Four Decades of Array Signal Processing Research: An Optimization Relaxation Technique Perspective

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Prof. Alex B. Gershman (1962-2011)

Great scientist, teacher and friend.

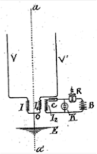
Direction-of-Arrival (DoA) Estimation

- Objective: Determine directions of multiple superimposed signals in the presence of noise from signals at sensor arrays.
- Closely related to fundamental problems: harmonic retrieval, frequency estimation, and time-delay estimation.
- Numerous classical and recent applications:
  - Radar, sonar (source localization, military, automotive).
  - Communications (directed transmission, satellite communication).
  - Radio Astronomy (high resolution imaging).
  - Medical Imaging (ultrasound, tomography).
  - Geophysical Exploration (seismic, oil exploration).
  - Biomedical (hearing aids, heart rate monitoring).
- More recent applications:
  - Drone localization at airports and public buildings.
  - Parametric channel estimation and user localization in Massive MIMO.

Introduction
Motivation

Direction-of-Arrival (DoA) Estimation

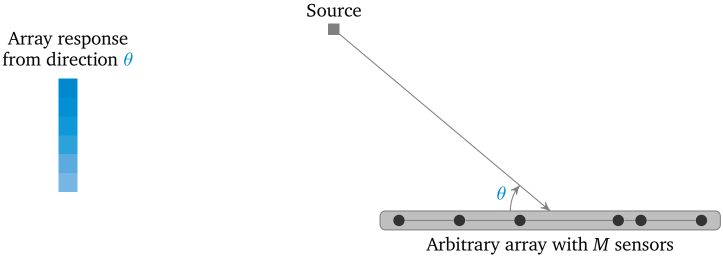
- A mature topic with long history of development.
  - Patent by Stone Stone in 1902 for RF-based direction finding using a two element array with less than half wavelength [Stone'1902], [Stone'1906-2].



  - Later improved upon by De Forest [de Forest'1904], Marconi [Marconi'1906], Bellini and Tosi [Bellini'1909], [Bellini'1910], and Adcock [Adcock'1919].
  - See [Schantz'11] for an overview on the origin of RF-based direction finding
  - Trend toward digital processing in the 60s by [Capon'66], [Capon'67]
  - Development of "super resolution" algorithms since the late 70s, including [Schmidt'79], [Schmidt'81],[Bienvenu'79],[Barabell'83],[Böhme'84],[Ziskind'88],[Stoica'89],[Böhme'86],[Viberg'91
- In this tutorial, we revisit several aspects in the last four decades of "super-resolution" DoA estimation from a unified perspective.

Introduction

DoA Estimation Setup



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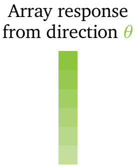
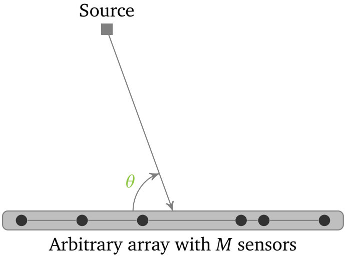
DoA Estimation Setup





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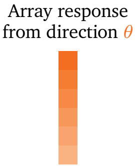
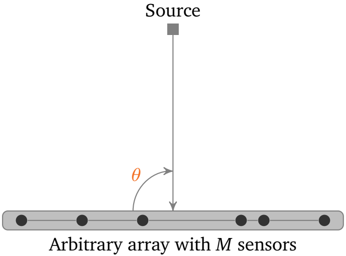
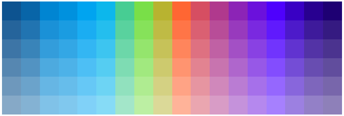
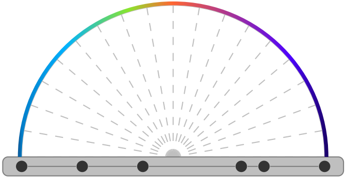




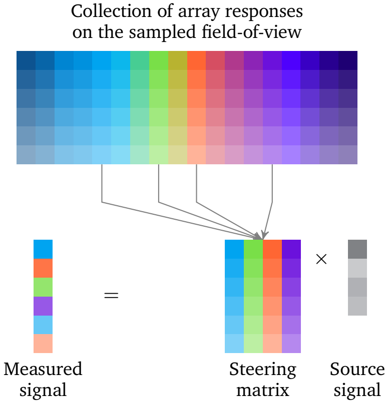
Introduction
DoA Estimation Setup

Collection of array responses
on the sampled field-of-view

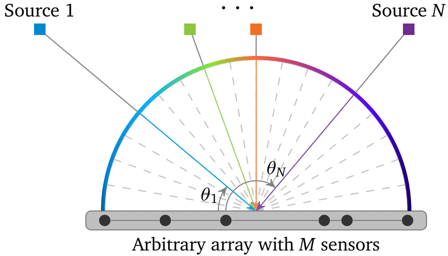




Arbitrary array with M sensors

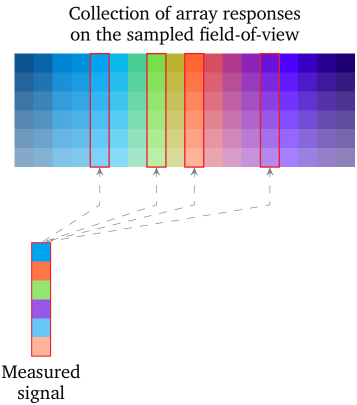


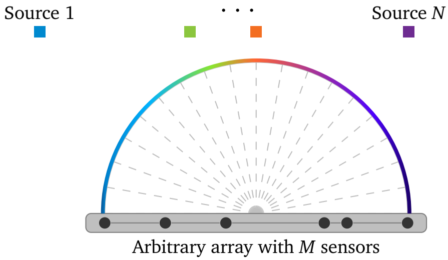
Collection of array responses
on the sampled field-of-view
=
×
Measured
signal
Steering
matrix
Source
signal



Source 1
. . .
Source N
θ1
θN
Arbitrary array with M sensors

Introduction
DoA Estimation Problem



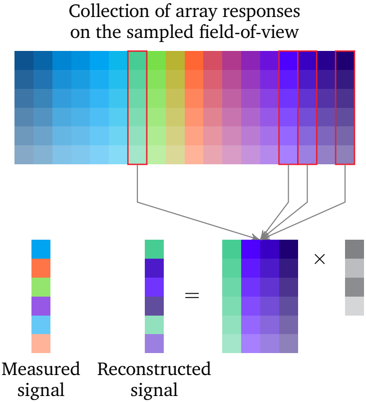


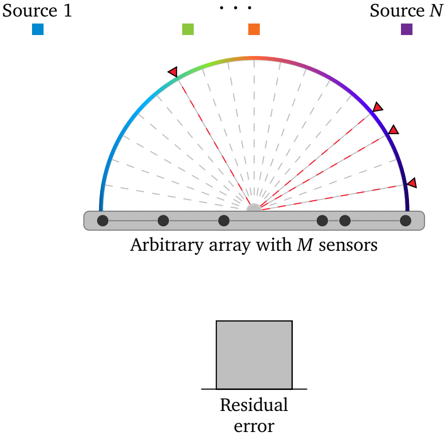
Multiple Classes of DoA Estimators:

- Maximum Likelihood Estimators,
- Spectral-based methods,
- Search-free methods,
- . . .

Goal of this Tutorial: Insight into Conventional and Modern DoA Estimators from the Perspective of Optimization Techniques

Introduction
Maximum Likelihood Estimators





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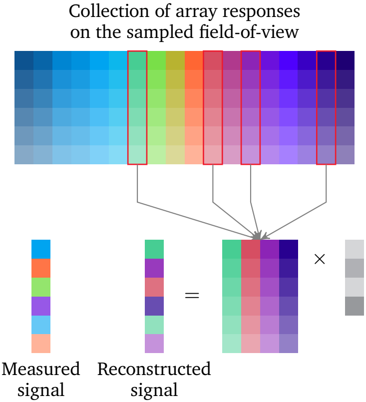

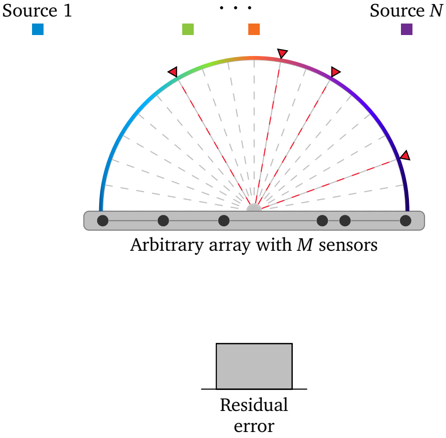

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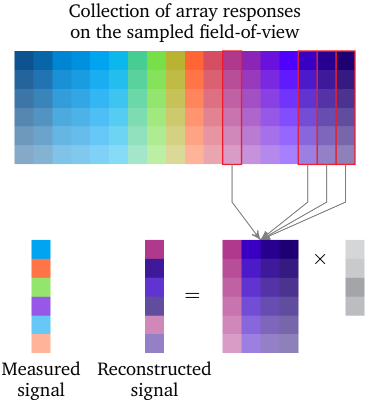
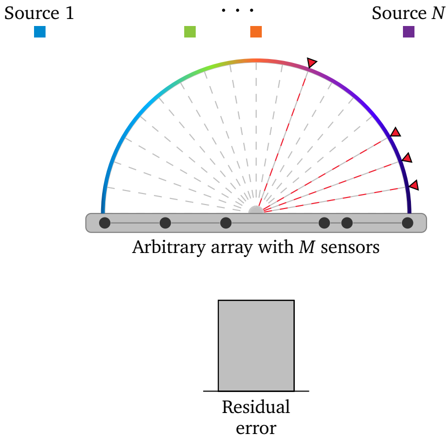
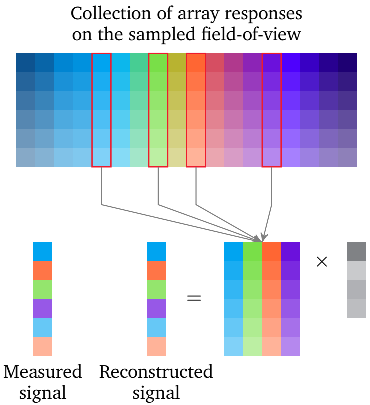
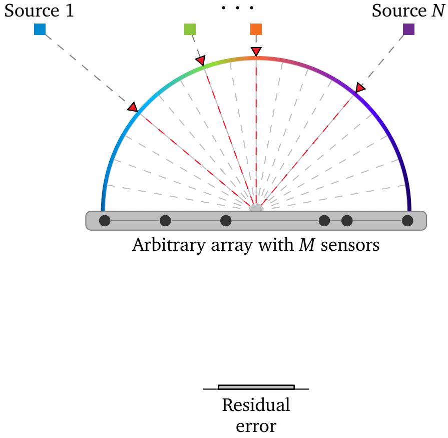




