Joint model order estimation for multiple tensors with a coupled mode and applications to the joint decomposition of EEG, MEG magnetometer, and gradiometer tensors

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Introduction and Motivation

- ❐ **Problem Statement**
  Model order (rank) estimation of multidimensional data (tensors) corrupted by additive noise is essential for analysis, i.e., CP decomposition.
- ❐ **Observations**
  - ⇒ Simultaneously collected data through heterogeneous sensors, i.e., biomedical studies, share coupled factors among multiple tensors.
  - ⇒ This coupling can lead to a better model order estimation.
- ❐ **Contributions**
  - ⇒ Extension of the rank estimation technique from single tensor (using HOSVD) to noise-corrupted coupled tensors that share one of their factor matrices (using coupled HOSVD).
  - ⇒ Improved performance in comparison with classical criteria.
  - ⇒ Application to EEG, MEG Magnetometer, and Gradiometer measurements.

State of the Art

- ❐ Akaike's Information Criteria (AIC)
- ❐ Minimum Description Length (MDL)
- ❐ CORe CONsistency DIAgnostic (CORCONDIA)
- ❐ LineAr Regression of Global Eigenvalues (LaRGE)

[1] Hirotugu Akaike. A new look at the statistical model identification. *IEEE Transactions on Automatic Control*, 19(6):716–723, 1974.

[2] Jorma Rissanen. Modeling by shortest data description. *Automatica*, 14(5):465–471, 1978.

[3] Rasmus Bro and Henk AL Kiers. A new efficient method for determining the number of components in PARAFAC models. *Journal of Chemometrics: A Journal of the Chemometrics Society*, 17(5):274–286, 2003.

[4] Alexey A. Korobkov, Marina K. Diugurova, Jens Haueisen, and Martin Haardt. Multi- dimensional model order estimation using linear regression of global eigenvalues (large) with applications to eeg and meg recordings. in Proc. *28th European Signal Processing Conference*, pages 1005–1009, 2021

HOSVD and Global EigenValues (GEVs)

- ❐ Given: $\mathcal{X} = \mathcal{X}_0 + \mathcal{N} \in \mathbb{C}^{M_1 \times \ldots \times M_N}$, where $\mathcal{X}_0 \in \mathbb{C}^{M_1 \times \ldots \times M_N}$ is the noiseless data of rank R, and $\mathcal{N} \in \mathbb{C}^{M_1 \times \ldots \times M_N}$ is an additive noise tensor.
- ❐ HOSVD of $\mathcal{X}$ given by
  $\mathcal{X} = \mathcal{S} \times_1 \boldsymbol{U}_1 \times_2 \boldsymbol{U}_2 \times_3 \ldots \times_N \boldsymbol{U}_N$

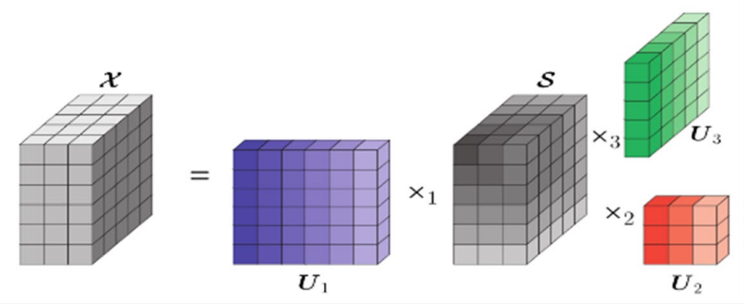

Fig. 1. HOSVD of a 3-D tensor $\mathcal{X} = \mathcal{S} \times_1 \boldsymbol{U}_1 \times_2 \boldsymbol{U}_2 \times_3 \boldsymbol{U}_3$

- ❐ Global EigenValues (GEVs) are computed as the product of the n-mode singular values $\sigma_i^{(n)}$ as
  $\tilde{\lambda}_i^{[G]} = \prod_{n=1}^{N} \left(\sigma_i^{(n)}\right)^2, i = 1, \ldots, M$, where $M = \min\{M_n\}, n \in \{1, \ldots, N\}$ is the smallest dimension in $\mathcal{X}$.

[5] Joao Paulo C. L. da Costa, Martin Haardt, and Florian Roemer. Robust methods based on the HOSVD for estimating the model order in PARAFAC models. In *Proc. 5th IEEE Sensor Array and Multichannel Signal Processing Workshop*, p. 510–514, 2008.

