Robust Signal Processing over Simplicial Complexes

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Outline

- Motivating remarks
- Impact of small topological perturbations on FIR filters robustness
- Robust simplicial FIR filters design
- Numerical results
- Conclusions

Motivating remarks

Motivations

- In many applications (brain networks, social and communication networks) the topology associated with data may undergo perturbations in an unknown manner

- How design FIR filters robust to graph and simplices perturbations

Motivations

- State of art:
  - FIR filters over graphs: A. Sandryhaila et al. (2014), S. Segarra et al. (2017), J. Liu et al. (2018)...
  - Filtering over simplicial complexes: J. Jia et al. (2019), M.T. Schaub et al. (2020), M. Yang et al. (2021), S. Sardellitti and S. Barbarossa (2022)...
  - Stability to perturbations of graph FIR filters: F. Gama et al. (2020), H. Kenlay et al. (2020)...
  - Small perturbation analysis in GSP: E. Ceci and B. Barbarossa (2020), J. Miettinen at al. (2021)...

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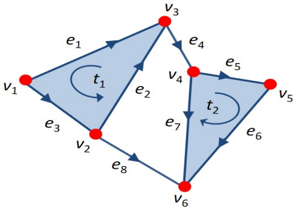
- Our novel contribution
    - Design of robust FIR filters over graphs and simplicial complexes hinging on small perturbation analysis of the Laplacian eigenvectors
    - Impact of perturbation on Laplacian kernel dimension

Introduction to TSP

- Topological signal processing (TSP) provides tools for the processing of signals defined over simplicial complexes

- An abstract simplicial complex is a finite collection of subsets (simplices) of various cardinality of the elements of a set of vertices $\mathcal{V}$ satisfying the inclusion property



- Combinatorial first-order Laplacian matrix

$$\mathbf{L}_1 = \underbrace{\mathbf{B}_1^T \mathbf{B}_1}_{\mathbf{L}_1^d} + \underbrace{\mathbf{B}_2 \mathbf{B}_2^T}_{\mathbf{L}_1^u}$$

with

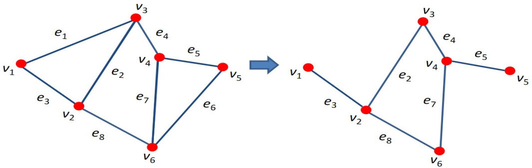
- $\mathbf{B}_1 \in \mathbb{R}^{N \times E}$: nodes-edges incidence matrix
- $\mathbf{B}_2 \in \mathbb{R}^{E \times T}$: edges-triangles incidence matrix

It holds $\mathbf{B}_1 \mathbf{B}_2 = \mathbf{0}$

Nodes, edges and triangles signals can be defined over a simplicial complex

Topological domain perturbation

- Graph perturbation: Given a nominal graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ with Laplacian $\mathbf{L}_0 = \mathbf{B}_1\mathbf{B}_1^T$ a few edges are added or removed



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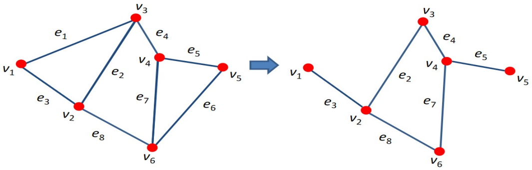
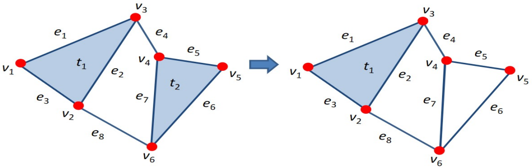


- Simplicial perturbation: Given a nominal simplicial complex $\mathcal{X} = \{\mathcal{V}, \mathcal{E}, \mathcal{T}\}$ with Laplacian $\mathbf{L}_1 = \mathbf{B}_1^T\mathbf{B}_1 + \mathbf{B}_2\mathbf{B}_2^T$ a few triangles are added or removed



