# Identification of Edge Disconnections in Networks Based on Graph Filter Outputs

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# Overview

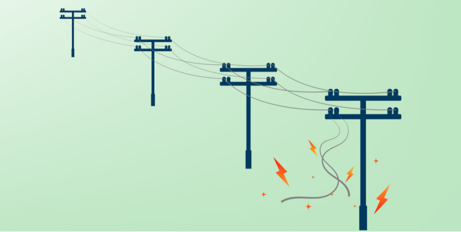
1. Motivation and Background
2. Measurements Model
3. Identification of edge disconnections
4. Greedy approaches for identifying edge disconnections
5. Simulations
6. Conclusions

# Identifying Edge Disconnections Using Graph Signal Processing

**Motivation:**

- GSP methods are based on known underlying topology
- Topology changes may degrade the performance of GSP tasks
- Edge disconnections are a common problem, especially in physical networks.

**Goal:** use graph signals to identify edge disconnections, where the original underlying network is known



**Example:** Identifying line outages in electrical networks, due to environmental factors, damages, aging, malicious attacks, etc.

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# Related works

- Numerous works in the literature have focused on (full) graph-topology learning.
  - Inefficient for identifying only a few specific disconnections
  - Suboptimal, since the nominal topology is known
- Other works detect topology changes based on graph data and not on "graph signals"
- Matched subspace detectors based on graph signals to decide which graph matches a given dataset, but does not use information on the nature of the change [1], [2]
- Edge exclusion tests for general graphical models [3]
- Laplacian learning in Gaussian Markov random field models with known connectivity [4]

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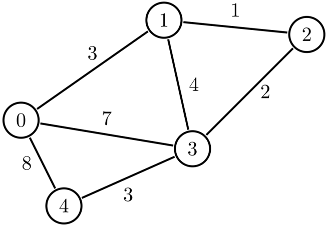
[1] C. Hu, J. Cheng, K. A. Sepulcre, G. E. Fakhri, Y. M. Lu, and K. Li, "Matched signal detection on graphs: Theory and application to brain imaging data classification", 2016.
[2] E. Isufi, A. S. U. Mahabir and G. Leus, "Blind Graph Topology Change Detection", 2018
[3] K. Tugnait, "Edge exclusion tests for graphical model selection: Complex Gaussian vectors and time series", 2019
[4] H. Egilmez, E. Pavez, and A. Ortega, "Graph Learning from Data under Structural and Laplacian Constraints", 2017

# Background: GSP Definitions

Given an undirected, connected, weighted graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathbf{W}\}$:

- $\mathcal{V}$ is set of vertices, where $N \triangleq |\mathcal{V}|$ and $\mathcal{E}$ is a set of edges.
- $\mathbf{W} \in \mathbb{R}^{N \times N}$ is the non-negative weighted *adjacency matrix* of the graph.
  If $(i,j) \in \mathcal{E}$, the entry $\mathbf{W}_{i,j} > 0$ represents the weight of the edge; otherwise, $\mathbf{W}_{i,j} = 0$.
- The *Laplacian matrix* is $\mathbf{L} \triangleq \mathrm{diag}(\mathbf{W1}) - \mathbf{W}$, where each entry satisfies

$$\mathbf{L}_{i,j} = \begin{cases} \sum_{k:\,(i,k)\in\mathcal{E}} \mathbf{W}_{i,k} \,, & i = j \,,\ i \in \mathcal{V} \\ -\mathbf{W}_{i,j} \,, & (i,j) \in \mathcal{E} \\ 0 \,, & \text{otherwise} \end{cases}$$




$$\underbrace{\begin{pmatrix} 18 & -3 & 0 & -7 & -8 \\ -3 & 8 & -1 & -4 & 0 \\ 0 & -1 & 3 & -2 & 0 \\ -7 & -4 & -2 & 16 & -3 \\ -8 & 0 & 0 & -3 & 11 \end{pmatrix}}_{\mathbf{L}}$$

- The singular value decomposition (SVD) is given by $\mathbf{L} = \mathbf{U}^{(\mathbf{L})} \Lambda^{(\mathbf{L})} \left(\mathbf{U}^{(\mathbf{L})}\right)^{\top}$.