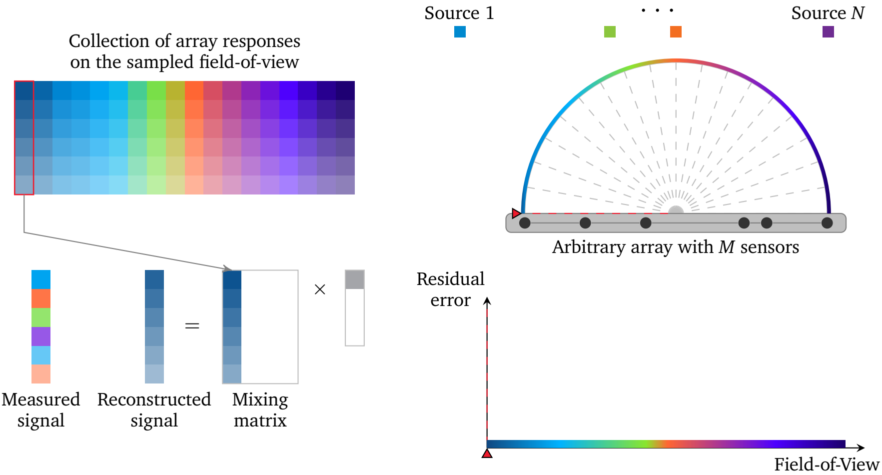


Collection of array responses
on the sampled field-of-view
=
×
Measured
signal
Reconstructed
signal



Source 1
. . .
Source N
Arbitrary array with M sensors
Residual
error

Introduction
Conventional Spectral-based Methods



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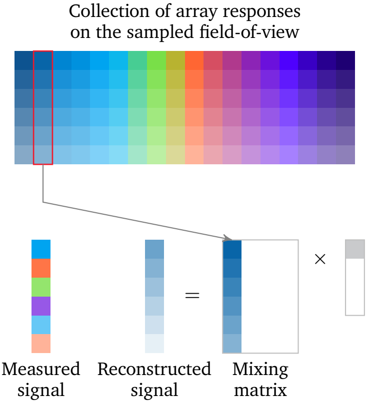

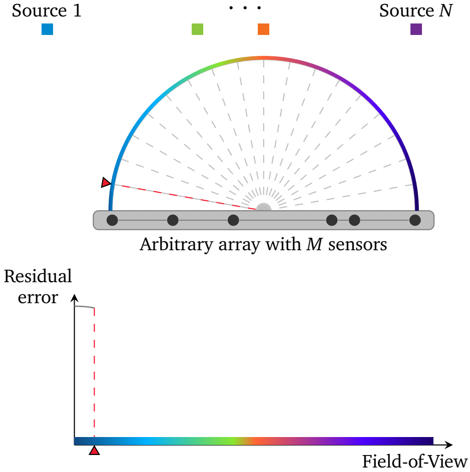

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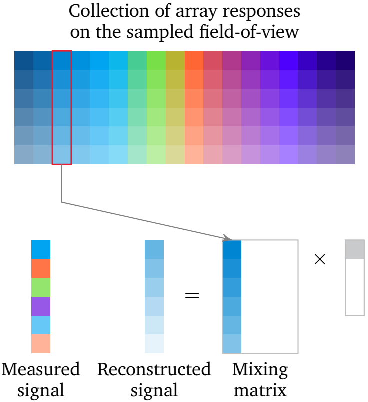

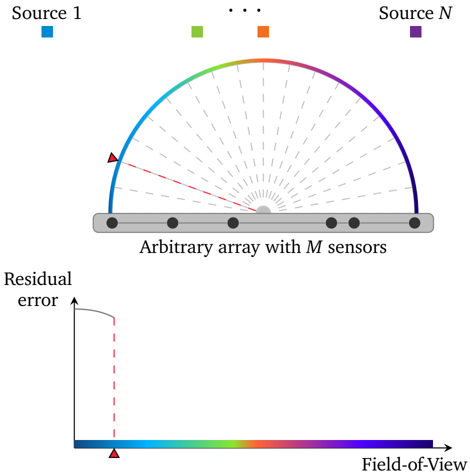

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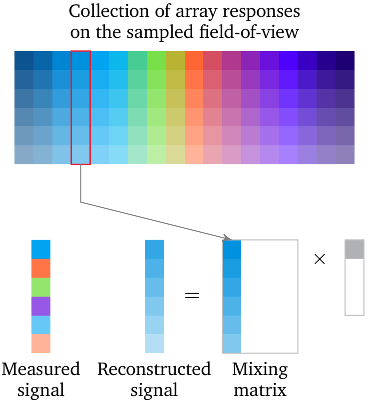

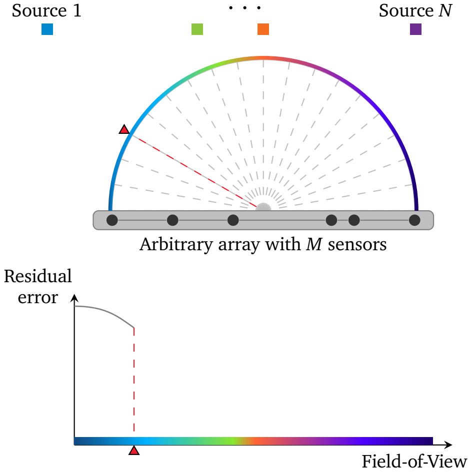

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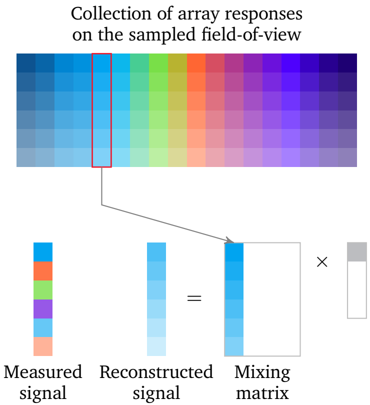

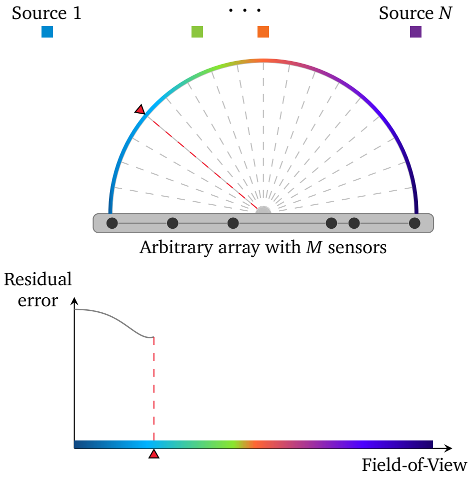

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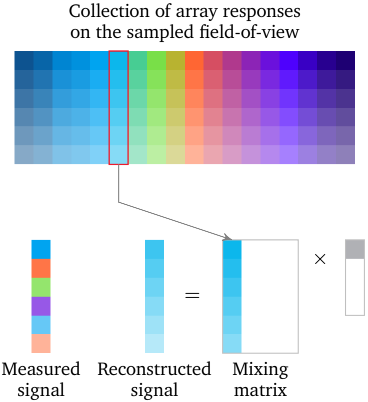

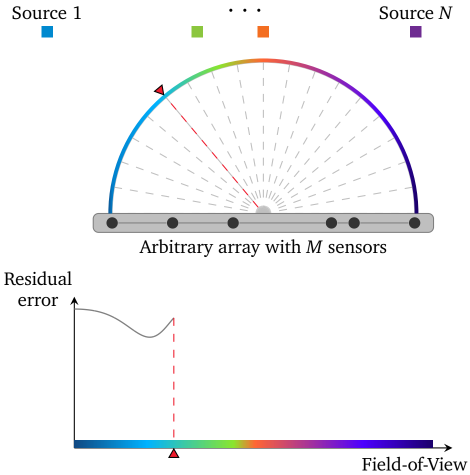

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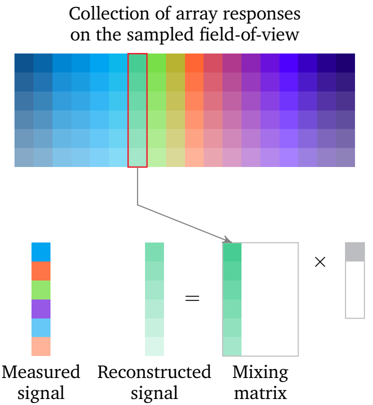

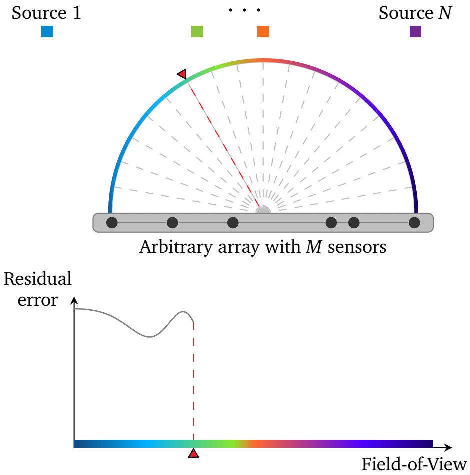

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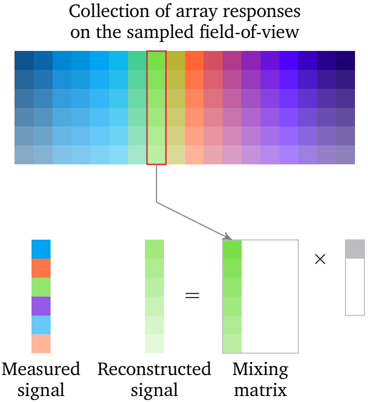

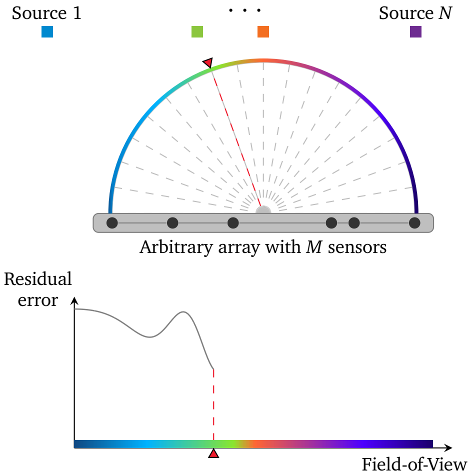

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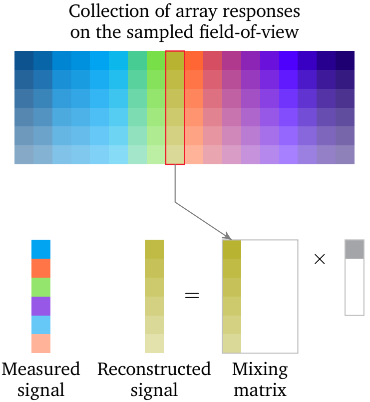

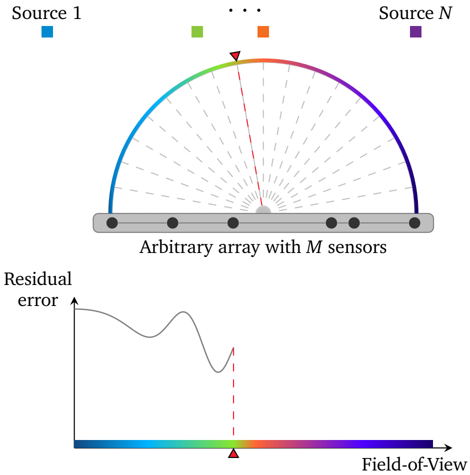

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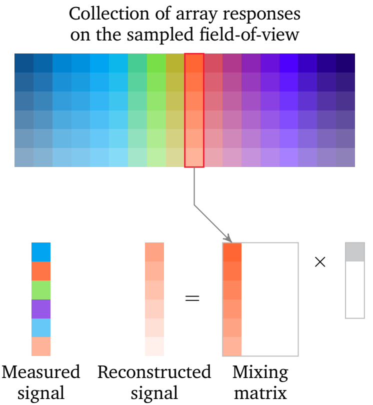