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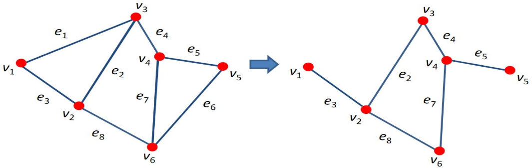

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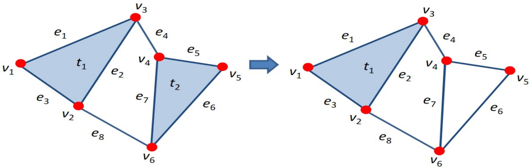

What is the impact of the domain perturbation on the perturbed Laplacian eigenpairs?

Graph perturbation

Perturbed graph Laplacian: $\tilde{\mathbf{L}}_0 = \mathbf{L}_0 + \Delta\mathbf{L}_0 = \tilde{\mathbf{U}}_0 \tilde{\mathbf{\Lambda}}_0 \tilde{\mathbf{U}}_0^T$

Perturbed eigenvectors and eigenvalues:

$$\tilde{\boldsymbol{u}}_i^0 = \boldsymbol{u}_i^0 + \delta \boldsymbol{u}_i^0, \quad \tilde{\lambda}_i^0 = \lambda_i^0 + \delta \lambda_i^0$$

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If all $\lambda_i(\mathbf{L}_0)$ are distinct, we may use the first-order analysis developed in *E. Ceci and B. Barbarossa (2020)*

Perturbation of the mth link: $\Delta\mathbf{L}_{0,m} = \sigma_m \boldsymbol{b}_m^1 \boldsymbol{b}_m^{1\,T}$

- $\sigma_m = 1, -1$ if the edge m is added or removed from graph
- $\boldsymbol{b}_m^1 \in \mathbb{R}^N$: column m of $\mathbf{B}_1$

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Approximate perturbations:

$$\delta\lambda_{i,m}^0 = \sigma_m \boldsymbol{u}_i^{0\,T} \boldsymbol{b}_m^1 \boldsymbol{b}_m^{1\,T} \boldsymbol{u}_i^0 = \sigma_m q_{i,m}$$

$$\delta\boldsymbol{u}_{i,m}^0 = \sigma_m \sum_{j=2, j\neq i}^{N} \frac{\boldsymbol{u}_j^{0\,T} \boldsymbol{b}_m^1 \boldsymbol{b}_m^{1\,T} \boldsymbol{u}_i^0}{\lambda_i^0 - \lambda_j^0} \boldsymbol{u}_j^0 = \sigma_m \sum_{j=2, j\neq i}^{N} c_{ji}^{(m)} \boldsymbol{u}_j^0,$$

where $q_{i,m} = \boldsymbol{u}_i^{0\,T} \boldsymbol{b}_m^1 \boldsymbol{b}_m^{1\,T} \boldsymbol{u}_i^0$ is the square norm of the gradient of $\boldsymbol{u}_i^0$

Graph perturbation

Review:

FIR graph filter $\mathbf{H}$ of order L:

$$\mathbf{H} = \sum_{n=0}^{L} a_n \mathbf{L}_0^n$$

GFT of the output vector: $\widehat{\boldsymbol{y}} = \sum_{n=0}^{L} a_n \boldsymbol{\Lambda}_0^n \widehat{\boldsymbol{x}} := \mathrm{diag}(\mathbf{h})\widehat{\boldsymbol{x}}$

where $\mathrm{diag}(\mathbf{h})$ is the desired frequency response of the filter

Filter coefficients: solutions of a least squares problem

$$\min_{\boldsymbol{a} \in \mathbb{R}^L} \quad \| \, \mathbf{h} - \boldsymbol{\Phi} \; \boldsymbol{a} \, \|_F^2$$

with $\boldsymbol{\Phi} = [\mathbf{1}, \boldsymbol{\lambda}_0, \ldots, \boldsymbol{\lambda}_0^L]$, $\boldsymbol{\lambda}_0^k = \{(\lambda_i^0)^k\}_{i=1}^N$, $k = 1, \ldots, L$