# Background: GSP Definitions

- A *graph signal* is a vector measured over the vertices, $\mathbf{a} : \mathcal{V} \to \mathbb{R}^N$.
- The *graph fourier transform (GFT)* w.r.t $\mathbf{L}$ is $\tilde{\mathbf{a}}^{(\mathbf{L})} = \left(\mathbf{U}^{(\mathbf{L})}\right)^\top \mathbf{a}$.
- The *inverse GFT (IGFT)* w.r.t $\mathbf{L}$ is $\mathbf{a} = \mathbf{U}^{(\mathbf{L})}\tilde{\mathbf{a}}^{(\mathbf{L})}$.
- The *smoothness* or *Dirichlet energy* is measured by

$$Q_{\mathbf{L}}(\mathbf{a}) \triangleq \frac{1}{2} \sum_{(i,j)\in\mathcal{E}} \mathbf{W}_{i,j}[\mathbf{a}_i - \mathbf{a}_j]^2 = \mathbf{a}^\top \mathbf{L}\mathbf{a}.$$

> *Smooth graph signals* are signals with "small" Dirichlet energy.
> Intuitively, a smooth graph signal is considered to be a "good match" with the graph if the signal values are close to their neighbors' values.

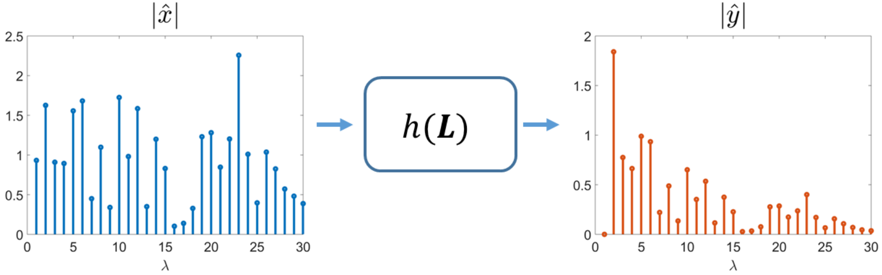
# Background: Graph Filter

Filtering in graph Fourier space can be represented by

$$\tilde{\mathbf{a}}_{\text{out}}^{(\mathbf{L})} = \text{diag}\big([h(\lambda_1^{(\mathbf{L})}), \ldots, h(\lambda_N^{(\mathbf{L})})]^T\big)\tilde{\mathbf{a}}_{\text{in}}^{(\mathbf{L})}.$$

Then,the *graph filter* is a linear operator relates to input-output $\mathbf{a}_{\text{out}} = h(\mathbf{L})\mathbf{a}_{\text{in}}$, where

$$h(\mathbf{L}) \triangleq \mathbf{U}^{(\mathbf{L})}\text{diag}\big([h(\lambda_1^{(\mathbf{L})}), \ldots, h(\lambda_N^{(\mathbf{L})})]^T\big)\big(\mathbf{U}^{(\mathbf{L})}\big)^T.$$



# Smooth Graph Filter: $\mathcal{E}[\mathrm{Q}_{\mathbf{L}}(\mathbf{a}_{\text{out}})] < \mathrm{E}[Q_{\mathbf{L}}(\mathbf{a}_{\text{in}})]$

| Graph Filter | $h(\lambda)$ | For $\mathbf{a}_{in} \sim \mathcal{N}(0, \mathbf{I})$ |
|---|---|---|
| Gaussian Markov random field (GMRF) with a Laplacian precision matrix | $\begin{cases} \frac{1}{\sqrt{\lambda}} & \lambda \neq 0 \\ 0 & \lambda = 0 \end{cases}$ | $\mathbf{a}_{out} \sim \mathcal{N}(0, \mathbf{L}^{\dagger})$ |
| Regularized Laplacian by Tikhonov Regularization | $\frac{1}{1+\alpha\lambda}, \ \alpha > 0$ | $\mathbf{a}_{out} \sim \mathcal{N}(0, (\mathbf{I} + \alpha\mathbf{L})^{-2})$ |
| Heat Diffusion Kernel | $\exp(-\tau\lambda), \ \tau > 0$ | $\mathbf{a}_{out} \sim \mathcal{N}(0, \exp(-2\tau\mathbf{L}))$ |

# Outline

# Measurement Model

- We consider the measurement model as an output of a smooth graph filter:

$$\mathbf{y}[m] = h(\mathbf{L})\mathbf{x}[m] + \mathbf{w}[m], \; m = 1 \dots M.$$