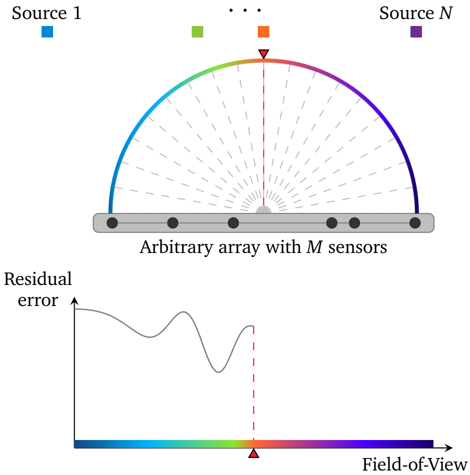

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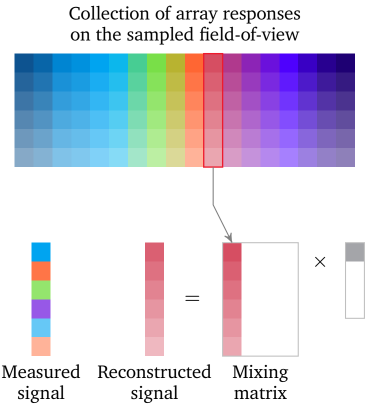

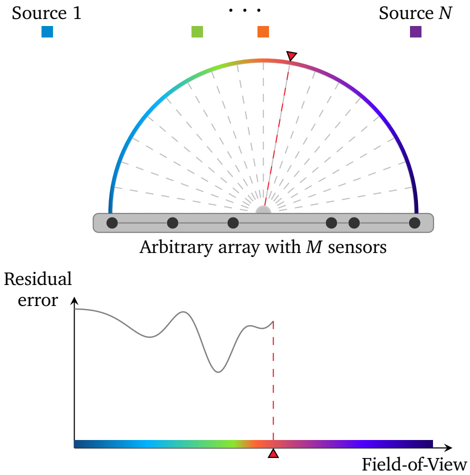

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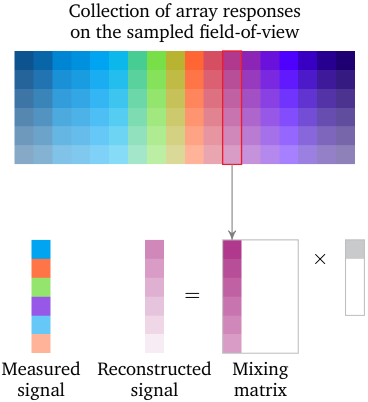

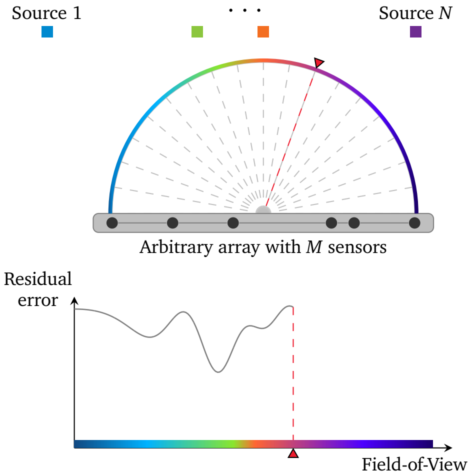

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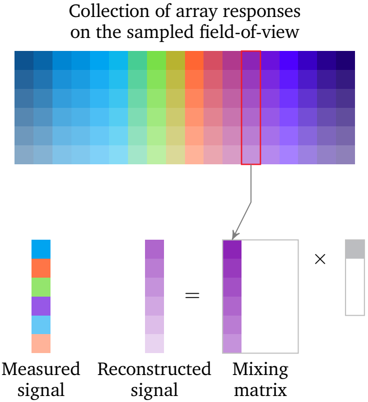

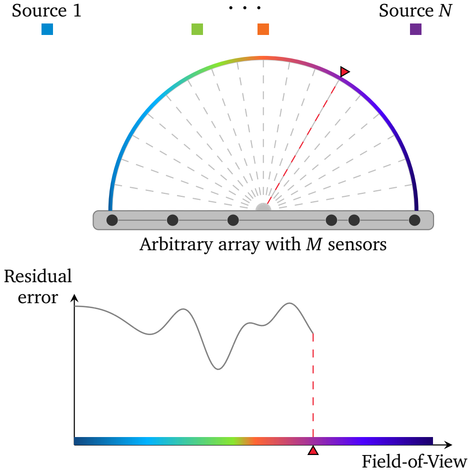

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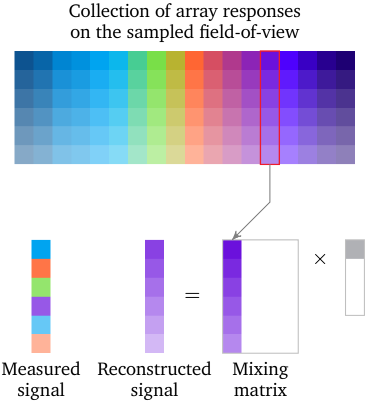

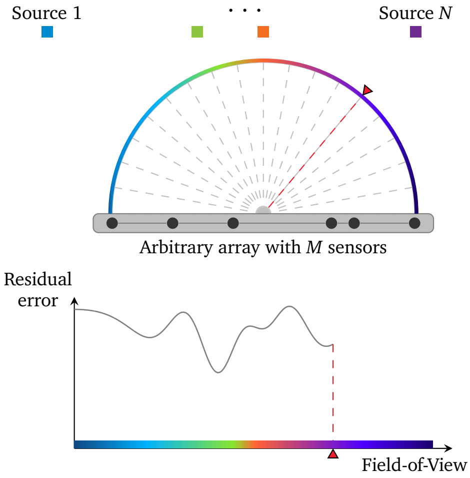

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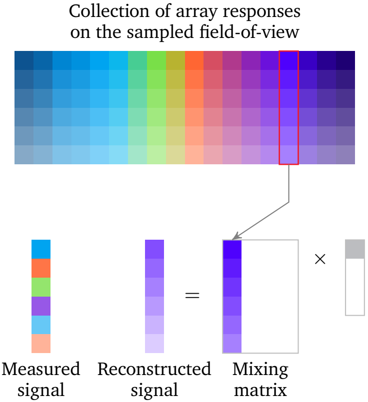

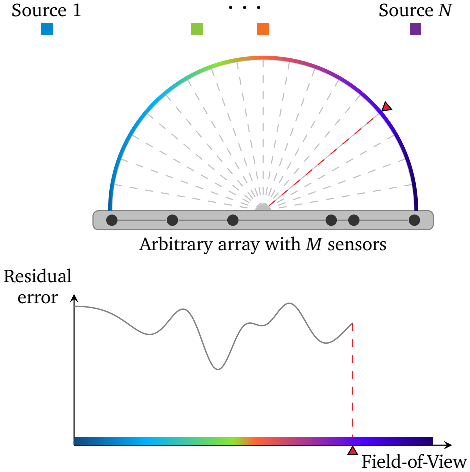

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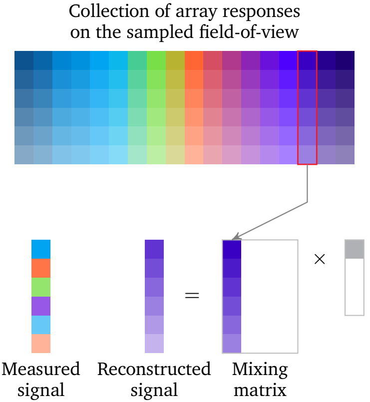

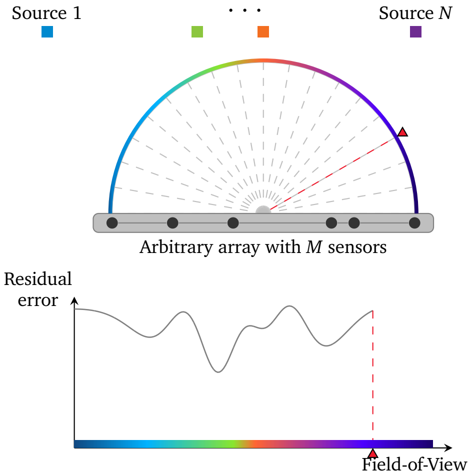

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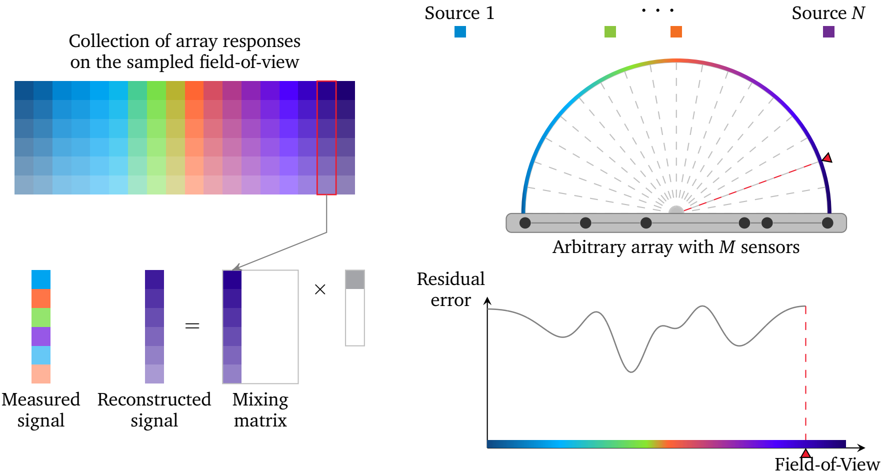

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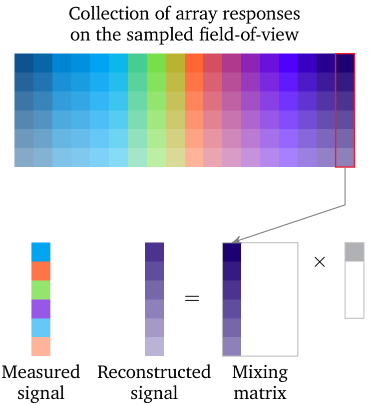