[6] Joao Paulo C. L. da Costa, Florian Roemer, Martin Haardt, and Rafael Timoteo de Sousa. Multi-dimensional model order selection. *EURASIP Journal on Advances in Signal Processing*, 2011(1):26, 2011.

Coupled HOSVD

- ❐ Coupled HOSVD of L tensors is expressed as follows

$$\mathcal{X}^{(l)} = \mathcal{S}^{(l)} \times_1 \boldsymbol{U}_1 \times_2 \boldsymbol{U}_2^{(l)} \times_3 \ldots \times_N \boldsymbol{U}_N^{(l)}$$

- ❐ $\boldsymbol{U}_1 \in \mathbb{C}^{M_1 \times M_1}$ comes from the SVD of the concatenated 1-mode unfoldings of $\mathcal{X}^{(l)}$.

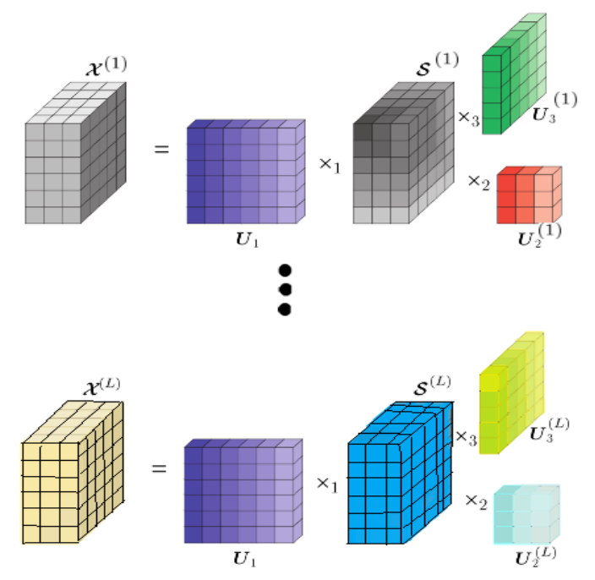

Fig. 2. Coupled HOSVD

Canonical Polyadic Decomposition (CPD)

- ❐ The CP decomposition of the 3-D tensor $\mathcal{X} \in \mathbb{C}^{I_1 \times I_2 \times I_3}$ with rank R is defined as

$$\mathcal{X} = \sum_{r=1}^{R} \boldsymbol{f}_1^{(r)} \circ \boldsymbol{f}_2^{(r)} \circ \boldsymbol{f}_3^{(r)} = \mathcal{I}_{3,R} \times_1 \boldsymbol{F}_1 \times_2 \boldsymbol{F}_2 \times_3 \boldsymbol{F}_3,$$

  where matrices $\boldsymbol{F}_1 \in \mathbb{C}^{I_1 \times R}$, $\boldsymbol{F}_2 \in \mathbb{C}^{I_2 \times R}$, $\boldsymbol{F}_3 \in \mathbb{C}^{I_3 \times R}$ are called factor matrices
- ❐ $\mathcal{I}_{3,R} \in \mathbb{C}^{R \times R \times R}$ has 1's on its super-diagonal and rest of the elements are zero.

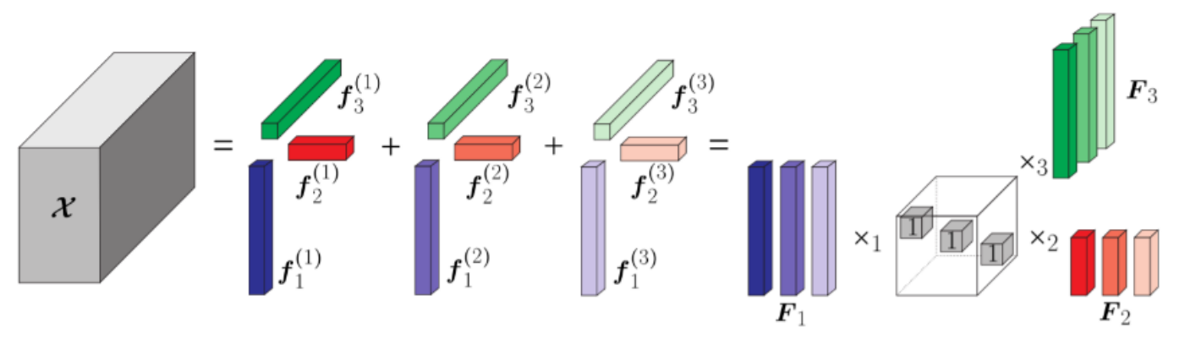

Fig. 3. CPD of a 3-D tensor $\mathcal{X} = \mathcal{I}_{3,R} \times_1 \boldsymbol{F}_1 \times_2 \boldsymbol{F}_2 \times_3 \boldsymbol{F}_3, R = 3$