- The log-likelihood of the augmented output vector of $M$ time samples, $\mathbf{y} \triangleq [\mathbf{y}^T[1], \dots, \mathbf{y}^T[M]]^T$, is

$$\log f(\mathbf{y}; \mathbf{L}) = -\frac{M}{2} \log \left((2\pi)^N |\sigma_{\mathbf{x}}^2 h^2(\mathbf{L}) + \sigma_{\mathbf{w}}^2 \mathbf{I}|_+\right) - \frac{M}{2} \mathrm{Tr}\left(\left(\sigma_{\mathbf{x}}^2 h^2(\mathbf{L}) + \sigma_{\mathbf{w}}^2 \mathbf{I}\right)^\dagger \mathbf{S}_{\mathbf{y}}\right),$$

where the sample covariance matrix

$$\mathbf{S}_{\mathbf{y}} \triangleq \frac{1}{M} \sum_{m=1}^{M} \mathbf{y}[m]\mathbf{y}^T[m].$$

$\Rightarrow$ the graph filter, $h(\mathbf{L})$, "colors" the input graph signal using the network connectivity.

# Problem formulation

Identification of edge disconnections:

$$\mathcal{H}_k : \quad \mathbf{L} = \mathbf{L}^{(k)}, \quad k = 0, 1, \ldots, K$$

based on the graph signals $\mathbf{y}$, where

- $\mathbf{L}^{(0)}$ is the Laplacian of the original, known topology (set of edges: $\mathcal{E}$),
- $\mathbf{L}^{(k)}$ is the Laplacian matrix after disconnections at the edges in $\mathcal{C}^{(k)} \subset \mathcal{E}$.

$$\mathbf{L}^{(k)} \triangleq \mathbf{L}^{(0)} - \underbrace{\sum_{(i,j)\in\mathcal{C}^{(k)}} \mathbf{E}^{(i,j)}}_{\mathbf{E}^{(k)}}$$

$$\mathbf{E}^{(i,j)} \triangleq \left[\mathbf{L}^{(0)}\right]_{i,j} \left(\mathbf{e}_i\mathbf{e}_j^T + \mathbf{e}_j\mathbf{e}_i^T - \mathbf{e}_j\mathbf{e}_j^T - \mathbf{e}_i\mathbf{e}_i^T\right).$$

corresponds to a single-edge disconnection at $(i,j) \in \mathcal{E}$.

$$\underbrace{\begin{pmatrix} 11 & -3 & 0 & 0 & -8 \\ -3 & 8 & -1 & -4 & 0 \\ 0 & -1 & 3 & -2 & 0 \\ 0 & -4 & -2 & 9 & -3 \\ -8 & 0 & 0 & -3 & 11 \end{pmatrix}}_{\mathbf{L}^{(k)}} = \underbrace{\begin{pmatrix} 18 & -3 & 0 & -7 & -8 \\ -3 & 8 & -1 & -4 & 0 \\ 0 & -1 & 3 & -2 & 0 \\ -7 & -4 & -2 & 16 & -3 \\ -8 & 0 & 0 & -3 & 11 \end{pmatrix}}_{\mathbf{L}^{(0)}} - \underbrace{\begin{pmatrix} 7 & 0 & 0 & -7 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -7 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}}_{\mathbf{E}^{(k)}}$$

# Outline

# Maximum likelihood decision rule

The maximum likelihood decision rule for this problem is given by

$$T(\mathbf{y}) = \underset{0 \le k \le K}{\operatorname{argmax}} \frac{\log f(\mathbf{y}; \mathbf{L}^{(k)})}{\log f(\mathbf{y}; \mathbf{L}^{(0)})} = \underset{0 \le k \le K}{\operatorname{argmax}} \, l(\mathbf{y}|\mathbf{L}^{(k)}) - \rho(\mathbf{L}^{(k)}).$$

where

- $l(\mathbf{y}|\mathbf{L}^{(k)}) \triangleq \mathrm{Tr}\left(\left(\left(\sigma_{\mathbf{x}}^2 h^2(\mathbf{L}^{(0)}) + \sigma_{\mathbf{w}}^2 \mathbf{I}\right)^{\dagger} - \left(\sigma_{\mathbf{x}}^2 h^2(\mathbf{L}^{(k)}) + \sigma_{\mathbf{w}}^2 \mathbf{I}\right)^{\dagger}\right)\mathbf{S}_{\mathbf{y}}\right)$ - data term
- $\rho(\mathbf{L}^{(k)}) \triangleq \log\left(\frac{|\sigma_{\mathbf{x}}^2 h^2(\mathbf{L}^{(k)}) + \sigma_{\mathbf{w}}^2 \mathbf{I}|_+}{|\sigma_{\mathbf{x}}^2 h^2(\mathbf{L}^{(0)}) + \sigma_{\mathbf{w}}^2 \mathbf{I}|_+}\right)$ - penalty term
- $\mathbf{S}_{\mathbf{y}} \triangleq \frac{1}{M} \sum_{m=1}^{M} \mathbf{y}[m]\mathbf{y}^T[m]$ - sample covariance matrix

