r-local Sensing: Improved algorithm and Applications

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SUMMARY

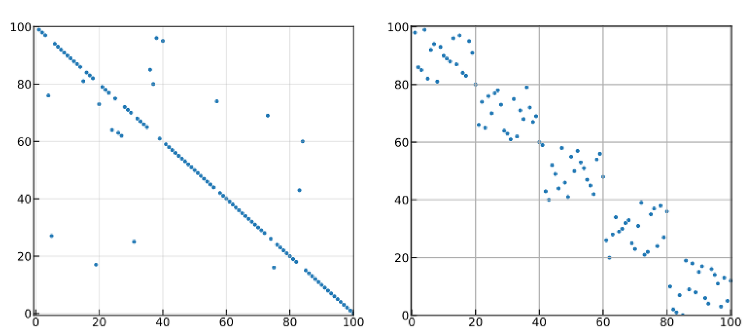
The unlabeled sensing problem is to solve a noisy linear system of equations under unknown permutation of the measurements. We study a particular case of the problem where the permutations are restricted to be r-local, i.e. the permutation matrix is block diagonal with r×r blocks. We propose a proximal alternating minimization algorithm that provably converges to a first order stationary point. We validate the algorithm on synthetic and real datasets. We also formulate the 1-d unassigned distance geometry problem as an unlabeled sensing problem with a structured measurement matrix.

PROBLEM STATEMENT

Unlabeled sensing problem. Given permuted measurement matrix $\mathbf{Y} \in \mathbb{R}^{n \times m}$, estimate signal matrix $\mathbf{X} \in \mathbb{R}^{d \times m}$ from

$$\mathbf{Y} = \mathbf{PBX} + \mathbf{W},$$

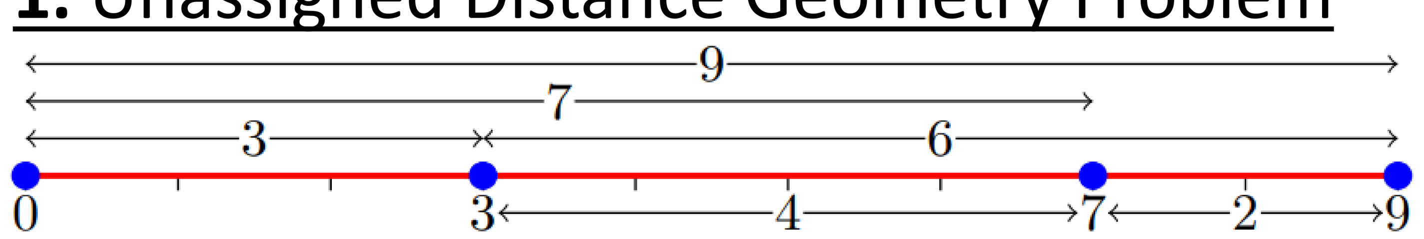
where $\mathbf{P} \in \mathbb{P}_{n \times n}$ is an unknown permutation matrix, $\mathbf{W} \in \mathbb{R}^{n \times m}$ is i.i.d. Gaussian noise with per-entry variance σ^2.



Left. Sparse permutation. Right. Proposed r-local permutation.