Coupled CPD

- ❐ Next, let us consider the coupled CPD of L coupled N-dimensional tensors $\mathcal{X}_0^{(l)}, l \in \{1, \ldots, L\}$, with the first mode in common

$$\mathcal{X}_0^{(l)} = \mathcal{I}_{3,R} \times_1 \boldsymbol{F}_1 \times_2 \boldsymbol{F}_2^{(l)} \times_3 \ldots \times_N \boldsymbol{F}_N^{(l)} \in \mathbb{C}^{M_1 \times M_2^{(l)} \times \ldots \times M_N^{(l)}},$$

where $\boldsymbol{F}_n^{(l)} \in \mathbb{C}^{M_n^{(l)} \times R}$, $n \in \{1, \ldots, N\}$, M_1 denotes the size of the common dimension for all tensors, R is a CPD rank satisfying $R \leq \min\{M_1, M_2^{(l)}, \ldots, M_N^{(l)}\}$.

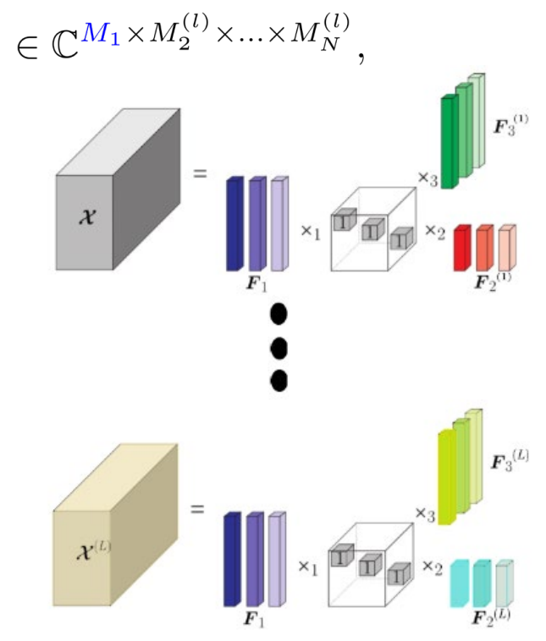

Fig. 4. Coupled CPD of L 3-D tensors,
$\mathcal{X}^{(l)} = \mathcal{I}_{3,R} \times_1 \boldsymbol{F}_1 \times_2 \boldsymbol{F}_2^{(l)} \times_3 \boldsymbol{F}_3^{(l)}, R = 3$

LaRGE, Global EigenValues (GEVs)

- ❐ Profile of the noise Global EigenValues fits an exponential law.
- ❐ LaRGE calculates linear approximation of the GEVs on a logarithmic scale, starting from the smallest noise Global EigenValue.
- ❐ The deviation from this linear regression i.e., the significant gap between noise GEVs and signal GEVs helps estimating the rank.

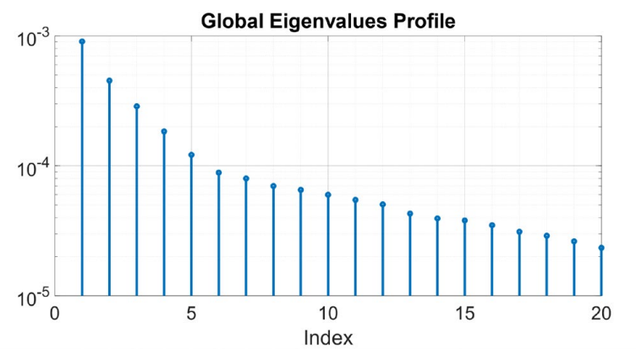

Fig. 5. Global eigenvalues profile at low SNR. Simulation parameters: $\mathcal{X} \in \mathbb{C}^{20 \times 25 \times 30}$, **rank** $= 5$, SNR $= -7\,\text{dB}$.

