A test for conditional correlation between random vectors based on weighted U-Statistics

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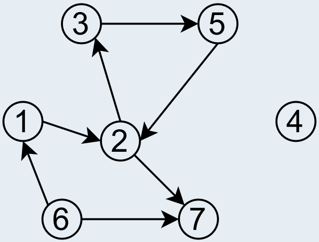
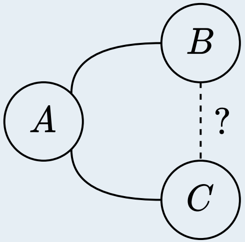
Graphical models and *the curse of dimensionality*

Graphical models



Graphical models and *the curse of dimensionality*

Conditional correlation



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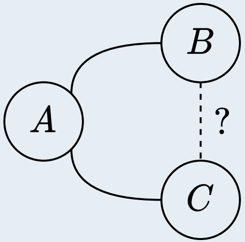
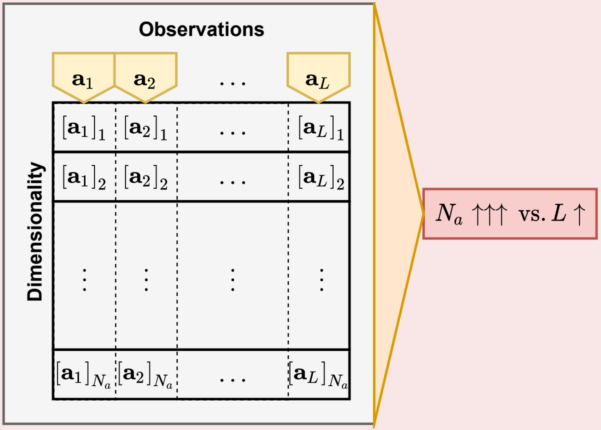


The curse of dimensionality



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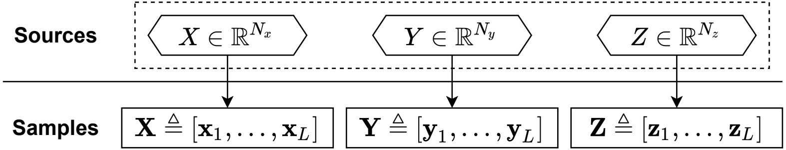
Definitions

	Random vector U	Random vector V
Statistical mean	$\boldsymbol{\mu}_U \triangleq \mathrm{E}\left[\mathbf{u}_l\right]$	$\boldsymbol{\mu}_V \triangleq \mathrm{E}\left[\mathbf{v}_l\right]$
Sample mean	$\widehat{\boldsymbol{\mu}}_U \triangleq \frac{1}{L}\sum_{l=1}^{L}\mathbf{u}_l$	$\widehat{\boldsymbol{\mu}}_V \triangleq \frac{1}{L}\sum_{l=1}^{L}\mathbf{v}_l$
Statistical covariance matrix	$\mathbf{C}_{UV} \triangleq \mathrm{E}\left[\mathbf{u}_l\mathbf{v}_l^T\right] - \boldsymbol{\mu}_U\boldsymbol{\mu}_V^T$	
Sample covariance matrix	$\widehat{\mathbf{C}}_{UV} \triangleq \frac{1}{L-1}\sum_{l=1}^{L}\left(\mathbf{u}_l - \widehat{\boldsymbol{\mu}}_U\right)\left(\mathbf{v}_l - \widehat{\boldsymbol{\mu}}_V\right)^T$	
Average conditional cross-covariance matrix	$\mathbf{C}_{UV\|Z} \triangleq \mathrm{E}_Z\left[\mathbf{C}_{U,V\|Z=\mathbf{z}}\right] = \int_{\mathbb{R}^{N_z}} \mathbf{C}_{UV\|Z=\mathbf{z}} dF_Z(\mathbf{z})$	