Optimal least squares solution: $\boldsymbol{a} = \boldsymbol{\Phi}^\dagger \mathbf{h}$

Graph perturbation

Our goal: Use small perturbation analysis to design robust FIR filters for signals defined over perturbed graphs

Assume a small subset $\mathcal{E}_p$ of edges are added to the graphs and the random variables σ_m are i.i.d. with $\sigma_m = 1$ w.p. p_m and $\sigma_m = 0$ w.p. $1 - p_m$, we find

$$E[\delta\lambda_i^0] = \textstyle\sum_{m\in\mathcal{E}_p} p_m q_{i,m}, \quad E[\Delta\mathbf{U}_0] = \sum_{m\in\mathcal{E}_p} p_m \mathbf{U}_0\mathbf{C}_m$$
$$E[(\delta\lambda_i^0)^2] = \sum_{m\in\mathcal{E}_p} p_m (q_{i,m})^2 + \sum_{m,n,m\neq n\in\mathcal{E}_p} p_m p_n q_{i,m} q_{i,n}$$

Robust FIR filter coefficients as solutions of the problem

$$\min_{\tilde{\boldsymbol{a}}\in\mathbb{R}^L} \quad E[\| \tilde{\mathbf{h}} - \tilde{\boldsymbol{\Phi}}\tilde{\boldsymbol{a}} \|_F^2]$$

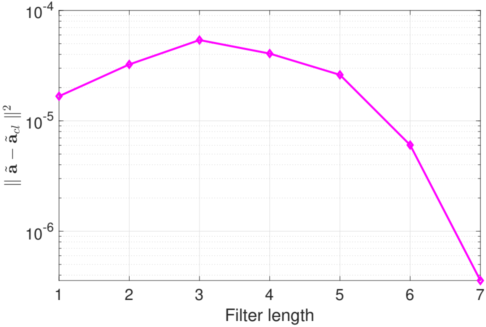
with $\tilde{\boldsymbol{h}} = h(\tilde{\boldsymbol{\lambda}_0})$, $h(\lambda) : \mathbb{R} \to \mathbb{R}$ the frequency response of the filter, $\tilde{\boldsymbol{\Phi}} = [\mathbf{1}, \tilde{\boldsymbol{\lambda}}_0, \ldots, \tilde{\boldsymbol{\lambda}}_0^L]$

Optimal solution $\tilde{\boldsymbol{a}} = E[\tilde{\boldsymbol{\Phi}}^T\tilde{\boldsymbol{\Phi}}]^{-1}E[\tilde{\boldsymbol{\Phi}}^T\tilde{\mathbf{h}}]$

We derived approximated closed form for the average matrix $\mathbf{G}_1 = E[\tilde{\boldsymbol{\Phi}}^T\tilde{\boldsymbol{\Phi}}]$ and the vector $\boldsymbol{g}_1 = E[\tilde{\boldsymbol{\Phi}}^T\tilde{\mathbf{h}}]$

Numerical results

Averaged error in the estimation of the coefficients of a FIR filter, assuming the perturbation perfectly known or by using the closed form solutions



Simulation setting: the edges of a graph with two clusters, each with 20 nodes, are perturbed w.p. $p = 0.01$

The closed form solutions provide an accurate estimation of the filter coefficients

Graph perturbation

In the above analysis we assumed:

- the eigenvalues are all distinct
- the graph undergoes small perturbations of its edges by preserving its connectivity, i.e. the Laplacian kernel dimension

If the removed edges disconnect the graph, we use the approach in U. Von Luxburg (2007):

- Instead of completely removing critical edges we assign them a low weight
- Measure the distance between the subspaces associated with the smallest eigenvalues of the nominal and perturbed Laplacians