[7] A. Korobkov, M. K. Diugurova, J. Haueisen, and M. Haardt, "Multi-dimensional model order estimation using LineAr Regression of Global Eigenvalues (LaRGE) with applications to EEG and MEG recordings," in *Proc. 28th European Signal Processing Conference (EUSIPCO 2020), Amsterdam, Netherlands*, Jan. 2021, p. 1005–1009

Coupled-LaRGE Algorithm

```
Require: $N$-way tensors $\mathcal{T}^{(l)} \in \mathbb{C}^{M_1 \cdots \times M_n^{(l)} \times \cdots \times M_N^{(l)}}$
    $\sigma_{l,i}^{(n)} \longleftarrow$ coupled HOSVD of $\mathcal{T}^{(l)}$s (mode-1 is coupled),
    $M = \min\{M_1, \ldots, M_N^{(l)}\}\ \forall l$
    for $l = 1, \ldots, L$ do
        for $i = 1, \ldots, M$ do
            $\tilde{\lambda}_{l,i}^{[G]} = \prod_{n=1}^{N}\left(\sigma_{l,i}^{(n)}\right)^2$
        end for
    end for
    for $i = 1, \ldots, M$ do
        $\tilde{\lambda}_i^{[G]} = \sqrt[L]{\tilde{\lambda}_{1,i}^{[G]} \cdot \ldots \cdot \tilde{\lambda}_{l,i}^{[G]} \cdot \ldots \cdot \tilde{\lambda}_{L,i}^{[G]}}$
        $\lambda_i^{(G)} = \ln \tilde{\lambda}_i^{(G)}$
    end for
```

Singular values from coupled HOSVD

Global Eigenvalues (GEV's) from singular values

Coupled Global EigenValues (C-GEV's)

Coupled-LaRGE Algorithm (continued)