APPLICATIONS

1. Unassigned Distance Geometry Problem

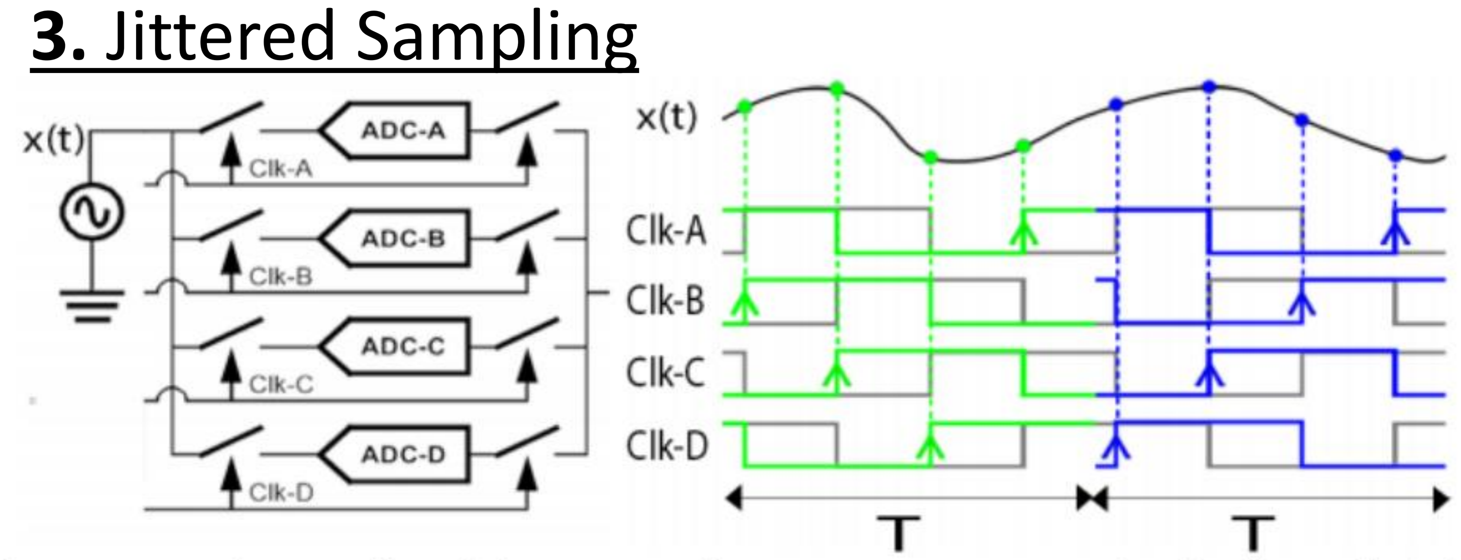


The 1-d unassigned distance geometry problem (uDGP) is to recover the point coordinates $(0, 3, 7, 9)$ from their unlabeled pairwise distances $\{3, 2, 4, 6, 7, 9\}$. uDGP can be formulated as an unlabeled sensing problem with a deterministic measurement matrix.

2. Scrambled Image Recovery

Left. Input image. **Middle**. Scrambled image. **Right**. Reconstruction by algorithm.

3. Jittered Sampling



4 permuted samples (blue, green) are output at each clock cycle $\mathbf{T}$.

PROPOSED ALGORITHM

- Given $\mathbf{Y} = \mathbf{PBX} + \mathbf{W}$, consider the following optimization problem.

$$\text{minimize}_{\mathbf{X},\mathbf{P}} \ \|\mathbf{Y} - \mathbf{PBX}\|_2^2 \qquad \text{subject to } \mathbf{P} \in \mathbb{P}_r \tag{2}$$

- Several existing works consider one-step estimators for $\mathbf{P}, \mathbf{X}$.
- We propose Proximal Alternating Minimization **(PAM)** algorithm for (2). For $\lambda > 0$,

$$\mathbf{P}^{(t+1)} = \underset{\mathbf{P} \in \mathbb{P}_r}{\text{argmin}} \ \|\mathbf{Y} - \mathbf{PBX}^{(t)}\|^2 + \lambda \|\mathbf{P} - \mathbf{P}^{(t)}\|^2, \tag{3}$$

$$\mathbf{X}^{(t+1)} = \underset{\mathbf{X} \in \mathbb{R}^{d \times m}}{\text{argmin}} \ \|\mathbf{Y} - \mathbf{P}^{(t+1)}\mathbf{BX}\|^2 + \lambda \|\mathbf{X} - \mathbf{X}^{(t)}\|^2 \tag{4}$$

- The proximal λ terms in (3),(4) regularize the alternating minimization updates on $\mathbf{P}, \mathbf{X}$ and guarantee convergence to stationary point.