Problem under study



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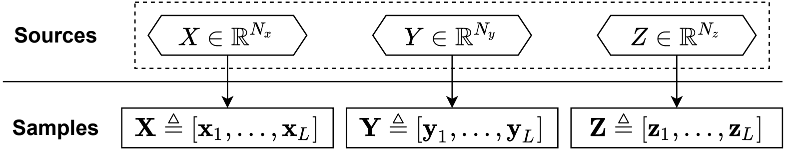
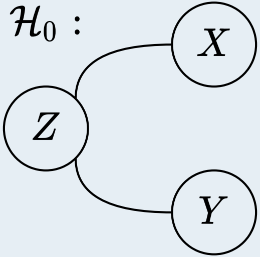
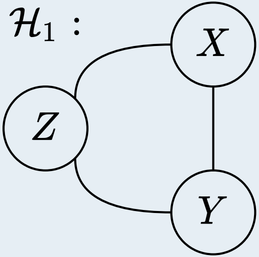


Hypotheses





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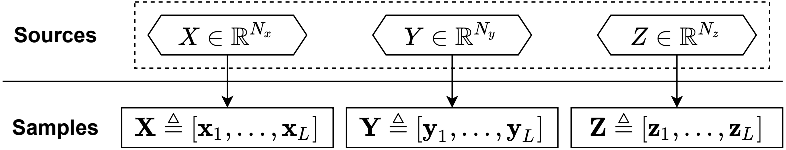

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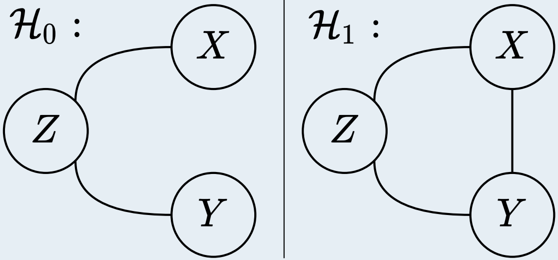

Hypothesis test

$$\left.\begin{array}{ll} \mathcal{H}_0: & \mathbf{C}_{XY|Z} = \mathbf{0} \\ \mathcal{H}_1: & \mathbf{C}_{XY|Z} \neq \mathbf{0} \end{array}\right\} \quad (1)$$

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Unconditioned test

$$\left.\begin{array}{ll}\mathcal{H}_0: & \mathbf{C}_{XY} = \mathbf{0} \\ \mathcal{H}_1: & \mathbf{C}_{XY} \neq \mathbf{0}\end{array}\right\} \longrightarrow \frac{\max_{\mathbf{C}_{WW}} f(\mathbf{W}|\mathbf{C}_{WW})}{\max_{\mathbf{C}_{XX}} f(\mathbf{X}|\mathbf{C}_{XX}) \max_{\mathbf{C}_{YY}} f(\mathbf{Y}|\mathbf{C}_{YY})} \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \lambda, \quad W \triangleq \begin{bmatrix} X \\ Y \end{bmatrix} \quad (2)$$

Unconditioned test

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Test for jointly Gaussian sources → *Hadamard Ratio Test (HRT)*

$$\mathrm{T}_{\mathrm{HAD}}(\mathbf{X}, \mathbf{Y}) \triangleq \frac{\det[\widehat{\mathbf{C}}_{WW}]}{\det[\widehat{\mathbf{C}}_{XX}] \det[\widehat{\mathbf{C}}_{YY}]} \in [0, 1], \quad \mathrm{T}_{\mathrm{HAD}} \uparrow \Rightarrow \mathcal{H}_0 \tag{3}$$

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Small correlation regime → *1st Order Taylor approximation*

$$\mathrm{T}_{\mathrm{FRO}}(\mathbf{X}, \mathbf{Y}) \triangleq \|\widehat{\mathbf{C}}\|_F^2 \geq 0, \quad \mathrm{T}_{\mathrm{FRO}} \downarrow \Rightarrow \mathcal{H}_0 \quad (4)$$

$$\widehat{\mathbf{C}} \triangleq \widehat{\mathbf{C}}_{XX}^{-1/2} \widehat{\mathbf{C}}_{XY} \widehat{\mathbf{C}}_{YY}^{-1/2}$$ *(sample coherence matrix)*

Unconditioned test

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RV Coefficient

$$\mathrm{T}_{\mathrm{RV}}(\mathbf{X}, \mathbf{Y}) \triangleq \frac{\|\widehat{\mathbf{C}}_{XY}\|_F^2}{\|\widehat{\mathbf{C}}_{XX}\|_F \|\widehat{\mathbf{C}}_{YY}\|_F} \tag{5}$$

