FAST LEARNING OF FAST TRANSFORMS, WITH GUARANTEES

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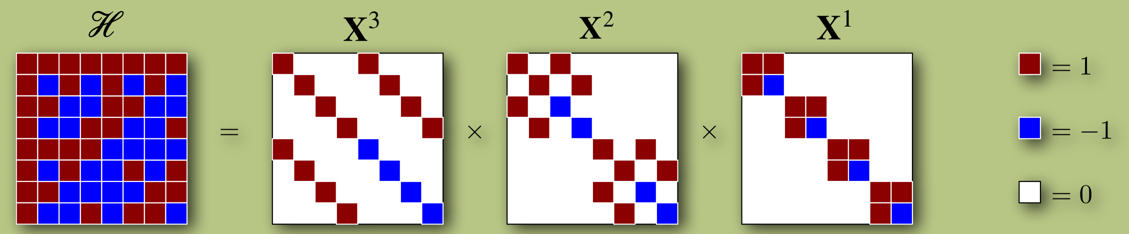
MULTI-LAYER BUTTERFLY MATRIX FACTORIZATION

PROBLEM: Approximate $\mathbf{A} \in \mathbb{C}^{2^J \times 2^J}$ by a product of J butterfly factors.

BUTTERFLY STRUCTURE: J factors $\mathbf{X}^J, \ldots, \mathbf{X}^1$ are called butterfly factors (**BF**) if we have $\mathbf{X}^\ell \in \mathbb{C}^{2^J \times 2^J}$ and $\text{supp}(\mathbf{X}^\ell) \subseteq \text{supp}(\mathbf{S}^\ell), \forall 1 \leq \ell \leq J$ where $\text{supp}(\mathbf{Z}) = \{(i,j) \mid \mathbf{Z}_{i,j} \neq 0\}$ and $\mathbf{S}^\ell := \mathbf{I}_{N/2^\ell} \otimes \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \otimes \mathbf{I}_{2^{\ell-1}} \in \{0,1\}^{2^J \times 2^J}$.

Example:



The Hadamard transform $\mathscr{H}$ and its butterfly factors

MATHEMATICAL FORMULATION:

$$\underset{\mathbf{X}^J,\ldots,\mathbf{X}^1}{\text{Minimize}} \; \|\mathbf{A} - \prod_{\ell=1}^{J} \mathbf{X}^\ell\|^2 \quad \text{such that } \mathbf{X}^J, \ldots, \mathbf{X}^1 \text{ are BF} \qquad (1)$$

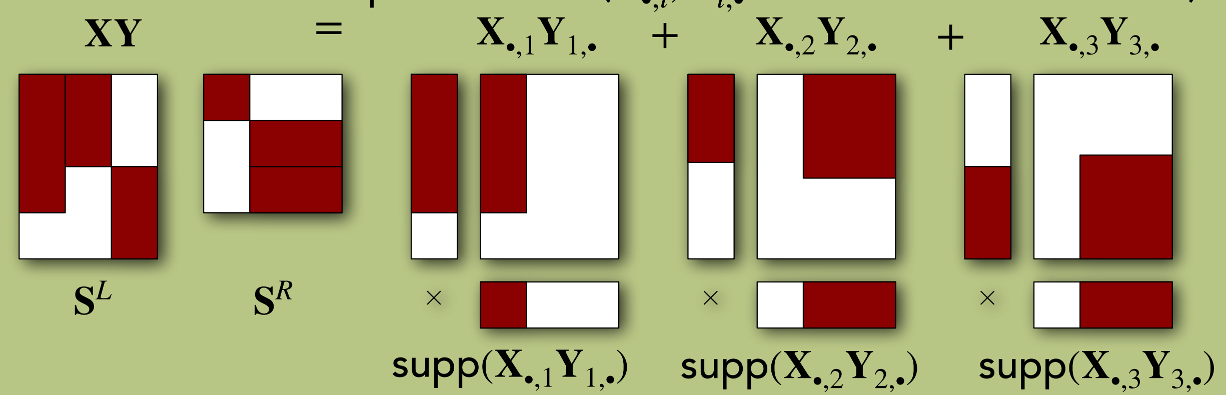
OBJECTIVE: An algorithm which is more efficient than classical gradient descent, with theoretical guarantee.

BACKGROUND: FIXED SUPPORT MATRIX FACTORIZATION

PROBLEM: Given $\mathbf{A} \in \mathbb{R}^{m \times n}, \mathbf{S}^L \in \{0,1\}^{m \times r}$ and $\mathbf{S}^R \in \{0,1\}^{r \times n}$:

$$\underset{(\mathbf{X},\mathbf{Y})}{\text{Minimize}} \quad \|\mathbf{A} - \mathbf{XY}\|_F^2, \text{ s.t } \text{supp}(\mathbf{X}) \subseteq \text{supp}(\mathbf{S}^L), \text{supp}(\mathbf{Y}) \subseteq \text{supp}(\mathbf{S}^R) \quad (2)$$