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$\hat{R}_{\text{LaRGE}} = 1$
for $k = 1, \ldots, M-1$ do
    linear approximation $\hat{\lambda}_{M-k}^{[G]}$
    $\delta_{M-k} = \frac{\Delta_{M-k}^{[G]}}{|\hat{\lambda}_{M-k}^{[G]}|} = \frac{\lambda_{M-k}^{[G]} - \hat{\lambda}_{M-k}^{[G]}}{|\hat{\lambda}_{M-k}^{[G]}|}$
    $\sigma_{M-k} = \sqrt{\frac{1}{k}\sum_{i=M}^{M-k}\left(\Delta_i^{[G]} - \frac{1}{k}\sum_{i=M}^{M-k}\Delta_i^{[G]}\right)^2}$
    $\text{PESDR}_k = \frac{\delta_{M-k}}{\sigma_{M-k-1}}$
    if $(\text{PESDR}_{k-1} < \rho) \wedge (\text{PESDR}_k \geq \rho)$ then
        $\hat{R}_{\text{LaRGE}} = M - k$
    end if
end for
```

Linear approximation of C-GEV's

Prediction error

Standard deviation

Prediction Error to Standard Deviation Ratio

Rank Estimation

C-LaRGE Algorithm

- ❐ C-LaRGE computes the geometric mean of the GEVs to create a set of Coupled GEVs (C-GEVs).
- ❐ When the PESDR (Prediction Error to Standard Deviation Ratio) exceeds the predefined threshold ρ for the first time, it indicates the detection of the rank.
- ❐ C-LaRGE PF (penalty function) ensures that the value of σ_{M-k} exceeds a certain threshold ε, that allows avoiding the outliers that may lead to wrong estimates.
- ❐ The C-GEVs can also be utilized for the extension of the well known MDL and AIC techniques for the coupled model order estimation.

Threshold Estimation, C-LaRGE

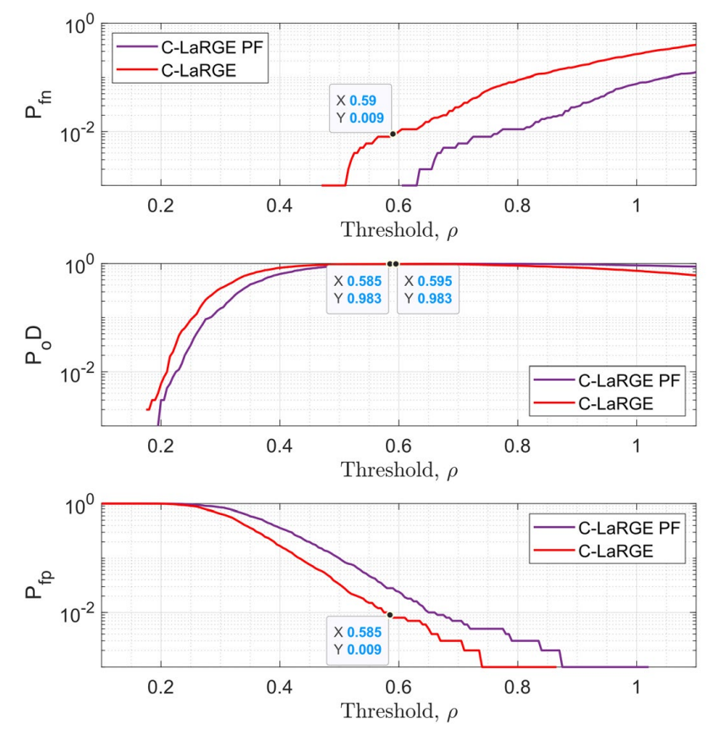

Fig. 6. Prob. of false negative $P_{\text{fn}}(\rho)$ (top), correct detection $P_{\text{cd}}(\rho)$ (middle), false positive $P_{\text{fp}}(\rho)$ (bottom).

Simulation Parameters

- $\mathcal{X}^{(1)}, \mathcal{X}^{(2)}, \mathcal{X}^{(3)}, \mathcal{X}^{(4)} \in \mathbb{C}^{103 \times 50 \times 30}$, with second factor matrix in common.