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Conditional RV Coefficient

$$\mathrm{T}_{\mathrm{RV}}(\mathbf{X}, \mathbf{Y}|\mathbf{Z}) \triangleq \frac{\|\widehat{\mathbf{C}}_{XY|Z}\|_F^2}{\|\widehat{\mathbf{C}}_{XX|Z}\|_F \|\widehat{\mathbf{C}}_{YY|Z}\|_F} \quad (6)$$

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Jointly Gaussian sources

$$\widehat{\mathbf{C}}_{UV|Z} \triangleq \widehat{\mathbf{C}}_{UV} - \widehat{\mathbf{C}}_{UZ}\widehat{\mathbf{C}}_{ZZ}^{-1}\widehat{\mathbf{C}}_{ZV} \quad (7)$$

(Schur complement)

RV Coefficient

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(Schur complement)

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Covariance estimation using U-Statistics

U-Covariance Matrix

$$\widehat{\mathbf{C}}_{UV} = \frac{2}{L(L-1)} \sum_{i=1}^{L-1} \sum_{j=i+1}^{L} \mathring{\mathbf{u}}_{i,j} \mathring{\mathbf{v}}_{i,j}^T, \quad \mathring{\mathbf{u}}_{i,j} \triangleq \frac{\mathbf{u}_i - \mathbf{u}_j}{\sqrt{2}}, \quad \mathring{\mathbf{v}}_{i,j} \triangleq \frac{\mathbf{v}_i - \mathbf{v}_j}{\sqrt{2}} \tag{8}$$

Covariance estimation using U-Statistics

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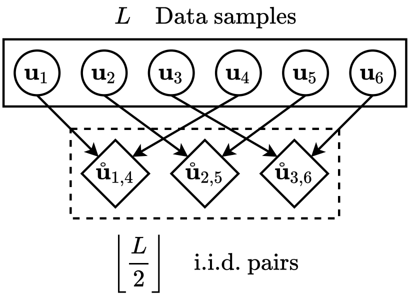

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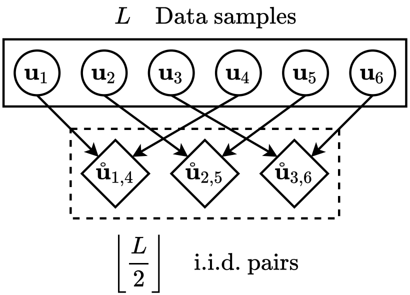
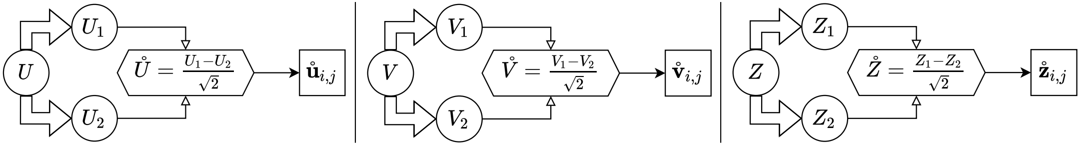


Incomplete U-Covariance Matrix

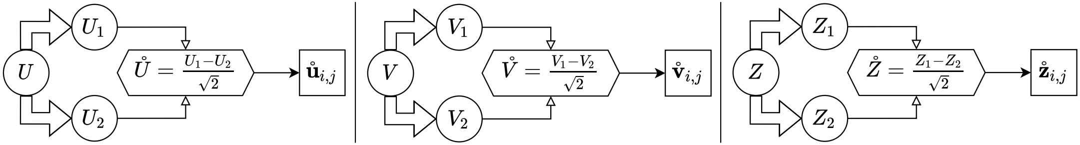
$$\widehat{\mathbf{C}}'_{UV} = \frac{1}{\lfloor L/2 \rfloor} \sum_{i=1}^{\lfloor L/2 \rfloor} \mathring{\mathbf{u}}_{i,i+\lfloor L/2 \rfloor} \mathring{\mathbf{v}}_{i,i+\lfloor L/2 \rfloor}^T \tag{9}$$

Unused pairs: $\Delta L = \frac{L(L-1)}{2} - \left\lfloor \frac{L}{2} \right\rfloor$