$\Rightarrow$The problem of testing *structured* covariance matrix of random Gaussian vectors

# Remark 1: Penalty function interpretation

**proposition:** Consider two connected graphs, $\mathcal{G}^{(k_1)}$ and $\mathcal{G}^{(k_2)}$, with the Laplacian matrices, $\mathbf{L}^{(k_1)}$ and $\mathbf{L}^{(k_2)}$, respectively, and assume:

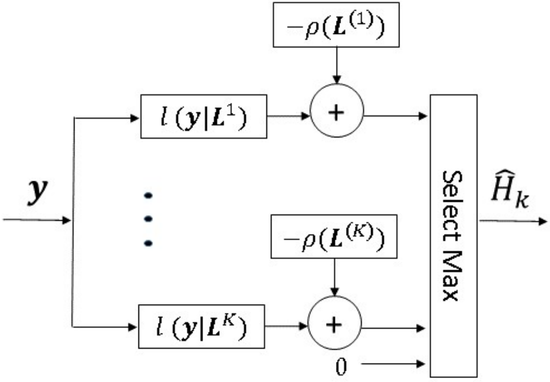
C.1) $\mathcal{C}^{(k_2)}$ is a proper subset of $\mathcal{C}^{(k_1)}$, i.e. $\mathcal{C}^{(k_2)} \subset \mathcal{C}^{(k_1)}$.

C.2) The graph filter, $h(\lambda)$, is a monotonic decreasing function of $\lambda$ for any $\lambda > 0$.

C.3) The covariance matrices are non-singular matrices.

Then,

$$\rho(\mathbf{L}^{(k_2)}) \leq \rho(\mathbf{L}^{(k_1)}).$$




⇒A larger penalty for more edge disconnections

# Remark 2: The sufficient statistics

The $k$th likelihood can be written in the graph spectral domain as

$$l(\mathbf{y}|\mathbf{L}^{(k)}) = \frac{\sigma_{\mathbf{x}}^2}{\sigma_{\mathbf{w}}^2}\left(\sum_{n=1}^{N} \frac{h^2(\lambda_n^{(\mathbf{L}^{(k)})})}{\sigma_{\mathbf{w}}^2 + \sigma_{\mathbf{x}}^2 h^2(\lambda_n^{(\mathbf{L}^{(k)})})}\psi_n^{(\mathbf{L}^{(k)})} - \frac{h^2(\lambda_n^{(\mathbf{L}^{(0)})})}{\sigma_{\mathbf{w}}^2 + \sigma_{\mathbf{x}}^2 h^2(\lambda_n^{(\mathbf{L}^{(0)})})}\psi_n^{(\mathbf{L}^{(0)})}\right),$$

where the sufficient statistics for the identification are the graph-frequency energy levels, i.e.

$$\psi_n^{(\mathbf{L}^{(l)})} \triangleq \frac{1}{M}\sum_{m=1}^{M}\left(\left[\tilde{\mathbf{y}}^{(\mathbf{L}^{(l)})}[m]\right]_n\right)^2 \quad n = 1, \ldots, N,\ l = 0, k.$$

⇒the maximum likelihood decision rule only requires the evaluation of the $NK$ scalars

⇒ it can be shown that is governed by the low-graph frequencies

# Remark 3: GMRF model with a Laplacian precision

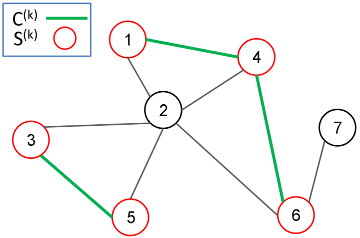
In this case, the $k$th likelihood term satisfies

$$l(\mathbf{y}|\mathbf{L}^{(k)}) = -\frac{1}{\sigma_{\mathbf{x}}^2 M} \sum_{\boxed{(i,j)\in \mathcal{C}^{(k)}}} L_{i,j}^{(0)} \sum_{m=1}^{M} (y_i[m] - y_j[m])^2$$