Perturbed simplicial complexes

Assumptions: The matrix $\mathbf{B}_1$ is perfectly known and there may be uncertainties about the presence of (filled) triangles, i.e. about the matrix $\mathbf{B}_2$

If a triangle is removed/added the upper Laplacian $\mathbf{L}_1^u$ is perturbed

$$\tilde{\mathbf{L}}_1^u = \mathbf{L}_1^u + \Delta\mathbf{L}_1^u = \mathbf{L}_1^u + \sum_{m \in \mathcal{T}_p} t_m \boldsymbol{b}_m^2 \boldsymbol{b}_m^{2\,T}$$

with $\boldsymbol{b}_m^2$ the column m of $\mathbf{B}_2$ and $t_m = -1$ (or 1) if the m-th triangle is removed (or added)

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Approximate perturbation of the eigenpairs

Let $\mathbf{L}_1^u = \mathbf{U}_1^u \boldsymbol{\Lambda}_u \mathbf{U}_1^{u\,T}$ with $\mathbf{U}_1^u$ containing the eigenvectors associated with the non-zeros eigenvalues λ_i^1

$$\delta\lambda_{i,m}^1 = t_m \left(\sum_{l=1}^{3} b_{m_l}^2 u_i^1(m_l)\right)^2 = t_m q_{i,m}^1, \quad \delta\boldsymbol{u}_{i,m}^1 = t_m \sum_{j=1, j\neq i}^{r_u} \frac{\boldsymbol{u}_j^{1\,T} \boldsymbol{b}_m^2 \boldsymbol{b}_m^{2\,T} \boldsymbol{u}_i^1}{\lambda_i^1 - \lambda_j^1} \boldsymbol{u}_j^1$$

where $q_{i,m}^1 = \left(\sum_{l=1}^{3} b_{m_l}^2 u_i^1(m_l)\right)^2$ is the square of the curl of $\boldsymbol{u}_i^1$ along the triangle m

Remark: The eigenvector $\boldsymbol{u}_i^1$ does not perturb the eigenpairs if its curl along the altered triangle is zero

Perturbed simplicial complexes

- Differently from graphs in simplicial complexes the removal/addition of any triangle increases/decreases the dimension of $\ker(\mathbf{L}_1)$
- If we remove/add a triangle that shares only vertices but no edges with the other triangles then the perturbed Laplacian $\tilde{\mathbf{L}}_1^u$ preserves the same eigenpairs of $\mathbf{L}_1^u$
- The perturbation depends on the number N_2 of 2-simplices (triangles) lower-adjacent with each triangle

Perturbed simplicial complexes

For large perturbation we generalize the approach proposed in U. Von Luxburg (2007) to simplicial complexes:

- To keep the homology of the complex approximately unaltered, the addition/removal of triangles is controlled by assigning them a small positive weight α

- The perturbation of the upper Laplacian is $\boldsymbol{\Delta}\mathbf{L}_1^u = \sum_{n \in \mathcal{T}_p} t_n \alpha \boldsymbol{b}_n^2 \boldsymbol{b}_n^{2\,T}$

- Measure the distance between the subspaces spanned by the eigenvectors of the nominal and perturbed Laplacians

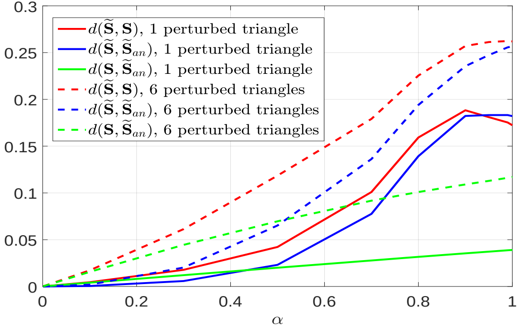
- The distance between the subspaces spanned by the orthonormal matrices $\mathbf{A}, \mathbf{P}$, is defined as

$$d(\mathbf{A}, \mathbf{P}) = \| \sin(\boldsymbol{\theta}) \|_F \, / c$$

  where $\theta_i = \text{acos}(\sigma_i)$ with σ_i the singular values of the matrix $\mathbf{A}^T\mathbf{P}$ and $c > 0$ a normalization coefficient