Weighted U-Statistics



Weighted U-Statistics

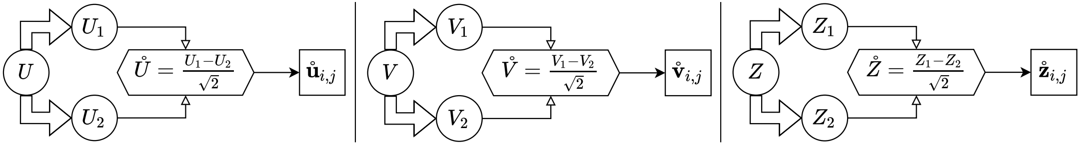


Fact[1]

$$\mathbf{C}_{UV} \equiv \mathbf{C}_{\mathring{U}\mathring{V}}$$

[1] A. J. Lee, *U-Statistics: Theory and Practice*, Routledge, 2019

Weighted U-Statistics



$$\mathbf{C}_{UV|Z} = \int_{\mathbb{R}^{N_z}} \mathbf{C}_{UV|Z=\mathbf{z}} dF_Z(\mathbf{z})$$

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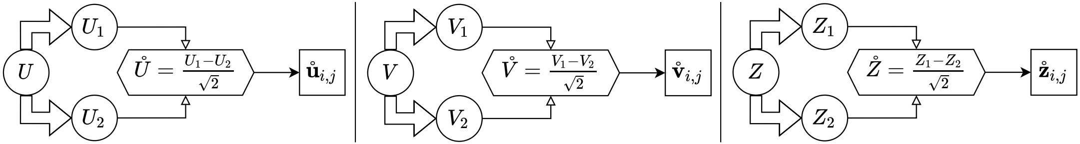
Weighted U-Statistics



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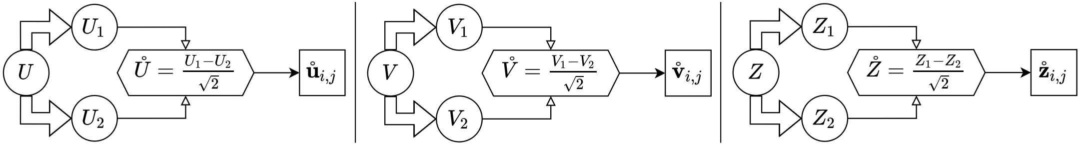

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$$= \iint_{\mathbb{R}^{N_z \times N_z}} \mathbf{C}_{\mathring{U}\mathring{V}|\mathring{Z}(Z_1,Z_2)=\mathbf{0}} dF_{Z_1,Z_2}(\mathbf{z}_1, \mathbf{z}_2)$$

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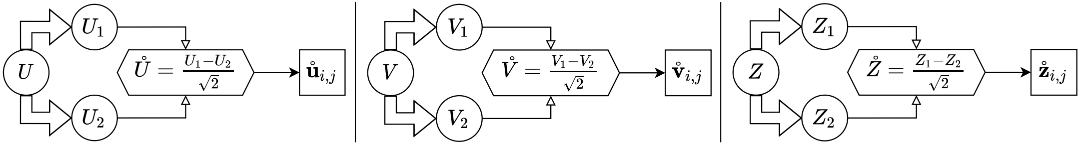

$$
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\mathbf{C}_{UV|Z} &= \int_{\mathbb{R}^{N_z}} \mathbf{C}_{\mathring{U}\mathring{V}|Z=\mathbf{z}} dF_Z(\mathbf{z}) \\
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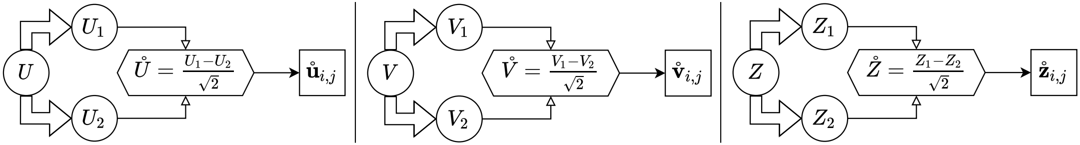