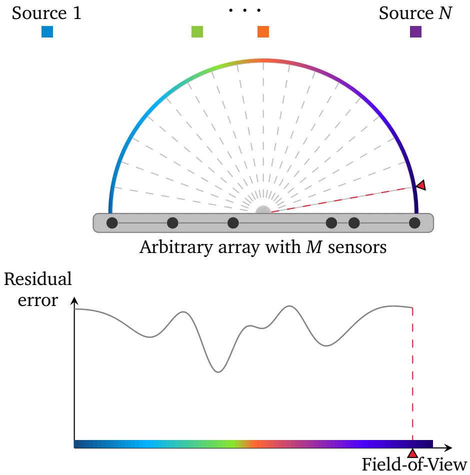

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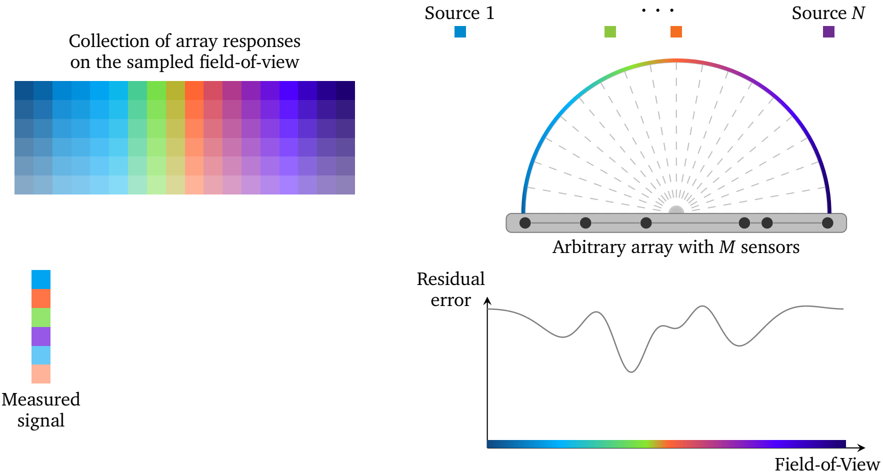

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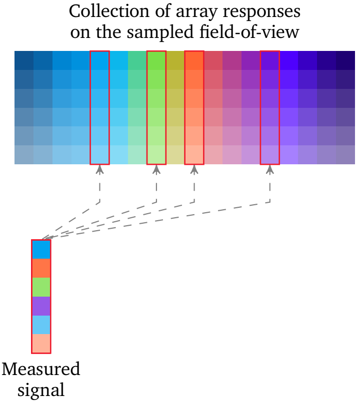

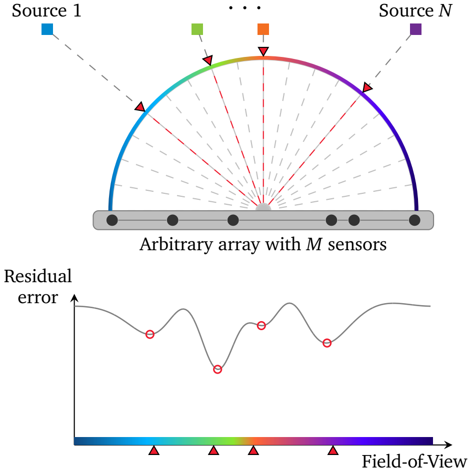

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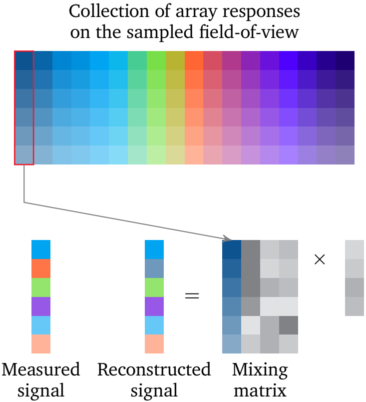
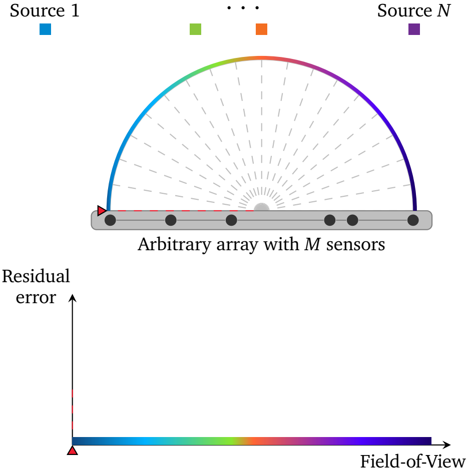
Partial Relaxation Estimators





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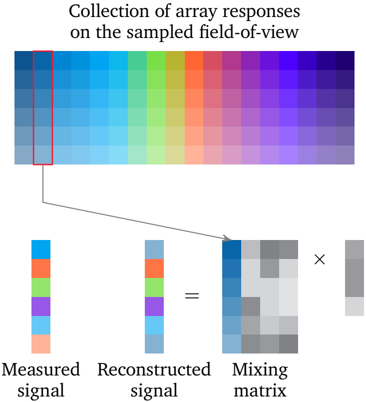

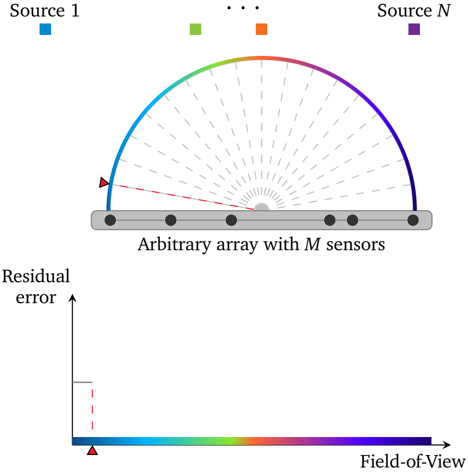

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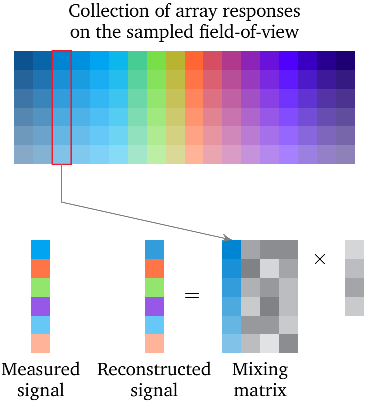

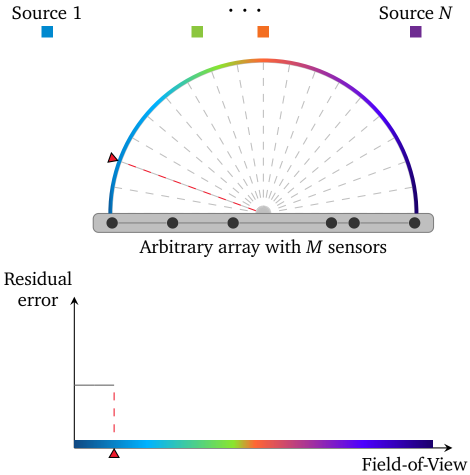

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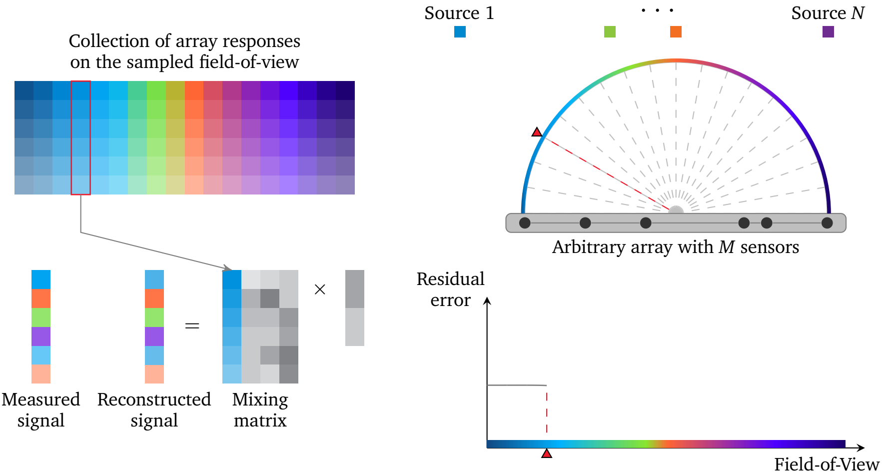
Partial Relaxation Estimators



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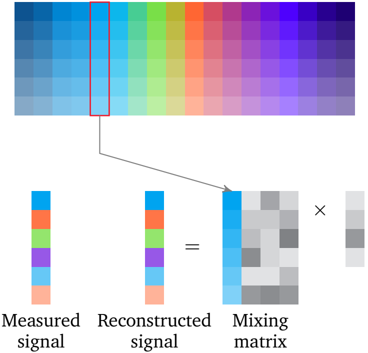
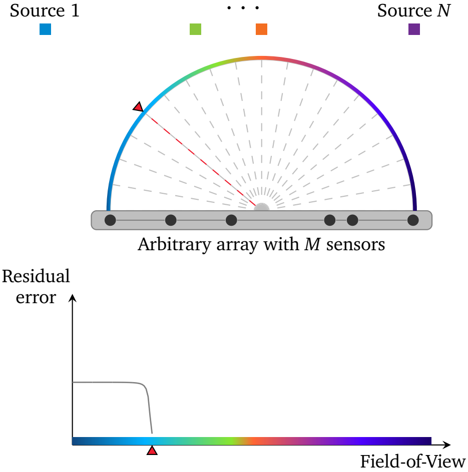
Partial Relaxation Estimators

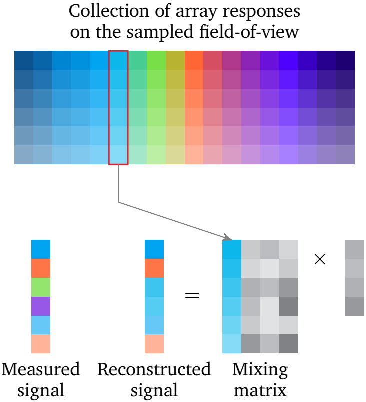
Collection of array responses on the sampled field-of-view

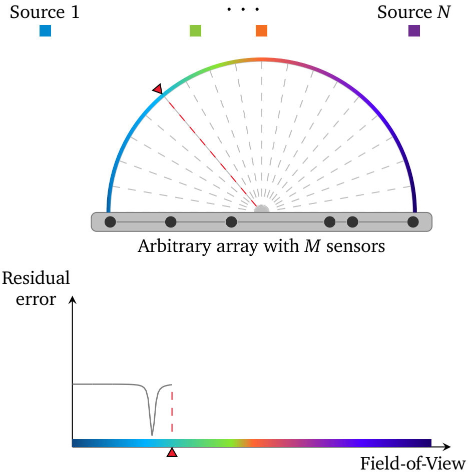




Introduction

Partial Relaxation Estimators





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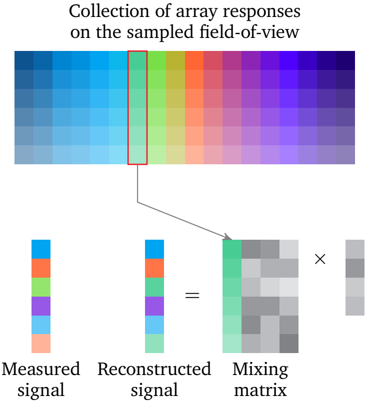