**proposition:** Consider a connected graph, $\mathcal{G}^{(k)}$, with a Laplacian matrix, $\mathbf{L}^{(k)}$. Then, for the noiseless GMRF model with a Laplacian precision matrix, the $k$th term

$$\underbrace{l(\mathbf{y}|\mathbf{L}^{(k)})}_{\text{measurements + 2nd-order statistics}} - \underbrace{\rho(\mathbf{L}^{(k)})}_{\text{2nd-order statistics}}$$

is only a function of the vertices in $\mathcal{S}^{(k)}$.




$\mathcal{G}^{(k)}$ is obtained by removing the edges in $\mathcal{C}^{(k)} = \{(1,4),(4,6),(3,5)\}$, associated with the vertices in $\mathcal{S}^{(k)} = \{1,3,4,5,6\}$.

# Computational complexity

- Number of hypotheses in the general case:

$$K = \sum_{r=1}^{r_{\max}} \binom{|\mathcal{E}|}{r},$$

where $r_{\max}$ is the maximum number of possible edge disconnections

- The calculation of $l(\mathbf{y}|\mathbf{L}^{(k)})$ and $\rho(\mathbf{L}^{(k)})$ requires an inversion of $N \times N$ matrix
- The computational complexity of the maximum likelihood decision rule grows exponentially with the graph size and is impractical for large networks
- We develop efficient low-complexity methods based on:
  - Greedy approach
  - Local properties of smooth graph filters

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# Greedy approach

**Algorithm 1:** Greedy identification

- Input: $\mathbf{S}_{\mathbf{y}}$, $\mathbf{L}^{(0)}$, $\mathcal{E}$, $\sigma^2_{\mathbf{x}}$, $\sigma^2_{\mathbf{w}}$, $h(\cdot)$, Optional: $r_{\max}$.
- Output: Estimated edge disconnections set, $\hat{\mathcal{C}}$.

Initialize $\hat{\mathcal{C}}^0 = \emptyset$, $\hat{\mathcal{E}}^0 = \mathcal{E}$, $\hat{\mathbf{L}}^0 = \mathbf{L}^{(0)}$, and $l = 0$.
Find the maximal edge, $\hat{k} \in \hat{\mathcal{E}}^l$, by

$$\hat{k} = \underset{k=(i,j)\in\hat{\mathcal{E}}^l}{\operatorname{argmax}} \; l(\mathbf{y}|\hat{\mathbf{L}}^l - \mathbf{E}^{(k)}) - \rho(\hat{\mathbf{L}}^l - \mathbf{E}^{(k)}),$$

**if** $l(\mathbf{y}|\hat{\mathbf{L}}^l - \mathbf{E}^{(\hat{k})}) - \rho(\hat{\mathbf{L}}^l - \mathbf{E}^{(\hat{k})}) > 0$ **then**

  Update $\hat{\mathcal{C}}^{l+1} = \hat{\mathcal{C}}^l \cup \{\hat{k}\}$, $\hat{\mathcal{E}}^{l+1} = \hat{\mathcal{E}}^l \setminus \{\hat{k}\}$, $\hat{\mathbf{L}}^{l+1} = \hat{\mathbf{L}}^l - \mathbf{E}^{(\hat{k})}$, and $l \leftarrow l + 1$

  **if** $|\hat{\mathcal{C}}^l| = r_{\max}$ **then**

    **Return:** $\hat{\mathcal{C}}^l$.

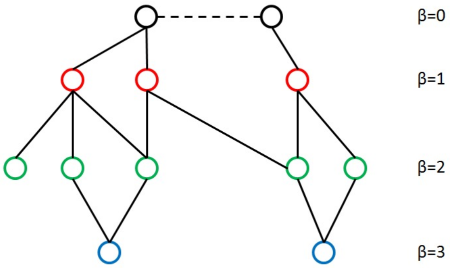
  **Repeat** to step 1.