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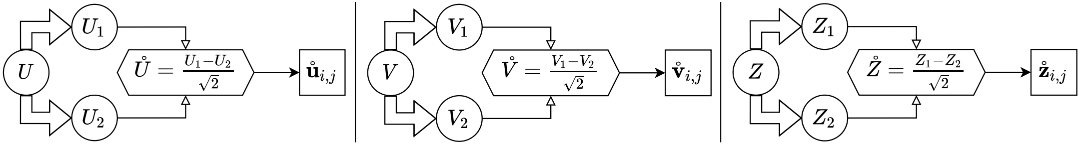

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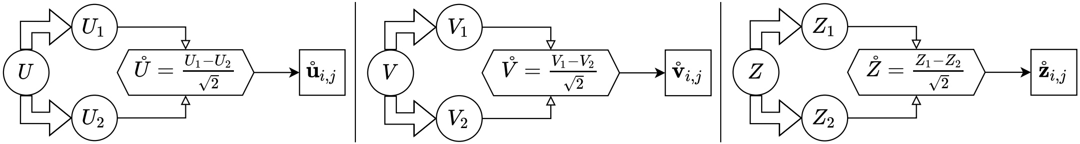

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$$\boxed{\mathbf{C}_{UV|Z} = \mathbf{C}_{\mathring{U}\mathring{V}|\mathring{Z}=\mathbf{0}}} \quad (10)$$

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Relaxation

$$\Pr\{\mathring{Z} = 0\} = 0$$

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Indicator function

$$I_\epsilon(\lambda) \triangleq \begin{cases} 1, & 0 \leq \lambda \leq \epsilon \\ 0, & \text{otherwise} \end{cases} \quad (11)$$

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Conditional covariance matrix estimator

$$\breve{\mathbf{C}}_{UV|Z} \triangleq \frac{\sum_{i=1}^{L-1} \sum_{j=i+1}^{L} \mathring{\mathbf{u}}_{i,j} \mathring{\mathbf{v}}_{i,j}^T I_\epsilon(\|\mathbf{z}_i - \mathbf{z}_j\|)}{\sum_{i=1}^{L-1} \sum_{j=i+1}^{L} I_\epsilon(\|\mathbf{z}_i - \mathbf{z}_j\|)} \quad (12)$$

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Conditional RV Coefficient

$$\breve{\mathrm{T}}_{\mathrm{RV}}(\mathbf{X}, \mathbf{Y}|\mathbf{Z}) = \frac{\|\breve{\mathbf{C}}_{XY|Z}\|_F^2}{\|\breve{\mathbf{C}}_{XX|Z}\|_F \|\breve{\mathbf{C}}_{YY|Z}\|_F} \quad (13)$$

Order statistics

$$\sum_{i=1}^{L-1} \sum_{j=i+1}^{L} I_\epsilon(\|\mathbf{z}_i - \mathbf{z}_j\|) \triangleq L_p \in \left[1, \frac{L(L-1)}{2}\right] \tag{14}$$

Order statistics

$$\sum_{i=1}^{L-1} \sum_{j=i+1}^{L} I_\epsilon(\|\mathbf{z}_i - \mathbf{z}_j\|) \triangleq L_p \in \left[1, \frac{L(L-1)}{2}\right] \tag{14}$$

Selection of pairs from sorting

$$\boxed{\left\{ \begin{array}{c} \|\mathring{\mathbf{z}}_{1,2}\| \\ \|\mathring{\mathbf{z}}_{1,3}\| \\ \vdots \\ \|\mathring{\mathbf{z}}_{1,L}\| \\ \|\mathring{\mathbf{z}}_{2,3}\| \\ \vdots \\ \|\mathring{\mathbf{z}}_{L-1,L}\| \end{array} \right\}} \tag{15}$$

Order statistics

$$\sum_{i=1}^{L-1} \sum_{j=i+1}^{L} I_\epsilon(\|\mathbf{z}_i - \mathbf{z}_j\|) \triangleq L_p \in \left[1, \frac{L(L-1)}{2}\right] \quad (14)$$