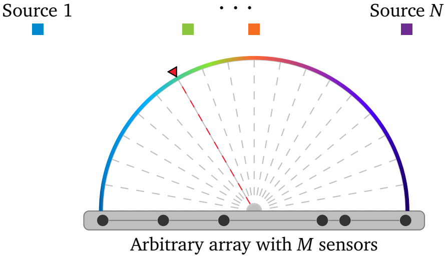

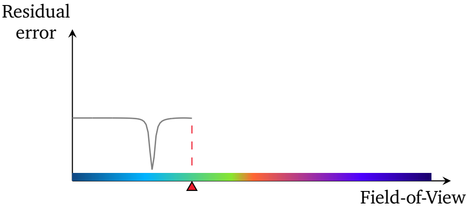

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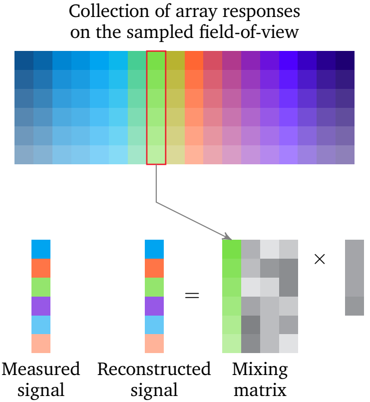

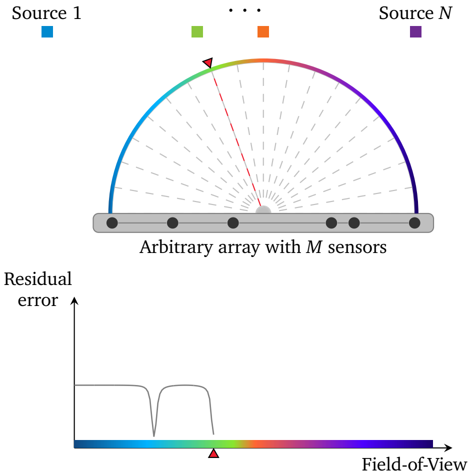

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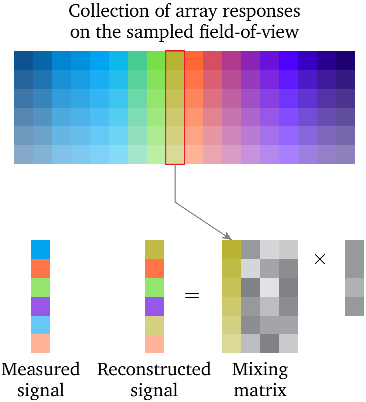

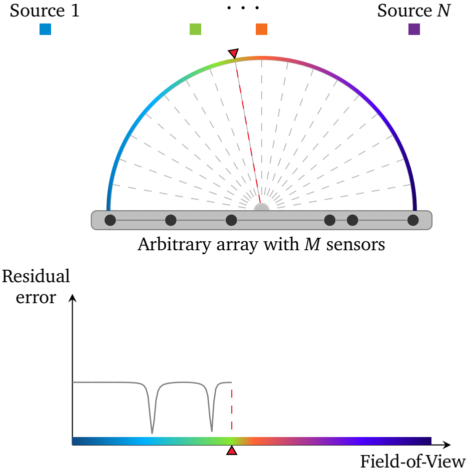

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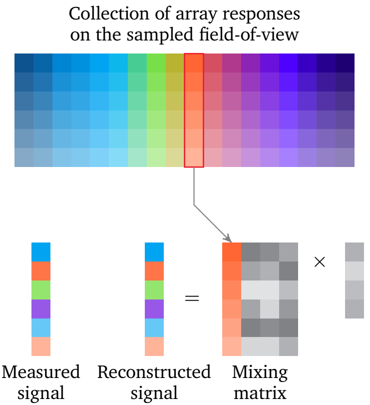

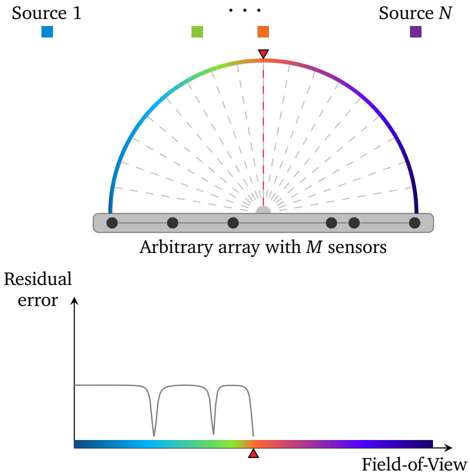

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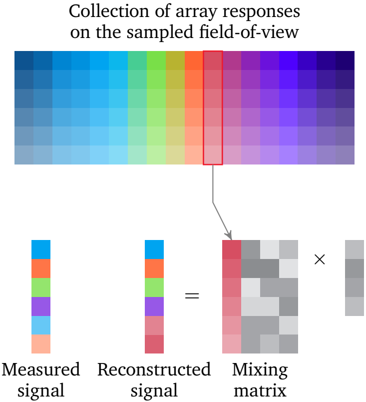

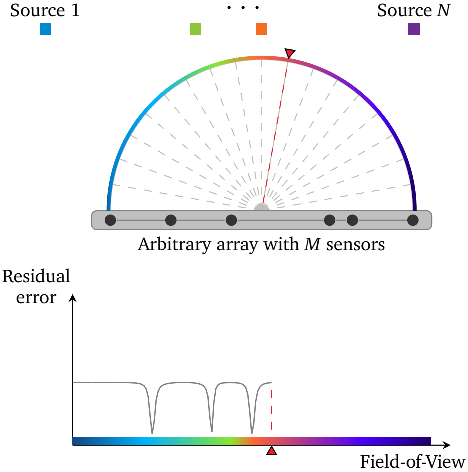

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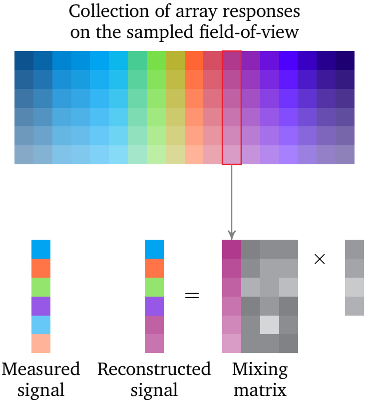

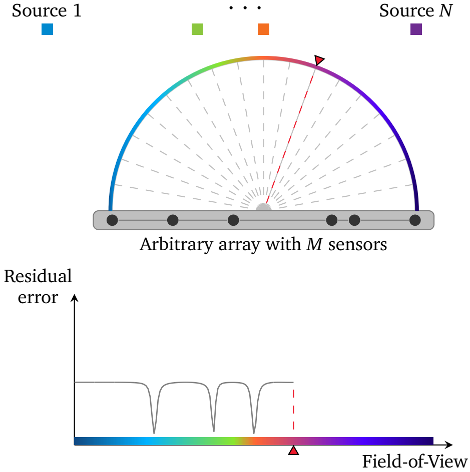

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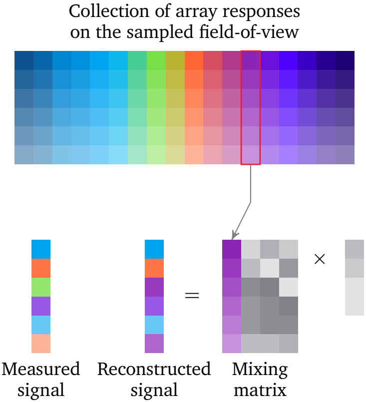

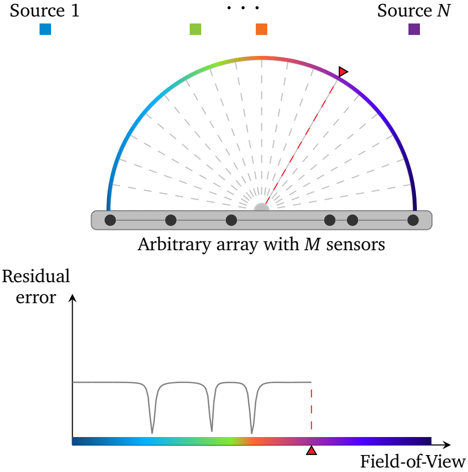

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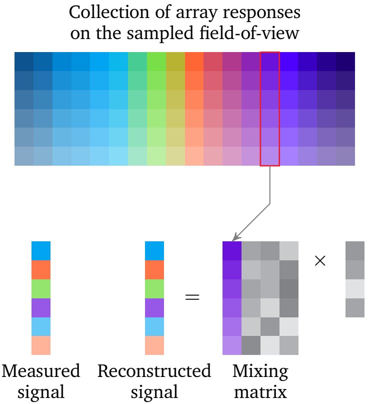

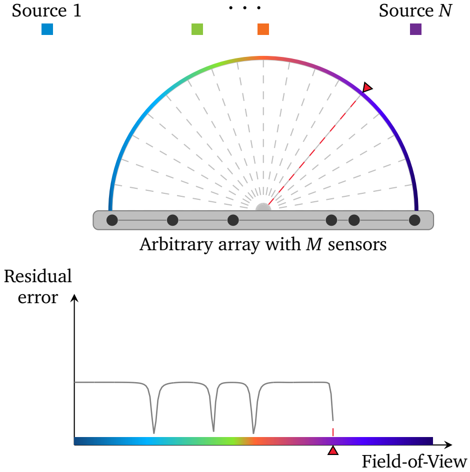

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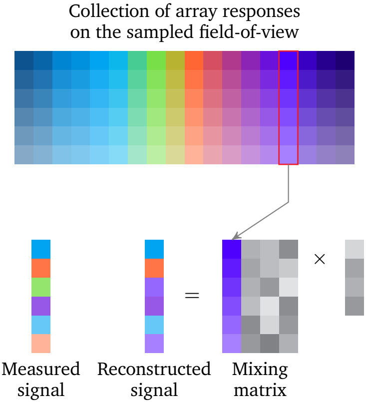

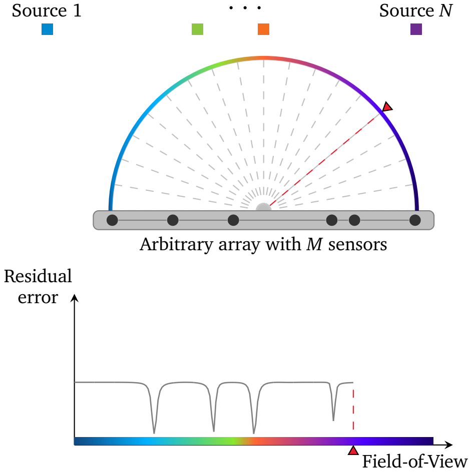

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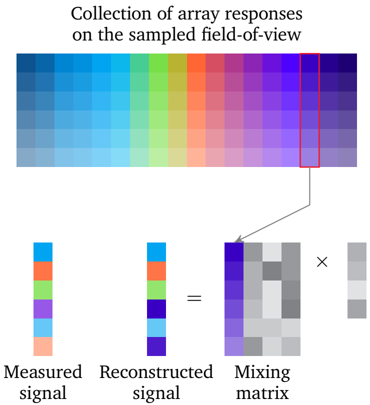