Proposition

PAM iterates $\{(\mathbf{P}^{(t)}, \mathbf{X}^{(t)})\}$ converge to first order stationary point of the objective in (2).

INITIALIZATION

$$\mathbf{Y} = \mathbf{P}_{n \times n}\mathbf{B}\mathbf{X}_{d \times m}, \ \text{where } \mathbf{P} = \ \text{blkdiag}(\mathbf{P}_1, \cdots, \mathbf{P}_{n/r})$$

- Each block $\mathbf{Y}_k \in \mathbb{R}^{r \times m}$ can be expressed as

$$\mathbf{P}_k \mathbf{B}_k \mathbf{X} = \mathbf{Y}_k$$

- Sum shuffled measurements in each block $[\mathbf{P}_k \mathbf{B}_k \vdots \mathbf{Y}_k]$ to obtain n/r measurements

$$\mathbb{1}_r^{\intercal}[\mathbf{P}_k \mathbf{B}_k \vdots \mathbf{Y}_k] \rightarrow [\tilde{\mathbf{b}} \vdots \tilde{\mathbf{y}}] \ \ \forall k \in [n/r]$$

- Initialization $\widehat{\mathbf{X}}^{(0)}$ is given by minimum norm solution to the n/r equations

$$\widehat{\mathbf{X}}^{(0)} = \widetilde{\mathbf{B}}^{\dagger}\widetilde{\mathbf{Y}}$$

BENCHMARKS

Let $\mathbb{P}_r$ denote the set of $r \times r$ permutation matrices. Let R be the number of blocks in $\mathbf{P}$. For $k \in [R]$,

- One-step estimator [Zhang et al., '21].

$$\widehat{\mathbf{P}}_k = \underset{\mathbf{P} \in \mathbb{P}_{r \times r}}{\text{argmin}} \ -\langle \mathbf{P}, \mathbf{YY}^{\intercal}\mathbf{BB}^{\intercal} \rangle$$

- Biconvex relaxation [Zhang et al., '20].

$$\widehat{\mathbf{P}}_k = \underset{\mathbf{P}_1, \mathbf{P}_2 \in \mathbb{P}_{r \times r}}{\text{argmin}} \ -\langle \mathbf{P}_1, \mathbf{BB}^{\dagger}\mathbf{P}_2\mathbf{YY}^{\intercal} \rangle \quad \text{s.t. } \mathbf{P}_1 = \mathbf{P}_2$$

- One-dimensional Levsort [Pananjady et al, '17].

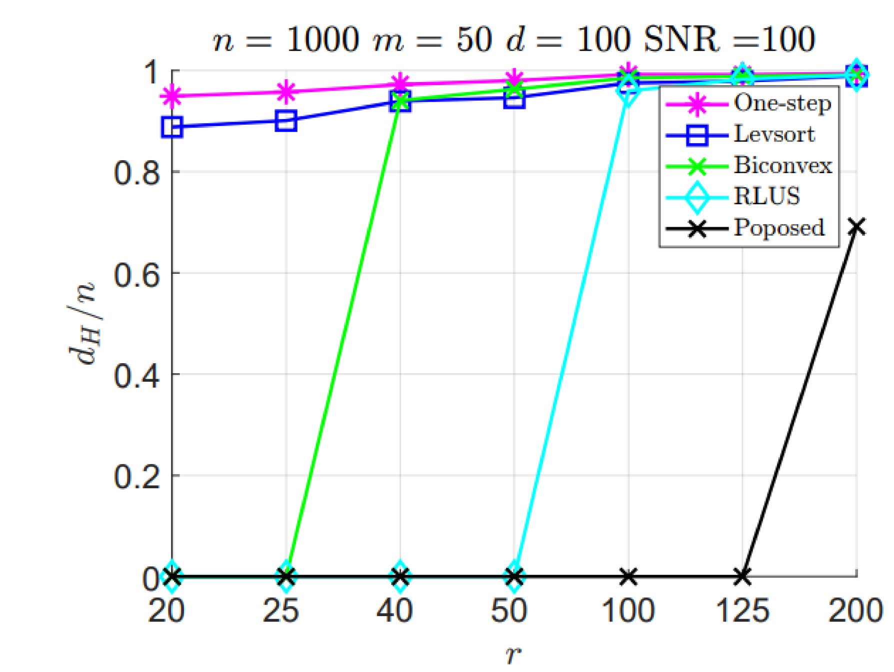
$$\widehat{\mathbf{P}}_k = \underset{\mathbf{P} \in \mathbb{P}_{r \times r}}{\text{argmin}} \ \|\text{diag}(\mathbf{U_Y}\mathbf{U_Y}^{\intercal}) - \mathbf{P}\text{diag}(\mathbf{U_B}\mathbf{U_B}^{\intercal})\|_2^2,$$