Selection of pairs from sorting

$$\left\{ \begin{array}{c} \|\mathring{\mathbf{z}}_{1,2}\| \\ \|\mathring{\mathbf{z}}_{1,3}\| \\ \vdots \\ \|\mathring{\mathbf{z}}_{1,L}\| \\ \|\mathring{\mathbf{z}}_{2,3}\| \\ \vdots \\ \|\mathring{\mathbf{z}}_{L-1,L}\| \end{array} \right\} \xrightarrow{\text{sort}} \mathring{\mathbf{z}}_{\text{sort}} \triangleq \begin{bmatrix} \|\mathring{\mathbf{z}}_{i(1),j(1)}\| \text{ (min)} \\ \vdots \\ \|\mathring{\mathbf{z}}_{i(l),j(l)}\| \\ \vdots \\ \|\mathring{\mathbf{z}}_{i(L_p),j(L_p)}\| \\ \vdots \end{bmatrix} \quad (15)$$

Order statistics

$$\sum_{i=1}^{L-1}\sum_{j=i+1}^{L} I_\epsilon(\|\mathbf{z}_i-\mathbf{z}_j\|) \triangleq L_p \in \left[1, \frac{L(L-1)}{2}\right] \tag{14}$$

Selection of pairs from sorting

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$$\breve{\mathbf{C}}_{UV|Z} = \frac{1}{2L_p} \sum_{l=1}^{L_p} (\mathbf{u}_{i(l)} - \mathbf{u}_{j(l)})(\mathbf{v}_{i(l)} - \mathbf{v}_{j(l)})^T \tag{16}$$

$$\boxed{\breve{\mathbf{C}}_{UV|Z} = \frac{1}{2L_p}\sum_{l=1}^{L_p}(\mathbf{u}_{i(l)} - \mathbf{u}_{j(l)})(\mathbf{v}_{i(l)} - \mathbf{v}_{j(l)})^T} \tag{16}$$

Efficient computation

$$\mathring{\mathbf{U}} \triangleq \left[\left[\mathbf{u}_{i(1)} - \mathbf{u}_{j(1)}\right], \ldots, \left[\mathbf{u}_{i(L_p)} - \mathbf{u}_{j(L_p)}\right]\right] \longrightarrow \mathbf{K}_{\mathbf{U}} \triangleq \mathring{\mathbf{U}}^T\mathring{\mathbf{U}}$$

$$\breve{\mathrm{T}}_{\mathrm{RV}}(\mathbf{X},\mathbf{Y}|\mathbf{Z}) = \frac{\mathrm{Tr}[\mathring{\mathbf{X}}^T\mathring{\mathbf{X}}\mathring{\mathbf{Y}}^T\mathring{\mathbf{Y}}]}{\sqrt{\mathrm{Tr}[\mathring{\mathbf{X}}^T\mathring{\mathbf{X}}\mathring{\mathbf{X}}^T\mathring{\mathbf{X}}]\,\mathrm{Tr}[\mathring{\mathbf{Y}}^T\mathring{\mathbf{Y}}\mathring{\mathbf{Y}}^T\mathring{\mathbf{Y}}]}} = \boxed{\frac{\|\mathbf{K}_{\mathbf{X}}\mathbf{K}_{\mathbf{Y}}\|_F^2}{\sqrt{\|\mathbf{K}_{\mathbf{X}}^2\|_F^2\|\mathbf{K}_{\mathbf{Y}}^2\|_F^2}}} \tag{17}$$

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Test setting

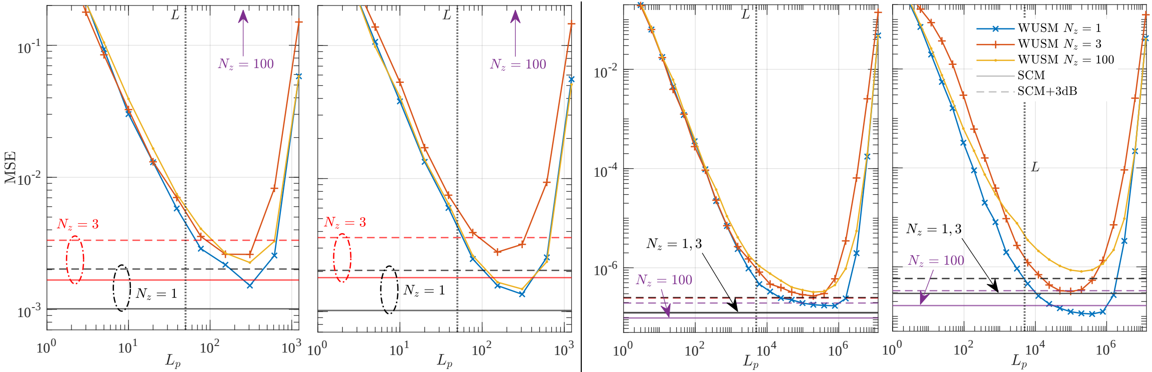
- Techniques:
  - Weighted U-Statistics Method *(WUSM)*
  - Schur Complement Method *(SCM)*
- Confounder dimensions (N_z):
  - 1
  - 3
  - 100
- Power: $\mathrm{E}\left[X^2\right] = \mathrm{E}\left[Y^2\right] = \mathrm{E}\left[Z_n^2\right] = 1$
- RV Coefficient (T_{RV}):
  - $\mathrm{T}_{\mathrm{RV}}\left(X, Y\right) > 0.2$
  - $\mathrm{T}_{\mathrm{RV}}\left(X, Y | Z\right) \approx 0$