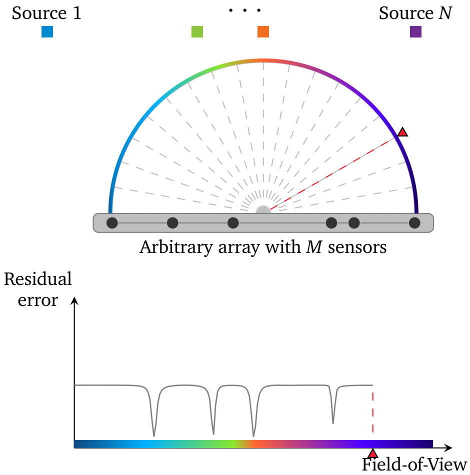

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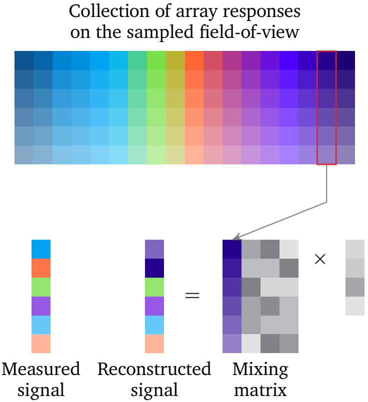

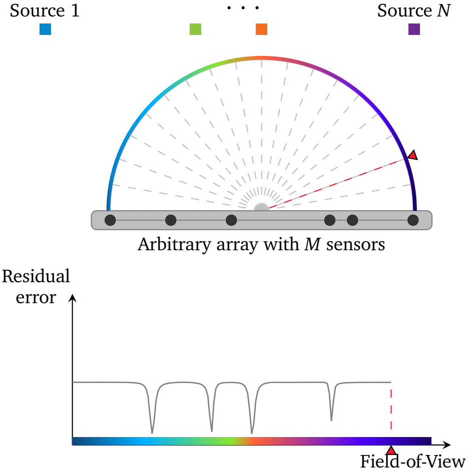

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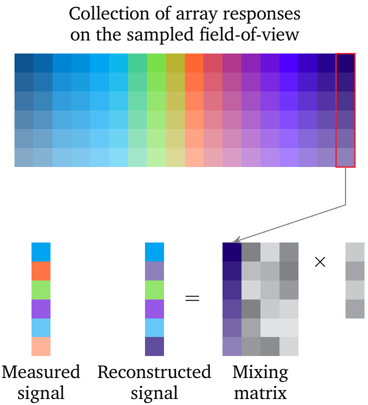
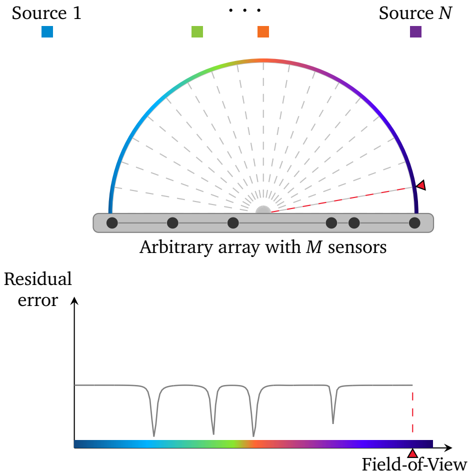
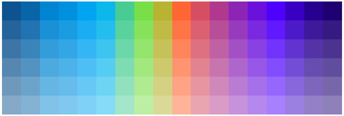


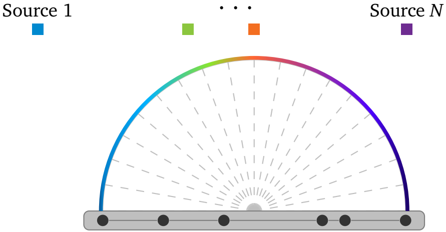


Introduction
Partial Relaxation Estimators

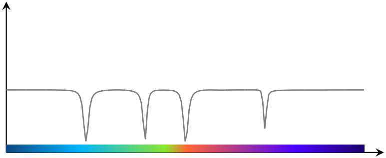
Collection of array responses on the sampled field-of-view



Measured signal



Arbitrary array with M sensors



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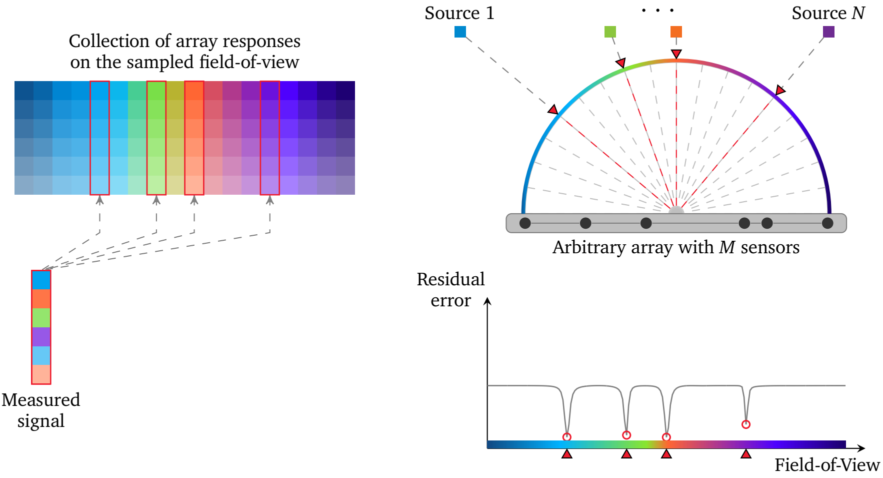

Motivation
Tutorial Overview

- The tutorial addresses both,
  - experienced researchers in sensor array processing, as well as,
  - newcomers to the field.
- We approach classical and novel DoA estimation methods from a modern optimization (problem approximation/relaxation) perspective.
- We highlight, how problem approximation and relaxation have always played an important role in developing efficient algorithms:
  - sometimes explicitly in the design ...
  - ... often implicitly, as the consequence of proposed (ad-hoc) algorithms.
- We show novel derivations for existing algorithms that explicitly highlight the use of relaxation of prior knowledge ...
- ... and introduce a framework for designing novel algorithms under partial relaxation.

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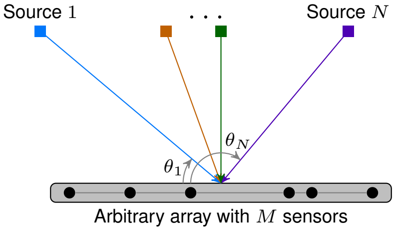
Conventional Signal Model
Assumptions and Signal Model

- Sensor array composed of M sensors.
- N sources in the far-field of the array. $\left(\text{distance} \gg \frac{2\times(\text{diameter of array})^2}{\text{wavelength}}\right)$
- N plane wave narrow-band signals impinge on array.
- We assume that the number of sensors M exceeds the number of source signals N, hence $M > N$.



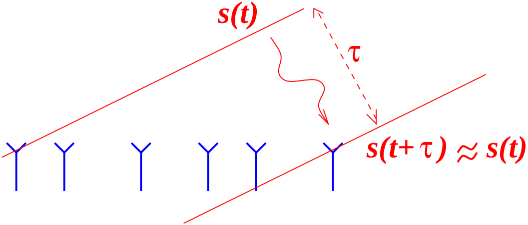
Conventional Signal Model
Assumptions and Signal Model

Narrowband condition:

- The relative bandwidth of the signals is small.

$$\text{relative bandwidth} = \frac{\text{signal bandwidth}}{\text{carrier frequency}} \ll \frac{1}{\pi M}$$

- The maximal traveling time $\tau_{\max}$ across the array is substantially smaller than the effective correlation time of signal waveforms.



Conventional Signal Model
Assumptions and Signal Model

Array measurement (snapshot) at time instant t

$$\boldsymbol{x}(t) = \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{s}(t) + \boldsymbol{n}(t)$$

- $\boldsymbol{\theta} = [\theta_1, \ldots, \theta_N]^\mathsf{T}$: DOAs of N source signals.
- W.l.o.g. we consider only azimuth angle estimation $\theta \in \Theta = [0, 180^\circ)$.
- $\boldsymbol{A}(\boldsymbol{\theta}) = [\boldsymbol{a}(\theta_1), ..., \boldsymbol{a}(\theta_N)] \in \mathbb{C}^{M \times N}$: Steering matrix.
- $\boldsymbol{a}(\theta)$: Steering vector from the direction θ.
  - Dependent on the geometry of the sensor array and the direction θ.
  - Example: Uniform Linear Array (ULA) with baseline d:

$$\boldsymbol{a}(\theta) = [1, e^{-\mathrm{j}\frac{2\pi}{\lambda} d \cos(\theta)}, \ldots, e^{-\mathrm{j}\frac{2\pi}{\lambda}(M-1) d \cos(\theta)}]^\mathsf{T}.$$

Array manifold

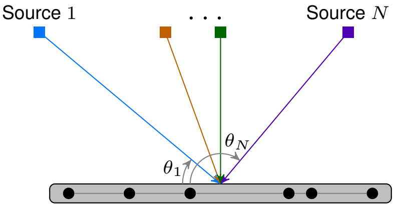
$$\mathcal{A}_N = \left\{\boldsymbol{A} \in \mathbb{C}^{M \times N} \,|\, \boldsymbol{A} = [\boldsymbol{a}(\vartheta_1), \ldots, \boldsymbol{a}(\vartheta_N)] \text{ with } 0 \leq \vartheta_1 < \ldots < \vartheta_N < 180^\circ\right\}$$

Conventional Signal Model
Assumptions and Signal Model

Array measurement (snapshot) at time instant t

$$\boldsymbol{x}(t) = \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{s}(t) + \boldsymbol{n}(t)$$

- $\boldsymbol{x}(t) = [x_1(t), \ldots, x_M(t)]^{\mathsf{T}} \in \mathbb{C}^{M\times 1}$: Receive signal vector of the M sensors.
- $\boldsymbol{s}(t) = [s_1(t), \ldots, s_N(t)]^{\mathsf{T}} \in \mathbb{C}^{N\times 1}$: Source signal vector of the N sources.
- $\boldsymbol{n}(t) = [n_1(t), \ldots, n_M(t)]^{\mathsf{T}} \in \mathbb{C}^{M\times 1}$: Sensor noise vector of the M sensors.



Conventional Signal Model

Assumptions and Signal Model

Sensor noise $\boldsymbol{n}(t)$ modeled as complex circular Gaussian random variable $\mathbf{n}(t)$, with:

- Identical noise variance (power) ν in all sensors (uniform).
- Independent noise in different antennas (spatially white).
- Independent noise in different time instants (temporally white).

Uniform spatially and temporally white noise

- Zero mean: $\mathbb{E}\left\{\mathbf{n}(t)\right\} = \mathbf{0}_M.$
- Covariance matrix: $\mathbb{E}\left\{\mathbf{n}(t)\mathbf{n}^{\mathsf{H}}(t)\right\} = \nu \boldsymbol{I}_M \in \mathbb{C}^{M \times M}.$

Conventional Signal Model
Assumptions and Signal Model

Multiple measurement version: T snapshots

$$\boldsymbol{X} = \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{S} + \boldsymbol{N}$$

- $\boldsymbol{X} = [\boldsymbol{x}(1), \boldsymbol{x}(2), \ldots, \boldsymbol{x}(T)] \in \mathbb{C}^{M \times T}$: Receive signal matrix.
- $\boldsymbol{S} = [\boldsymbol{s}(1), \boldsymbol{s}(2), \ldots, \boldsymbol{s}(T)] \in \mathbb{C}^{N \times T}$: Source signal matrix.
- $\boldsymbol{N} = [\boldsymbol{n}(1), \boldsymbol{n}(2), \ldots, \boldsymbol{n}(T)] \in \mathbb{C}^{M \times T}$: Sensor noise matrix.
- T : Number of available snapshots.