$\mathbf{U}$ denote singular vectors.

- RLUS/Quadratic Assignment Problem (QAP) [Abbasi et al., '21].

$$\widehat{\mathbf{P}}_k = \underset{\mathbf{P} \in \mathbb{P}_{r \times r}}{\text{argmin}} \ \|\mathbf{Y}_k\mathbf{Y}_k^{\intercal} - \mathbf{P}\widehat{\mathbf{Y}}_k\widehat{\mathbf{Y}}_k^{\intercal}\mathbf{P}^{\intercal}\|_2^2$$

RESULTS: SYNTHETIC DATA



$\mathbf{Y} = \mathbf{P}^*\mathbf{B}_{n \times d}\mathbf{X}^*_{d \times m} + \mathbf{W}$. The fractional Hamming distortion d_H/n (y-axis), i.e. the number of mismatches $d_H(\hat{\mathbf{P}}, \mathbf{P}) = \Sigma_i \mathbb{1}(\hat{\mathbf{P}}(i) \neq \mathbf{P}(i))$, is plotted against block size r (x-axis).

- **Data generation**. The entries of the sensing matrix $\mathbf{B}$ and the signal matrix $\mathbf{X}$ are sampled i.i.d. from the $\mathbf{N}(0, 1)$ distribution.
- **Results**. Proposed algorithm recovers $\mathbf{P}$ for block size $r \leq n/8$.

RESULTS: SCRAMBLED IMAGE RECOVERY

Left. Unscrambled input image $\mathbf{y}$ from the YALE B and MNIST dataset. **Middle**. Scrambled input image $\mathbf{Py}$. **Right**. PAM reconstruction $\hat{\mathbf{y}} = \widehat{\mathbf{P}}^{\intercal}\mathbf{Py}$.

- The sensing matrix $\mathbf{B}$ contains $d = 10$ principal components of the dataset.
- PAM reconstructs original image from unrecognizably scrambled input.

CONCLUSION AND FUTURE WORK

- We proposed a new algorithm for the r-local unlabeled sensing problem that outperforms existing algorithms.
- In future work, **i)** we will explore the second-order convergence properties of the proposed algorithm.
- **ii)** For what range of problem parameters (n, m, d, σ), does the proposed algorithm recover the true permutation?

CODE AND REFERENCES

1. MATLAB code for all experiments can be found at the first author's github account: https://github.com/aabbas02/Proximal-Alt-Min-for-ULS-UDGP

2. Ahmed Ali Abbasi, Abiy Tasissa, and Shuchin Aeron, "R-local unlabeled sensing: A novel graph matching approach for multiview unlabeled sensing under local permutations," IEEE Open Journal of Signal Process., vol. 2, pp. 309–317, 2021.

3. Hang Zhang and Ping Li, "Optimal estimator for unlabeled linear regression,"in Proceedings of the 37th Int. Conf. on Machine Learning (ICML), 2020.

4. H. Attouch, J. Bolte, P. Redont, and A. Soubeyran, "Proximal alternating minimization and projection methods for nonconvex problems: An approach based on the kurdyka-Łojasiewicz inequality," Mathematics of Operations Research, vol. 35, no. 2, pp. 438–457, 2010.