- SNR $= 10 \cdot \log_{10} \frac{\mathbb{E}\left\{\left\|\mathcal{X}_0\right\|^2\right\}}{\mathbb{E}\left\{\left\|\mathcal{N}\right\|^2\right\}} = 10\,\text{dB}$
- Correlation $= [0.2, 0.3, 0.4]$
- Correlation is added as $\boldsymbol{A} \leftarrow \boldsymbol{A} \cdot \boldsymbol{R}(\rho)$, where $\boldsymbol{R}(\rho) = (1-\rho) \cdot \boldsymbol{I}_{R \times R} + \frac{\rho}{R} \cdot \boldsymbol{1}_{R \times R}$.
- Rank (known)$= 5$
- Number of trials $= 1000$

Coupled LaRGE (C-LaRGE) performance

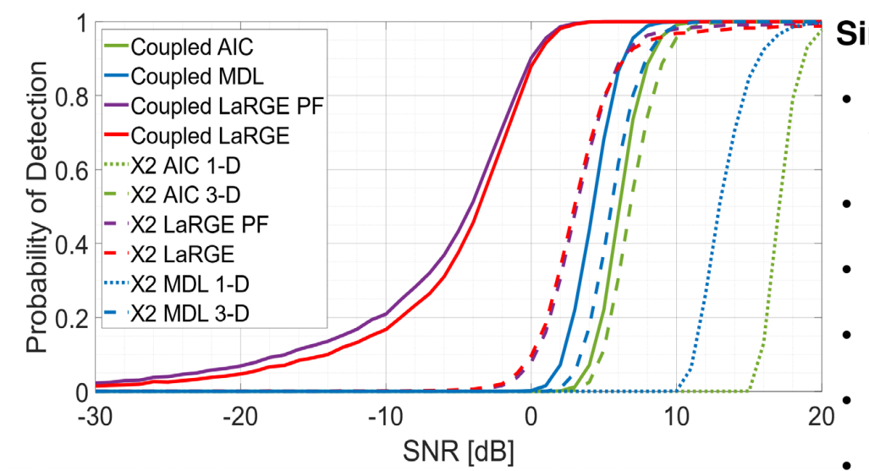

Fig.7. Probability of detection vs. SNR.

Simulation Parameters

- $\mathcal{X}^{(1)}, \mathcal{X}^{(2)}, \mathcal{X}^{(3)}, \mathcal{X}^{(4)} \in \mathbb{C}^{16 \times 16 \times 16}$, with same first factor matrix.
- SNR of $\mathcal{X}^{(1)} = 7\ \text{dB}$
- SNR of $\mathcal{X}^{(2)}, \mathcal{X}^{(3)}, \mathcal{X}^{(4)} = -30 : 20\,\text{dB}$
- Correlation $= [0.2, 0.3, 0.4]$
- Rank (known)$= 7$
- LaRGE threshold $= 0.57$
- C-LaRGE threshold $= 0.59$
- LaRGE PF/C-LaRGE PF threshold $= 0.7$
- Number of trials $= 10\,000$

Application on EEG, MEG, and Gradiometer data

- ❐ **Photic Stimulation (IPS):** simultaneously recorded 128 EEG electrodes and 306 MEG channels (MAG + GRAD-1 + GRAD-2).
- ❐ Measurement of individual α-rhythm followed by individual α-frequencies f_α calculation.
- ❐ **Closed eyes stimulation:** 30 trains of flickering light stimulation (40 periods of on/off light per train) different frequencies.

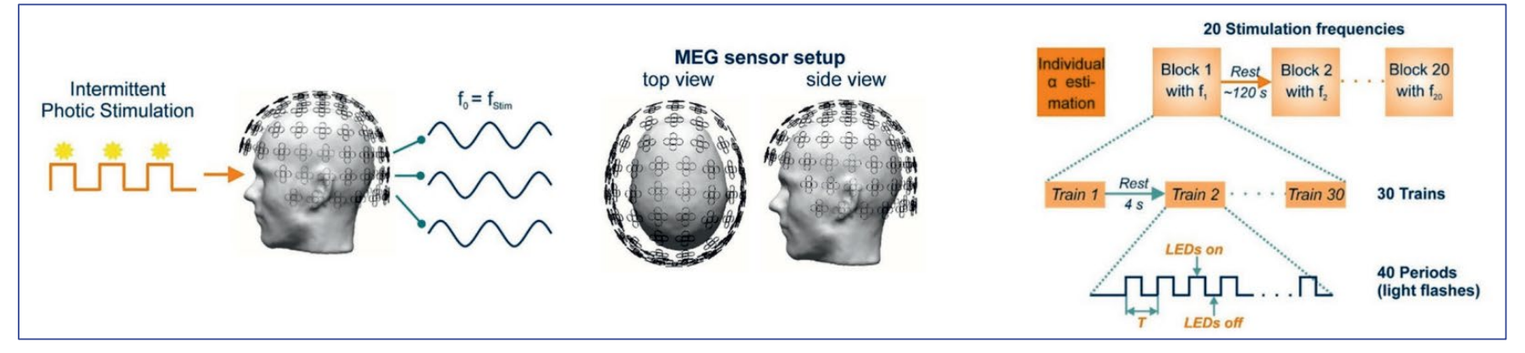

Fig. 8. Experimental Setup and data recording

[8] Salchow, Christina, Daniel Strohmeier, Sascha Klee, Dunja Jannek, Karin Schiecke, Herbert Witte, Arye Nehorai, and Jens Haueisen. "Rod driven frequency entrainment and resonance phenomena." *Frontiers in human neuroscience*, 2016.