Objective:

Given the receive signal $\boldsymbol{X}$ and the mapping $\boldsymbol{\theta} \mapsto \boldsymbol{A}(\boldsymbol{\theta})$, estimate the DOAs $\boldsymbol{\theta}$

Conventional Signal Model
Stochastic and Deterministic Covariance Model

Signal waveform $\boldsymbol{s}(t)$ modeled as complex circular Gaussian random variable $\mathbf{s}(t)$.

Stochastic (unconditional) signal model

- Zero mean: $\mathbb{E}\left\{\mathbf{s}(t)\right\} = \mathbf{0}_N$.
- Signal covariance matrix: $\boldsymbol{P} = \mathbb{E}\left\{\mathbf{s}(t)\mathbf{s}^{\mathsf{H}}(t)\right\} \in \mathbb{C}^{N\times N}$.
- Non-singularity: $\boldsymbol{P} \succ 0$ (not fully coherent signals).
- Gaussian measurements: $\mathbf{x}(t) \sim \mathcal{N}_{\mathsf{C}}\left(\mathbf{0}_M, \boldsymbol{R}\right)$.
- Receive correlation matrix: $\boldsymbol{R} = \mathbb{E}\left\{\mathbf{x}(t)\mathbf{x}^{\mathsf{H}}(t)\right\}$.
  $= \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{P}\boldsymbol{A}^{\mathsf{H}}(\boldsymbol{\theta}) + \nu \boldsymbol{I}_M \in \mathbb{C}^{M\times M}$.
- Parameter characterization: $\boldsymbol{\theta} \in \Theta^N, \boldsymbol{P} \in \mathbb{C}^{N\times N}, \nu \in \mathbb{R}_+$.

Number of parameters independent of number of observations T.

Conventional Signal Model
Stochastic and Deterministic Covariance Model

Signal waveform $\boldsymbol{s}(t)$ modeled as deterministic quantity.
Received signal $\boldsymbol{x}(t)$ modeled as random variable $\mathbf{x}(t) = \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{s}(t) + \mathbf{n}(t)$.

Deterministic (conditional) signal model

- Gaussian measurements: $\mathbf{x}(t) \sim \mathcal{N}_{\mathrm{C}}\big(\boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{s}(t), \nu \boldsymbol{I}\big)$.
- Parameter characterization: $\boldsymbol{\theta} \in \Theta^N$,
  $\boldsymbol{S} = [\boldsymbol{s}(1), \boldsymbol{s}(2), ..., \boldsymbol{s}(T)] \in \mathbb{C}^{N \times T}, \nu \in \mathbb{R}_+$.

Number of parameters grows with number of observations T.

Conventional Signal Model
Stochastic and Deterministic Covariance Model

- In practice, the true receive signal covariance matrix $\boldsymbol{R}$ is not available and must be estimated from finite samples.
- A commonly used sample covariance/correlation matrix estimator is given as:

Sample covariance/correlation matrix

$$\hat{\boldsymbol{R}} = \frac{1}{T} \sum_{t=1}^{T} \boldsymbol{x}(t)\boldsymbol{x}^{\mathsf{H}}(t) = \frac{1}{T} \boldsymbol{X}\boldsymbol{X}^{\mathsf{H}}$$

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Optimal Parametric Methods
Maximum Likelihood

General procedure [Lehmann'98]

- **Step 1:** Determine analytically a multivariate pdf $f\left(\mathbf{x}(1), \ldots, \mathbf{x}(T) | \boldsymbol{\alpha}\right)$ as a function of random observation model vectors and nonrandom parameters $\boldsymbol{\alpha}$.
- **Step 2:** Insert actual observations $\boldsymbol{x}(1), \ldots, \boldsymbol{x}(T)$ instead of "hypothetical" observation model vectors (random variables) $\mathbf{x}(1), \ldots, \mathbf{x}(T)$ to obtain the so-called likelihood function $f\left(\boldsymbol{x}(1), \ldots, \boldsymbol{x}(T) | \boldsymbol{\alpha}\right)$ from the pdf.
- **Step 3:** Maximize the likelihood function w.r.t. all unknown parameters and to ML parameter estimates, i.e.

$$\hat{\boldsymbol{\alpha}}_{\mathrm{ML}} = \arg \max_{\boldsymbol{\alpha}} f\left(\boldsymbol{x}(1), \ldots, \boldsymbol{x}(T) | \boldsymbol{\alpha}\right)$$

Why is Maximum Likelihood important?

- Maximum Likelihood achieves the Cramér-Rao lower-bound (under mild regularity conditions).

Optimal Parametric Methods
Maximum Likelihood

Concentration of ML function

- Use a specific partition $\boldsymbol{\alpha} = [\boldsymbol{\alpha}_1^\mathsf{T}, \boldsymbol{\alpha}_2^\mathsf{T}]^\mathsf{T}$ of the parameter vector.
- Maximize the likelihood function w.r.t. part of the variables, e.g., the partition $\boldsymbol{\alpha}_2$, while considering other variables as constant. Hence,

$$\max_{\boldsymbol{\alpha}} f\big(\boldsymbol{x}(1), \ldots, \boldsymbol{x}(T)|\boldsymbol{\alpha}\big) = \max_{\boldsymbol{\alpha}_1} \underbrace{\max_{\boldsymbol{\alpha}_2} f\big(\boldsymbol{x}(1), \ldots, \boldsymbol{x}(T)|\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2\big)}_{g\left(\boldsymbol{x}(1),\ldots,\boldsymbol{x}(T)|\boldsymbol{\alpha}_1\right)}$$

- If possible, find an analytic (closed-form) solution $\hat{\boldsymbol{\alpha}}_{2,\mathrm{ML}}(\boldsymbol{\alpha}_1)$ (as a function of $\boldsymbol{\alpha}_1$) for inner optimization problem

$$\begin{aligned} g\big(\boldsymbol{x}(1), \ldots, \boldsymbol{x}(T)|\boldsymbol{\alpha}_1\big) &= f\big(\boldsymbol{x}(1), \ldots, \boldsymbol{x}(T)|\boldsymbol{\alpha}_1, \hat{\boldsymbol{\alpha}}_{2,\mathrm{ML}}(\boldsymbol{\alpha}_1)\big), \\ \hat{\boldsymbol{\alpha}}_{1,\mathrm{ML}} &= \arg\max_{\boldsymbol{\alpha}_1} \; g\big(\boldsymbol{x}(1), \ldots, \boldsymbol{x}(T)|\boldsymbol{\alpha}_1\big). \end{aligned}$$

Optimal Parametric Methods
Deterministic Maximum Likelihood

Under the deterministic (unconditional) model [Böhme'84],[Wax'85],[Ziskind'88]

$$\mathbf{x}(t) \sim \mathcal{N}_{\mathrm{C}}\big(\boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{s}(t), \nu\boldsymbol{I}\big)$$

with parameter vector $\boldsymbol{\alpha} = [\boldsymbol{\theta}^{\mathsf{T}}, \boldsymbol{s}^{\mathsf{T}}(1), \ldots, \boldsymbol{s}^{\mathsf{T}}(T), \nu]^{\mathsf{T}}$.
Hence the corresponding likelihood is

$$f\big(\boldsymbol{x}(1), \ldots, \boldsymbol{x}(T)|\boldsymbol{\alpha}\big) = \prod_{t=1}^{T} \frac{1}{(\pi\nu)^M} \exp\left(-\frac{\|\boldsymbol{x}(t) - \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{s}(t)\|^2}{\nu}\right).$$

The negative log-likelihood is

$$\mathcal{L}\big(\boldsymbol{x}(1), \ldots, \boldsymbol{x}(T)|\boldsymbol{\alpha}\big) = \sum_{t=1}^{T} M \ln(\pi\nu) + \sum_{t=1}^{T} \frac{1}{\nu}\|\boldsymbol{x}(t) - \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{s}(t)\|^2.$$

Closed-form expressions for ML estimates for fixed $\boldsymbol{\theta}$

$$\hat{\boldsymbol{s}}_{\text{DML}}(t) = \left(\boldsymbol{A}^{\mathsf{H}}(\boldsymbol{\theta})\boldsymbol{A}(\boldsymbol{\theta})\right)^{-1}\boldsymbol{A}^{\mathsf{H}}(\boldsymbol{\theta})\boldsymbol{x}(t) = \boldsymbol{A}^{\dagger}(\boldsymbol{\theta})\boldsymbol{x}(t)$$

$$\hat{\nu}_{\text{DML}} = \frac{1}{M}\operatorname{Tr}\left(\boldsymbol{\Pi}^{\perp}_{\boldsymbol{A}(\boldsymbol{\theta})}\hat{\boldsymbol{R}}\right)$$

and where

$$\boldsymbol{A}^{\dagger}(\boldsymbol{\theta}) = \left(\boldsymbol{A}^{\mathsf{H}}(\boldsymbol{\theta})\boldsymbol{A}(\boldsymbol{\theta})\right)^{-1}\boldsymbol{A}^{\mathsf{H}}(\boldsymbol{\theta})$$

$$\boldsymbol{\Pi}_{\boldsymbol{A}(\boldsymbol{\theta})} = \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{A}^{\dagger}(\boldsymbol{\theta})$$

and $$\boldsymbol{\Pi}^{\perp}_{\boldsymbol{A}(\boldsymbol{\theta})} = \boldsymbol{I} - \boldsymbol{\Pi}_{\boldsymbol{A}(\boldsymbol{\theta})}$$

denote the pseudo-inverse of $\boldsymbol{A}(\boldsymbol{\theta})$, projectors onto the range space of $\boldsymbol{A}(\boldsymbol{\theta})$ and onto the nullspace of $\boldsymbol{A}^{\mathsf{H}}(\boldsymbol{\theta})$, respectively.

Optimal Parametric Methods

Deterministic Maximum Likelihood

Inserting $\hat{\boldsymbol{s}}_{\mathrm{DML}}(t)$ and $\hat{\nu}_{\mathrm{DML}}$ back into the negative log-likelihood

$$\mathcal{L}\big(\boldsymbol{x}(1), \ldots, \boldsymbol{x}(T) | \boldsymbol{\theta}\big) = TM \left(\ln\left(\mathrm{Tr}\big(\boldsymbol{\Pi}^{\perp}_{\boldsymbol{A}(\boldsymbol{\theta})} \hat{\boldsymbol{R}}\big)\right) + \ln(\pi) - \ln(M) + 1\right).$$

Minimization w.r.t. $\boldsymbol{\theta}$: [Böhme'84]

$$\hat{\boldsymbol{\theta}}_{\mathrm{DML}} = \arg\min_{\theta} \mathcal{L}\big(\boldsymbol{x}(1), \ldots, \boldsymbol{x}(T) | \boldsymbol{\theta}\big)$$
$$= \arg\min_{\boldsymbol{\theta}} \mathrm{Tr}\big(\boldsymbol{\Pi}^{\perp}_{\boldsymbol{A}(\boldsymbol{\theta})} \hat{\boldsymbol{R}}\big)$$

Interpretation: Find DoAs such that the total received energy in the noise subspace is minimized.

