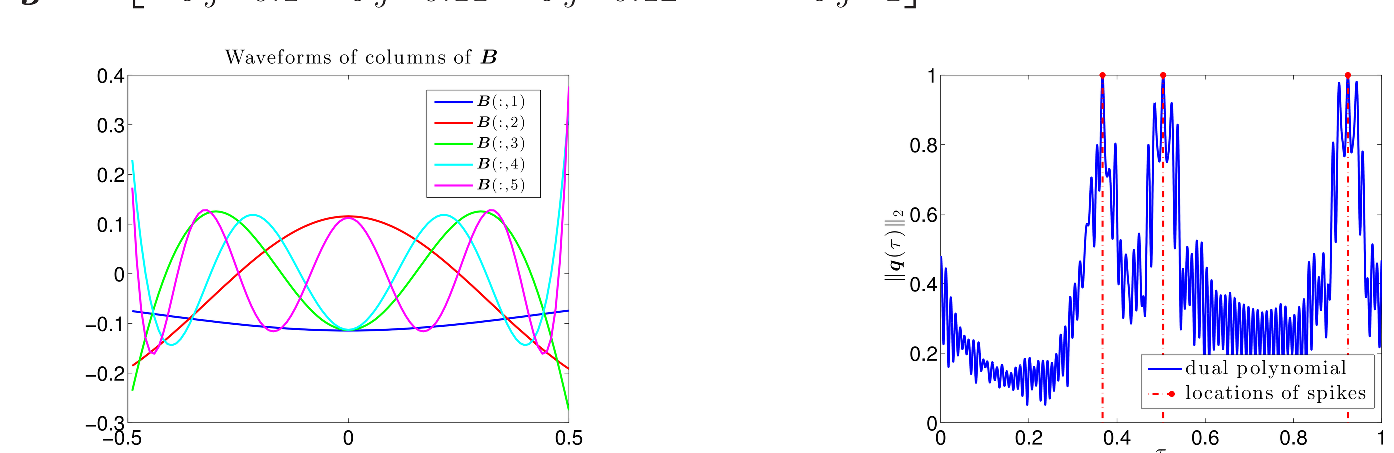


2. Phase transition



3. A practical example

- set $J=3$ and generate the locations of $\{\tau_j\}$ uniformly at random between 0 and 1 under the minimum separation $\Delta_\tau=\frac{1}{M}$
- $\boldsymbol{g}_j(n)$ are samples of the Gaussian waveform $g_J(t)=\frac{1}{\sqrt{2\pi\sigma_J^2}}e^{-\frac{t^2}{2\sigma_J^2}}$ with unknown variance $\sigma_J^2\in[0.1,1]$
- $\boldsymbol{B}$ is a rank-5 approximation of the dictionary $\boldsymbol{D}_g$, where $\boldsymbol{D}_g=\left[\boldsymbol{g}_{\sigma_J=0.1}\ \boldsymbol{g}_{\sigma_J=0.11}\ \boldsymbol{g}_{\sigma_J=0.12}\cdots\boldsymbol{g}_{\sigma_J=1}\right]$



Reference

Y. Chi, "Guaranteed blind sparse spikes deconvolution via lifting and convex optimization," *arXiv preprint arXiv:1506.02751*, 2015.

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