Application on EEG, MEG, and Gradiometer data

- ❐ **Preprocessing:** bandpass filtering from 3 to 40 Hz followed by frequency domain conversion with DFT.
- ❐ **Tensor construction:** – EEG, MEG-MAG, MEG-GRAD-1, MEG-GRAD-2 – tensors with dimensions Channels × Frequency × Trains.
- ❐ **Signal Processing Steps:** Coupled Rank Estimation with C-LaRGE followed by C-SECSI for coupled CPD.

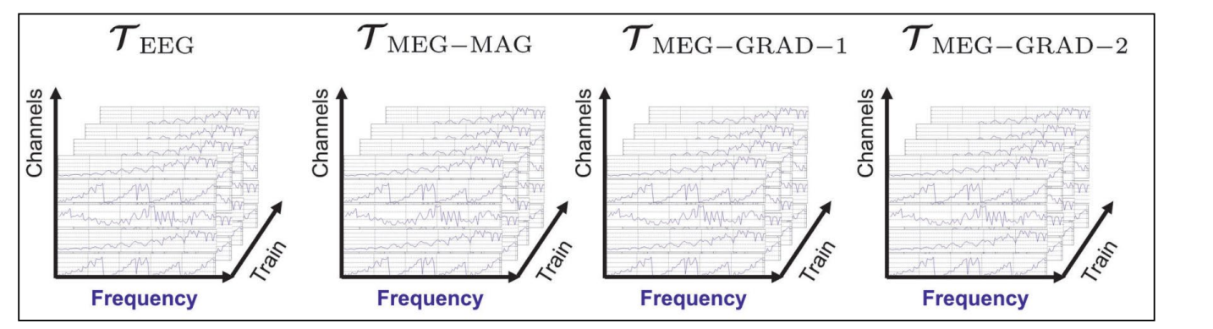

Fig. 9. Coupled tensors with dimensions Channels × Frequency × Trains.

[9] Alla Manina, Mikus Grasis, Liana Khamidullina, Alexey Korobkov, Jens Haueisen, and Martin Haardt. Coupled CP Decomposition of EEG and MEG Magnetometer and Gradiometer Measurements via the Coupled SECSI Framework. In *Proc. 55th Asilomar Conference on Signals, Systems, and Computers*, 2021.

Biomedical Application

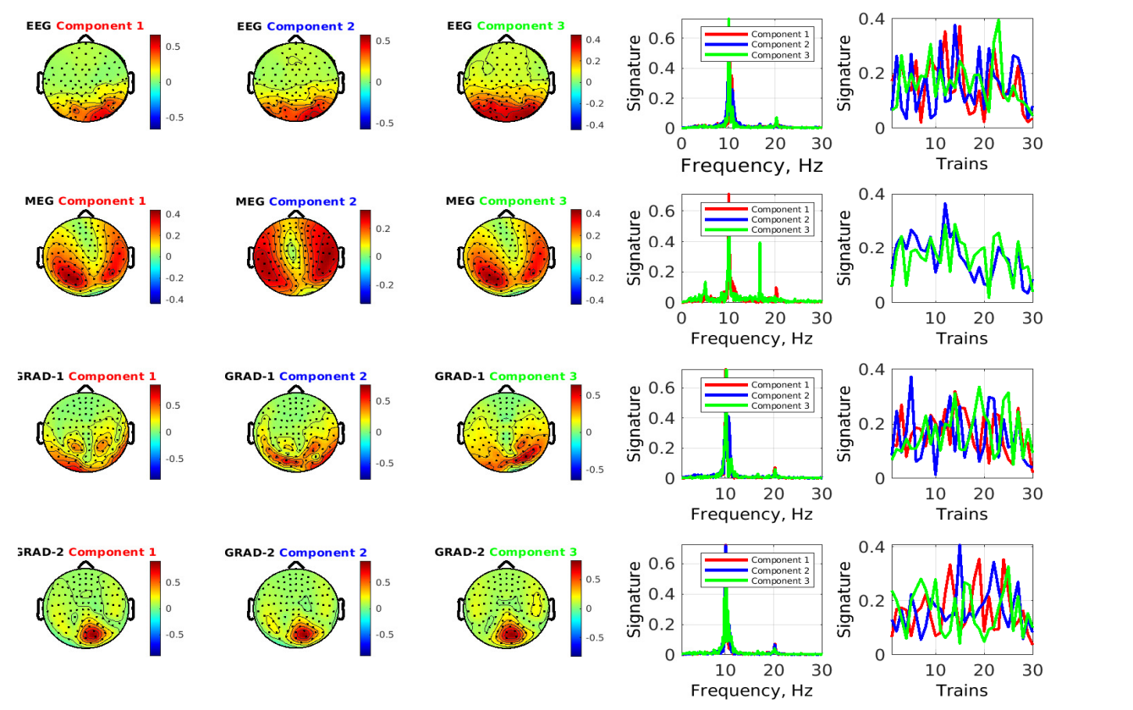

Fig. 10. The estimated factor matrices for volunteer No. 5 ($f_\alpha = 10.7$ Hz) at stimulation frequency 0.95 f_α and estimated rank $\hat{R} = 3$. The results of C-SECSI show that the MEG frequency mode estimate is not coupled.

Conclusions

- ❐ Extension of the LaRGE algorithm for multiple noise-corrupted coupled tensors has been proposed.
- ❐ The proposed coupled global eigenvalues are used to estimate the coupled rank with the C-LaRGE algorithm.
- ❐ Monte Carlo simulations have provided a suitable threshold to be used for C-LaRGE.
- ❐ C-LaRGE method outperforms its uncoupled counterpart as well as other classical methods, especially in low SNR scenarios.
- ❐ Four coupled tensors from a biomedical experiment have been used, and the dominant components have been extracted successfully.

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