**Return:** $\hat{\mathcal{C}}^l$.

# Neighboring strategy

**$\beta$-local maximum likelihood decision rule**

- For a given candidate edge, $(i, j)$, we calculate the likelihood ratio of the measurements in the $\beta$-neighborhood of $i$ and $j$, $\mathcal{N}(i,\beta)\bigcup\mathcal{N}(j,\beta)$, where $\mathcal{N}(i,\beta)$ is the set of vertices connected to vertex $i$ by a path of at most $\beta$ edges.
- For each iteration, building a new set of the suspicious edges for the next iteration.




$\Rightarrow$
The tunable parameter $\beta$ provides a trade-off between the identification accuracy and the computation cost.

# Computational complexity

| | ML rule | Greedy | Greedy + neighboring strategy |
|---|---|---|---|
| # Likelihood ratio calculations | $\sum_{r=1}^{r_{\max}} \binom{\lvert\mathcal{E}\rvert}{r}$ | $r_{\max} \times \lvert\mathcal{E}\rvert$ | $r_{\max} \times \lvert\mathcal{E}\rvert$ |
| Matrix inversion* | $\mathcal{O}(N^3)$ | $\mathcal{O}(N^3)$ | $\mathcal{O}(\lvert\mathcal{N}((i,j),\beta)\rvert^3)$ |
| Searching edge set size* | $\sum_{r=1}^{r_{\max}} \binom{\lvert\mathcal{E}\rvert}{r}$ | $\lvert\mathcal{E}\rvert$ | $\lvert\hat{\mathcal{E}}^{(l)}\rvert \ll \lvert\mathcal{E}\rvert$ |

*For sparse graphs, where $|\mathcal{E}| \ll \frac{N(N-1)}{2}$.

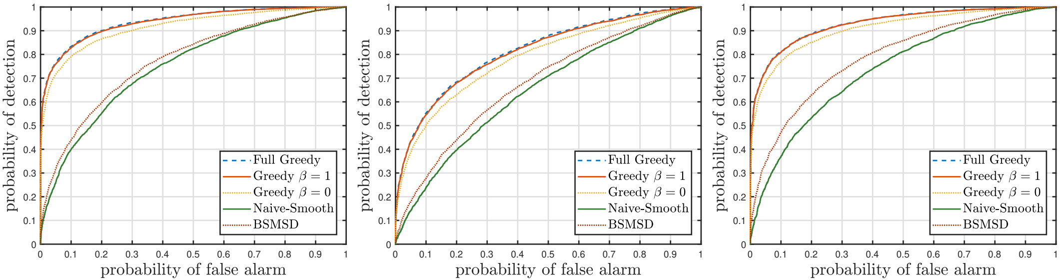
# Outline

# Simulations #1: synthetic data

- Smooth graph filters
- The initial graph was generated by using the Watts-Strogatz small-world graph model, with $N = 50$ vertices, mean degree of $d = 2$, and $|\mathcal{E}| = 100$
- The elements of $\mathbf{W}^{(0)}$ are independent, uniformly distributed weights in $[0.1, 5]$
- Topology change is obtained by removing an arbitrary set of $r$ edges from $\mathcal{E}$
- Comparison with
  - Blind simple-MSD (BSMD) detector [1, 2]
  - Smoothness detector
  - GGM-GLRT: uses the sample covariance matrix of the 1st-order neighbors of the edges [3]
  - Combinatorial graph Laplacian (CGL) method [4]
  - Constrained CGL (CCGL) method: CGL + information on the initial Laplacian matrix, $\mathbf{L}^{(0)}$

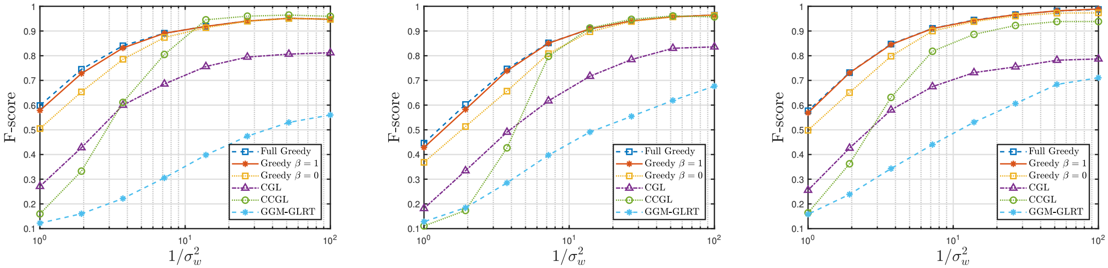
[1] C. Hu, J. Cheng, K. A. Sepulcre, G. E. Fakhri, Y. M. Lu, and K. Li, "Matched signal detection on graphs: Theory and application to brain imaging data classification", 2016.
[2] E. Isufi, A. S. U. Mahabir and G. Leus, "Blind Graph Topology Change Detection", 2018
[3] K. Tugnait, "Edge exclusion tests for graphical model selection: Complex Gaussian vectors and time series", 2019
[4] H. Egilmez, E. Pavez, and A. Ortega, "Graph Learning from Data under Structural and Laplacian Constraints", 2017