Perturbed simplicial complexes

Subspace distance versus α



$\mathbf{S}$, $\widetilde{\mathbf{S}}$ and $\widetilde{\mathbf{S}}_{an}$ are the subspaces spanned by the eigenvectors associated with the smallest non-zero eigenvalues of, respectively, $\mathbf{L}_1^u$, $\tilde{\mathbf{L}}_1^u$ and the upper Laplacian $\tilde{\mathbf{L}}_{1,a}^u$ derived from the formulas

The subspace distance is averaged over 100 random SCs

The derived formulas can still be used when the complex is near to be altered (α near to 1)

Perturbed simplicial complexes

Extension to Solenoidal FIR filter:

$$\hat{\mathbf{H}}_u = \sum_{n=1}^{L} \tilde{a}_n^u (\tilde{\mathbf{L}}_1^u)^n$$

The filter coefficients are derived as the solution of the mean squares problem

$$\min_{\tilde{\boldsymbol{a}}_u \in \mathbb{R}^L} \quad E[\| \, \tilde{\mathbf{h}}_u - \tilde{\boldsymbol{\Phi}}_u \, \tilde{\boldsymbol{a}}_u \, \|_F^2]$$

with $\tilde{\mathbf{h}}_u = h(\tilde{\boldsymbol{\lambda}}_u)$, $\tilde{\boldsymbol{\Phi}}_u = [\tilde{\boldsymbol{\lambda}}_u, \ldots, \tilde{\boldsymbol{\lambda}}_u^L]$

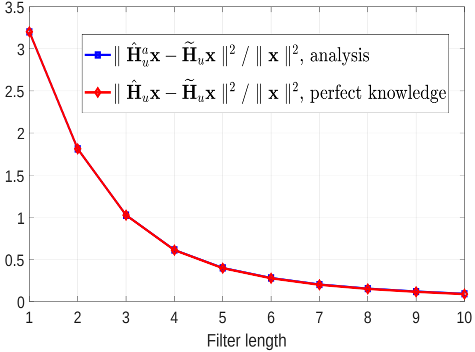
The constant vector $\mathbf{1}$ in $\tilde{\boldsymbol{\Phi}}_u$ is omitted to leave out the harmonic component from the filtering (S. Sardellitti and S. Barbarossa, 2022)

Optimal solution $\tilde{\boldsymbol{a}}_u = E[\tilde{\boldsymbol{\Phi}}_u^T \tilde{\boldsymbol{\Phi}}_u]^{-1} E[\tilde{\boldsymbol{\Phi}}_u^T \tilde{\mathbf{h}}_u]$

We derived closed formulas for $E[\tilde{\boldsymbol{\Phi}}_u^T \tilde{\boldsymbol{\Phi}}_u]$ and the vector $E[\tilde{\boldsymbol{\Phi}}_u^T \tilde{\mathbf{h}}_u]$

Numerical results

Error of the solenoidal filter output with respect to the ideal desired filter $\tilde{\mathbf{H}}_u$ with known perturbation, versus the filter length



Simulation settings: Starting from a nominal SC having all its $T = 80$ triangles filled, we removed some triangles w.p. $p = 0.01$

The proposed method provides the same performance of the case where the perturbation is perfectly known

Conclusions

- We investigated the impact of small perturbations of graphs and simplicial complexes on the robustness of FIR filters
- Hinging on small perturbation analysis we derived closed form expressions for the Laplacian eigenpairs which are useful to design robust FIR filters
- We showed how to deal with perturbations altering the dimension of the signal subspaces
- Future developments: improve performance by developing a second-order approximation of the eigenvectors of the perturbed Laplacian

Thanks for your attention!