Cramer-Rao bound for the Time-Varying Poisson

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Problems & results:

Periodic Time-varying Poisson new results:

1. the asymptotic Cramer-Rao bound,
2. a maximum likelihood parameter estimation.

Preliminaries

Time-varying Poisson (tvP):

•A sequence of random event times with independent increment, observed in period $[0, T]$.

•*Event times (ETs)* $T_1^n = \{T_1, \dots, T_n\}$.

•*Counting process* $N_t = N(0, t] = \#$ of events up to and including time t.

•*(deterministic) tvP Intensity function:*

$$\lambda(t) = \lim_{\delta \downarrow 0} \frac{1}{\delta} \Pr[N_{t+\delta} - N_t = 1],$$

•*Log-likelihood* [1]:

$$\mathcal{L}_T = \int_0^T \ln \lambda(t)\, dN_t - \int_0^T \lambda(u) du\,.$$

Cramer-Rao bound (CRB) for tvP:

•θ: parameter vector of dimension d.

Theorem 1 tvP CRB [1]. For an unbiased estimator $\hat{\theta}$,

$$V_T = \mathrm{var}[\hat{\theta}] \geq I_T^{-1}$$

where I_T is the Fisher Information Matrix (FIM)

$$I_T = \int_0^T \frac{1}{\lambda(t)} \frac{\partial \lambda(t)}{\partial \theta} \frac{\partial \lambda(t)}{\partial \theta^\top} dt\,,$$

and $A \geq B$ means $A - B$ is positive semi-definite.

Model assumption:

•We assume periodic tvP intensity

$$\lambda(t) = b + a\cos(\omega t)\,,$$

•So $\theta = [b, a, \omega]^\top$.

Reference:

[1] D. Snyder and M. Miller, Random Point Processes in Time and Space, Springer-Verlag, New York, 1991.

Asymptotic CRB

Averages of periodic functions:

•Suppose $\gamma(t) \geq 0$ has period $\frac{2\pi}{\omega}$.

•Then $\kappa(\tau) = \gamma(\tau/\omega)$ has period 2π

Result I Consider the integral

$$\mathcal{J}_T = \frac{1}{T^{p+1}} \int_0^T t^p \gamma(t)[1 - \alpha f_{q,r}(\omega t)] dt\,,$$

where $f_{q,r}(\tau) = \sin^q(\tau)\cos^r(\tau)$ and $p, q, r \geq 0$ are integers and $|\alpha| < 1$. Then,

$$\mathcal{J}_T \to \frac{1}{p+1} L_{q,r}(\alpha), \text{ as } T \to \infty$$

$$L_{q,r}(\alpha) = \int_0^{2\pi} \kappa(\xi)[1 - \alpha f_{q,r}(\xi)] \frac{d\xi}{2\pi}$$

Asymptotic CRB simplification:

• Introduce matrix $D_T = \mathrm{diag}(T^{\frac{1}{2}}, T^{\frac{1}{2}}, T^{\frac{3}{2}})$.

Result II Asymptotic CRB.

$$D_T V_T D_T \geq D_T I_T^{-1} D_T$$
$$= (D_T^{-1} I_T D_T^{-1})^{-1} \to V_*$$

$$V_* = \begin{bmatrix} bG & 0 \\ 0 & \frac{3R}{a\rho} \end{bmatrix} \;\&\; \rho = \frac{a}{b}$$

$$G = \begin{bmatrix} 1 & \rho \\ \rho & R \end{bmatrix} \;\&\; R = 1 + \sqrt{1 - \rho^2}$$

Remarks.

$\mathrm{var}[\hat{a}], \mathrm{var}[\hat{b}]$ decay at rate T^{-1} but $\mathrm{var}[\hat{\omega}]$ decays at T^{-3}: super efficiency of the frequency estimator.

MLE algorithm

MLE algorithm via cyclic ascent:

- First MLE for periodic tvP, surprisingly.
- Reparametrization: 2 positive components

$$\lambda(t) = b + a\cos(\omega t)$$
$$= c + a[1 + \cos(\omega t)]$$

- MLE via cyclic ascent:
  - Stage I: update $\hat{c}$ and $\hat{a}$ by EM
  - Stage II: update $\hat{\omega}$ by Newton-Raphson.
- Formulae to be found in the paper.

Simulations

Scale-free(SF) parameters and measure:

- *SF parameters:* $n_b = bT, m_T = \frac{\omega T}{2\pi}, \rho = \frac{a}{b}$
- *SF precision measure:* $\pi(\hat{\theta}) = \frac{\theta}{se(\hat{\theta})}$, where $se(\hat{\theta})$ is the standard error of the estimator.
- SF precision expressed in SF parameters:

$$\pi(\hat{b}) = \sqrt{n_b}, \qquad \pi(\hat{a}) = \sqrt{n_b}\sqrt{\gamma}, \qquad \pi(\hat{\omega}) = \frac{m_T\sqrt{n_b}\sqrt{\gamma}}{2\pi\sqrt{3}}$$

where $\sqrt{\gamma} = \frac{\rho}{\sqrt{R}}$.

- Interpretations:
  1. Precision only increases linearly with $\sqrt{n_b}$;
  2. But $\hat{\omega}$ precision also rises linearly with m_T: greater precision;
  3. If $\rho \approx 0$, precision for $\hat{a}$ and $\hat{\omega}$ very low.

Simulation setup:

- Parameters to be chosen: b, a, ω, T.
- One degree of freedom: fix $\omega = 3$.
- We vary SF parameters on a 8×8×8 grid:

$n_b = 15^2 : 30^2, \qquad m_T = 0.5:2, \qquad \rho = 0.3:0.6$

- Simulate 1,000 repeats on each grid point to estimate the precision: $\hat{\pi}(\hat{\theta}) = mean(\hat{\theta})/\widehat{se}(\hat{\theta})$

Result: Heatmaps of $\frac{\hat{\pi}(\hat{\theta})}{\pi(\hat{\theta})}$. Expect $\frac{\hat{\pi}(\hat{\theta})}{\pi(\hat{\theta})} \to 1$