# Detection performance




Receiver operating characteristic (ROC) curves of edge disconnection detection by the greedy algorithm, $\beta = 0, 1$-neighbors greedy algorithm, smoothness detector, and BMSD for: the GMRF (left), the regularized Laplacian (middle), and the heat diffusion (right) filters, with noise variance $\sigma_{\mathbf{w}}^2 = 0.5$, $M = 100$ edges, and $r = 5$ potential disconnections.

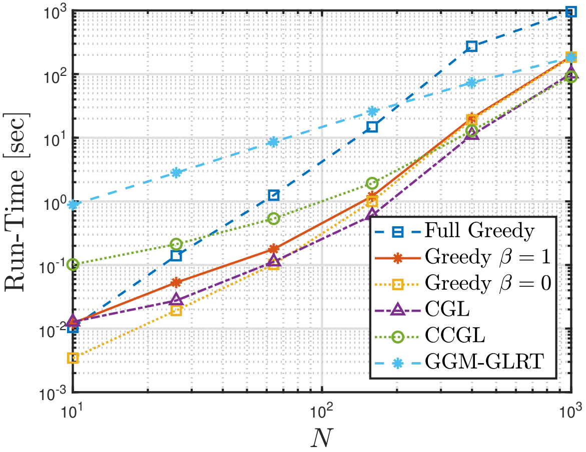
# Identification performance




The F-score measure for the GMRF (left), the regularized Laplacian (middle), and the heat diffusion (right) filters versus SNR for the greedy algorithm, $\beta = 0, 1$-neighbors greedy algorithm, CGL method, CCGL method, and the GGM-GLRT method with $M = 1,000$ and $r = 5$.

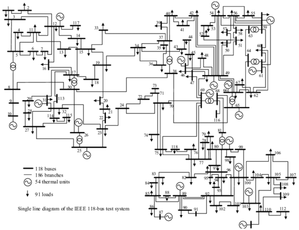
The F-score metric takes values between $0$ and $1$, where $1$ means perfect identification.

# Run-time



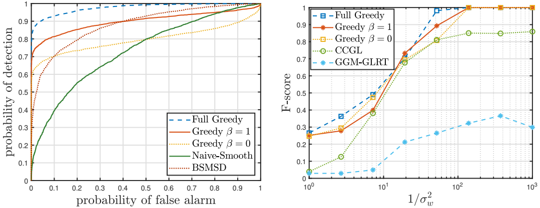

Run-time of the different methods for $\sigma_{\mathbf{w}}^2 = 0.1$, $M = 100,000$, $r = 2$, and $L = 100$

# Simulations #2: Identifying outages in power system dataset




- The vertices and the edges denote the buses (generators or loads), and the transmission lines between these buses, respectively. The branch susceptances determine the weights of the graph edges.
- We assume Phasor measurement units PMUs in the considered system that acquire noisy measurements of the voltage phases at all buses, and $|v_n| = 1$
- We tested random combinations of outages at the transmission lines,

Identifying outages in power system: ROC curves (left) of edge disconnections detector and the F-score measure (right) versus SNR by assuming the GMRF filter.



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# Conclusions

- We propose identifying edge disconnections in networks based on a graph filter model.
- Interpretations of the developed maximum likelihood decision rule:
  - Based on graph energy levels in the graph spectral domain
  - Has a penalty on models with a larger number of disconnections
  - A local smoothness detector for the noiseless GMRF filter with a Laplacian precision
- We propose two greedy algorithms that
  - converge to maximum likelihood decision rule for the noiseless GMRF filter
  - outperform state-of-the-art methods on the tested scenarios in terms of detection and identification performance, and computational complexity
  - The neighboring strategy is based on localization and smoothness properties
  - provide a good trade-off between performance and complexity
- Future research directions include
  1. Extension for “blind” scenarios with unknown graph filters
  2. Dynamically change detection
  3. Other typical topology changes