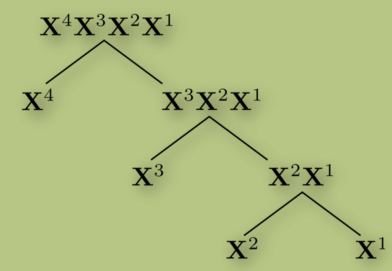
OBSERVATION: Decomposition $\mathbf{XY}$ ($\mathbf{Z}_{\bullet,i}, \mathbf{Z}_{i,\bullet}$ are ith column/row of $\mathbf{Z}$)



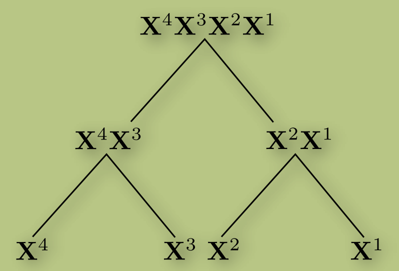
RESULT: If $\{\text{supp}(\mathbf{X}_{\bullet,i}\mathbf{Y}_{i,\bullet}) \mid i = 1, \ldots, r\}$ are pairwise disjoint, (2) is polynomially solvable and identifiable [1,2].

MAIN CONTRIBUTION: HIERARCHICAL METHOD

MAIN IDEA: Recursively use algorithms for (2) to factorize a matrix into two factors at each intermediate level of the hierarchy.



Unbalanced



Balanced

Two strategies to perform hierarchical factorization. Each corresponds to a tree

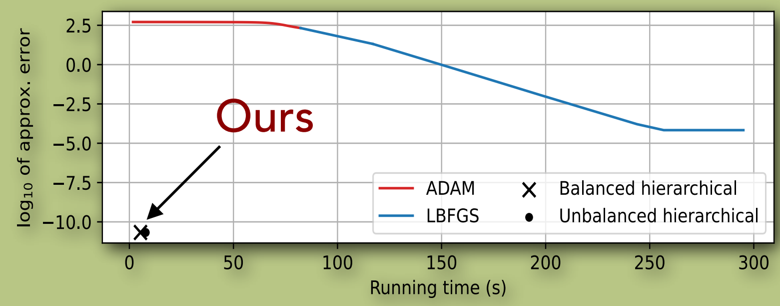
INPUT: A matrix $\mathbf{Z} = \mathbf{X}^J \ldots \mathbf{X}^1$ where $\mathbf{X}^J, \ldots, \mathbf{X}^1$ are butterfly factors.

THEORETICAL GUARANTEE: For any choice of tree, the hierarchical method will yield J factors $\bar{\mathbf{X}}^J, \ldots, \bar{\mathbf{X}}^1$ such that:

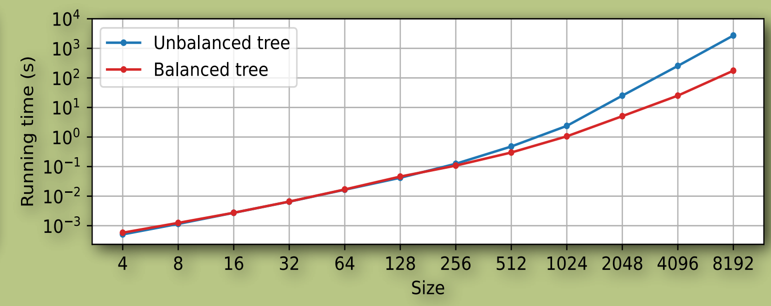
1) Exact factorization: $\mathbf{Z} = \bar{\mathbf{X}}^J \ldots \bar{\mathbf{X}}^1$

2) Recovery: $\mathbf{X}^\ell = \mathbf{D}^{\ell-1}\bar{\mathbf{X}}^\ell(\mathbf{D}^\ell)^{-1}$ where $\mathbf{D}^J, \ldots, \mathbf{D}^0$ are invertible diagonal matrices satisfying $\mathbf{D}^0 = \mathbf{D}^J = \mathbf{I}_{2^J}$.

EXPERIMENTAL RESULT: Comparison between the state-of-the-art method [3] (ADAM + LBFGS) and our methods (unbalanced and balanced).



Precision and running time of [3] and our methods in the factorization of the Discrete Fourier Transform of size 512 ($J = 9$)



Running time of *balanced* and *unbalanced* strategies factorizing a noisy version of a product of J butterfly factors, $1 \leq J \leq 13$.

REFERENCES

[1]: Q.T. Le, E. Riccietti and R. Gribonval, "Spurious Valleys, Spurious Minima and NP-hardness of Sparse Matrix Factorization With Fixed Support", arXiv preprint arXiv:2112.00386, 2021.

[2]: L.Zheng, E. Riccietti and R. Gribonval, "Efficient Identification of Butterfly Sparse Matrix Factorization", arXiv preprint, arXiv:2110.01235, 2022.

[3]: T. Dao, A. Gu, M. Eichhorn, A. Rudra and C. Re, "Learning Fast Algorithms for Linear Transforms Using Butterfly Factorization", 36th International Conference of Machine Learning, June 2019.