- Mean: $\mathrm{E}\left[X\right] = \mathrm{E}\left[Y\right] = \mathrm{E}\left[Z_n\right] = 0$
- Data models:
  - Jointly Normal
  - Gaussian Copula
- Nº of samples (L):
  - 50
  - 5000
- Nº of tests (M): 500

Estimation

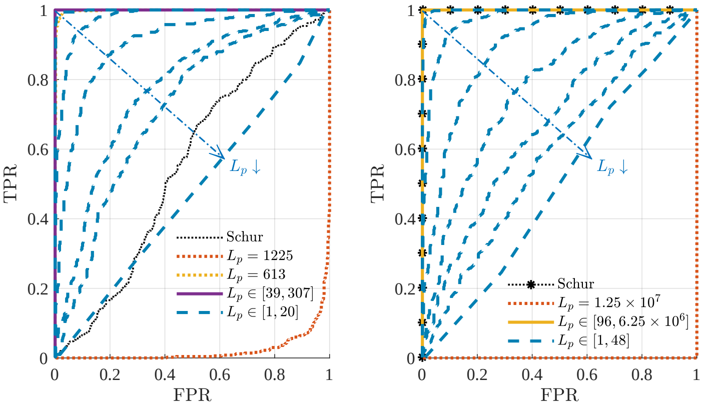
Mean Squared Error

$$\mathrm{MSE}(\breve{\mathrm{T}}_{\mathrm{RV}}) \triangleq \mathrm{var}(\breve{\mathrm{T}}_{\mathrm{RV}}) + \mathrm{bias}^2(\breve{\mathrm{T}}_{\mathrm{RV}})$$



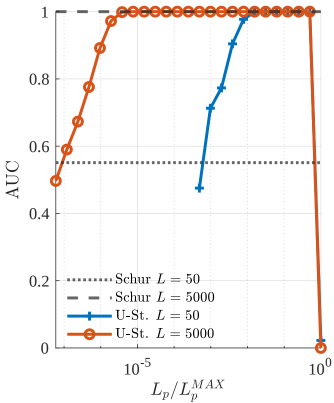
(a) Gaussian, $L = 50$ (b) Copula, $L = 50$ (c) Gaussian, $L = 5000$ (d) Copula, $L = 5000$

Detection



(a) $L = 50$ (b) $L = 5000$

Receiver Operation Characteristic *(ROC)*



Area Under Curve *(AUC)*

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Potential lines of research

Design aspects

- Alternative criteria for sorting (different norms).
- Soft indicator functions and data-driven weighting.

Applications

- Integration of information theoretic methods[2]: correlation $\longrightarrow$ dependence.

[2] J. Riba and F. de Cabrera, “Regularized Estimation of Information via High Dimensional Canonical Correlation Analysis,” 2020, arXiv:2005.02977 [cs.IT]

Thanks for your attention!

Main contributions of the article

Development of a test for conditional correlation based on weighted U-Statistics that:

- Is *data-driven* instead of *model-driven*.
- Does not involve matrix inverses nor determinants.
- Is robust to small-sample/high-dimensional regimes.
- Presents similar capabilities along a wide range of configurations, allowing for a good trade-off between complexity and performance.

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