Optimal Parametric Methods
Deterministic Maximum Likelihood

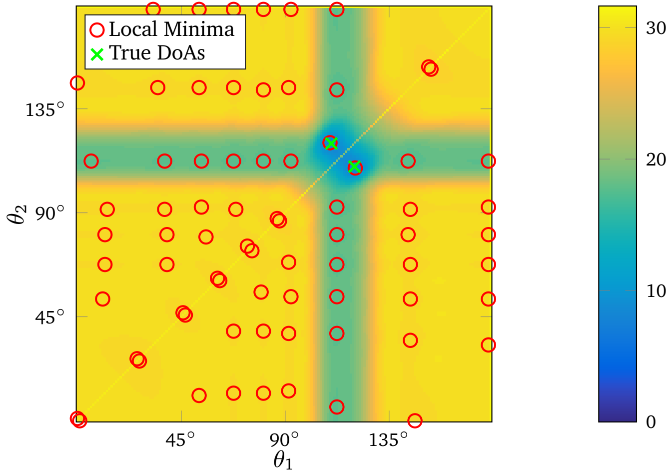
Minimization of the concentrated negative log-likelihood function

$$f_{\mathrm{DML}}(\boldsymbol{\theta}) = \mathrm{Tr}\left(\boldsymbol{\Pi}^{\perp}_{\boldsymbol{A}(\boldsymbol{\theta})}\hat{\boldsymbol{R}}\right)$$

- $f_{\mathrm{DML}}(\boldsymbol{\theta})$ is highly multi-modal, many local optima with cost close to global optimum.
- Minimum cannot be computed in closed form.
- Costly N dimensional search over field of view is required.
- Complexity grows exponentially with number of sources N.
- Generally, complexity becomes prohibitive if $N > 3$ sources.

Optimal Parametric Methods
Deterministic Maximum Likelihood

$$M = 10, \boldsymbol{\theta} = [110^\circ, 120^\circ]^{\mathrm{T}}, \mathrm{SNR} = 0 \text{ dB}, T = 100$$



Optimal Parametric Methods
Stochastic Maximum Likelihood

Under the stochastic (unconditional) model
[Böhme'86],[Bresler'88],[Jaffer'88],[Stoica'90-2]

$$\mathbf{x}(t) \sim \mathcal{N}_{\mathrm{C}}\big(\mathbf{0}_M, \boldsymbol{R}\big)$$

with $\boldsymbol{R} = \mathrm{E}\,\boldsymbol{x}(t)\boldsymbol{x}^{\mathsf{H}}(t) = \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{P}\boldsymbol{A}^{\mathsf{H}}(\boldsymbol{\theta}) + \nu\boldsymbol{I}_M$ and parameter vector $\boldsymbol{\alpha} = [\boldsymbol{\theta}^{\mathsf{T}}, \boldsymbol{p}^{\mathsf{T}}, \nu]^{\mathsf{T}}$.

Vector $\boldsymbol{p} \in \mathbb{R}^{N^2}$ contains the N elements on diagonal of matrix $\boldsymbol{P}$ and the $(N^2 - N)$ elements characterizing real and imaginary part of upper triangular of $\boldsymbol{P}$.
Hence the corresponding likelihood is

$$f\big(\boldsymbol{x}(1), \ldots, \boldsymbol{x}(T)|\boldsymbol{\alpha}\big) = \prod_{t=1}^{T} \frac{1}{\pi^M \det(\boldsymbol{R})} \exp\left(-\boldsymbol{x}^{\mathsf{H}}(t)\boldsymbol{R}^{-1}(\boldsymbol{\theta})\boldsymbol{x}(t)\right).$$

Optimal Parametric Methods
Stochastic Maximum Likelihood

The negative log-likelihood is

$$\mathcal{L}\big(\boldsymbol{x}(1),\ldots,\boldsymbol{x}(T)|\boldsymbol{\alpha}\big) = T\left(M\ln(\pi) + \ln\det\left(\boldsymbol{R}\right) + \mathrm{Tr}(\boldsymbol{R}^{-1}\hat{\boldsymbol{R}})\right)$$

Closed-form expressions for ML estimates for fixed $\boldsymbol{\theta}$

$$\hat{\nu}_{\mathrm{SML}} = \frac{1}{M-N}\mathrm{Tr}\left(\boldsymbol{\Pi}^{\perp}_{\boldsymbol{A}(\boldsymbol{\theta})}\hat{\boldsymbol{R}}\right)$$

$$\hat{\boldsymbol{P}}_{\mathrm{SML}} = \boldsymbol{A}^{\dagger}(\boldsymbol{\theta})\left(\hat{\boldsymbol{R}} - \hat{\nu}_{\mathrm{SML}}\boldsymbol{I}_M\right)\boldsymbol{A}^{\dagger\mathsf{H}}(\boldsymbol{\theta})$$

Inserting $\hat{\nu}_{\mathrm{SML}}$ and $\hat{\boldsymbol{P}}_{\mathrm{SML}}$ back and minimizing w.r.t. $\boldsymbol{\theta}$ yields

$$\hat{\boldsymbol{\theta}}_{\mathrm{SML}} = \arg\min_{\boldsymbol{\theta}} \det\left(\boldsymbol{\Pi}_{\boldsymbol{A}(\boldsymbol{\theta})}\hat{\boldsymbol{R}}\boldsymbol{\Pi}_{\boldsymbol{A}(\boldsymbol{\theta})} + \underbrace{\frac{1}{M-N}\mathrm{Tr}\left(\boldsymbol{\Pi}^{\perp}_{\boldsymbol{A}(\boldsymbol{\theta})}\hat{\boldsymbol{R}}\right)}_{\hat{\nu}_{\mathrm{SML}}}\boldsymbol{\Pi}^{\perp}_{\boldsymbol{A}(\boldsymbol{\theta})}\right).$$

Optimal Parametric Methods
Weighted Subspace Fitting

Eigendecomposition of the receive covariance matrix

$$\boldsymbol{R} = \mathrm{E}\, \mathbf{x}(t)\mathbf{x}^{\mathsf{H}}(t) = \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{P}\boldsymbol{A}^{\mathsf{H}}(\boldsymbol{\theta}) + \nu \boldsymbol{I}_M$$
$$= \sum_{m=1}^{M} \lambda_m \boldsymbol{u}_m \boldsymbol{u}_m^{\mathsf{H}}$$

where $\lambda_1 \geq \lambda_2 \ldots \geq \lambda_M \in \mathbb{R}_+$ are sorted eigenvalues of $\boldsymbol{R}$.

From the eigenanalysis of $\boldsymbol{R}$ we obtain that:

$$\lambda_m > \nu, \quad m = 1, \ldots, N \qquad \text{signal subspace eigenvalues}$$
$$\lambda_m = \nu, \quad m = N+1, \ldots, M \qquad \text{noise subspace eigenvalues}$$

with corresponding eigenvectors:

$$\boldsymbol{u}_1, \ldots, \quad \boldsymbol{u}_N, \qquad \text{signal eigenvectors}$$
$$\boldsymbol{u}_{N+1}, \ldots, \quad \boldsymbol{u}_M \qquad \text{noise eigenvectors.}$$

Optimal Parametric Methods
Weighted Subspace Fitting

Eigendecomposition in compact matrix notation:

$$\boldsymbol{R} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^{\mathsf{H}} = \boldsymbol{U}_{\mathrm{s}}\boldsymbol{\Lambda}_{\mathrm{s}}\boldsymbol{U}_{\mathrm{s}}^{\mathsf{H}} + \boldsymbol{U}_{\mathrm{n}}\boldsymbol{\Lambda}_{\mathrm{n}}\boldsymbol{U}_{\mathrm{n}}^{\mathsf{H}}$$

where we define

$\boldsymbol{U}_{\mathrm{s}} = [\boldsymbol{u}_1, \ldots, \boldsymbol{u}_N] \in \mathbb{C}^{M\times N}$	signal eigenvector matrix
$\boldsymbol{U}_{\mathrm{n}} = [\boldsymbol{u}_{N+1}, \ldots, \boldsymbol{u}_M] \in \mathbb{C}^{M\times (M-N)}$	noise eigenvector matrix
$\boldsymbol{\Lambda}_{\mathrm{s}} = \mathrm{diag}(\lambda_1, \ldots, \lambda_N) \in \mathbb{S}_+^{N\times N}$	diagonal matrix of signal eigenvalues
$\boldsymbol{\Lambda}_{\mathrm{n}} = \nu\, \boldsymbol{I}_{M-N} \in \mathbb{S}_+^{(M-N)\times (M-N)}$	diagonal matrix of noise eigenvalues

and

$\boldsymbol{U} = [\boldsymbol{U}_{\mathrm{s}}, \boldsymbol{U}_{\mathrm{n}}] \in \mathbb{C}^{M\times M}$	unitary matrix of eigenvectors
$\boldsymbol{\Lambda} = \mathrm{blkdiag}\left(\boldsymbol{\Lambda}_{\mathrm{s}}, \boldsymbol{\Lambda}_{\mathrm{n}}\right) \in \mathbb{S}_+^{M\times M}$	diagonal matrix of eigenvalues.

Optimal Parametric Methods
Weighted Subspace Fitting

- $\boldsymbol{U}$ is unitary, i.e. $\boldsymbol{U}^{\mathsf{H}}\boldsymbol{U} = \boldsymbol{I}_M$.
- The columns of the signal subspace eigenvectors $\boldsymbol{U}_{\mathrm{s}}$ span the signal subspace, i.e., the range space spanned by the columns of the steering matrix $\boldsymbol{A}(\boldsymbol{\theta})$ at the true DOAs $\boldsymbol{\theta}$, hence

$$\mathcal{R}(\boldsymbol{U}_{\mathrm{s}}) = \mathcal{R}(\boldsymbol{A}(\boldsymbol{\theta})).$$

- There exists a non-singular matrix $\boldsymbol{K} \in \mathbb{C}^{N \times N}$ such that $\boldsymbol{U}_{\mathrm{s}} = \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{K}$.
- The columns of the noise subspace eigenvectors $\boldsymbol{U}_{\mathrm{n}}$ span the noise-space, i.e., the null-space of the Hermitian of the true steering matrix $\boldsymbol{A}(\boldsymbol{\theta})$

$$\mathcal{R}(\boldsymbol{U}_{\mathrm{n}}) = \mathcal{N}(\boldsymbol{A}^{\mathsf{H}}(\boldsymbol{\theta})).$$

- Hence, the columns of the noise subspace eigenvectors $\boldsymbol{U}_{\mathrm{n}}$ are orthogonal to the column-space of the true steering matrix $\boldsymbol{A}(\boldsymbol{\theta})$, i.e.,

$$\boldsymbol{U}_{\mathrm{n}}^{\mathsf{H}}\boldsymbol{A}(\boldsymbol{\theta}) = \mathbf{0}_{(M-N)\times N}.$$

Optimal Parametric Methods
Weighted Subspace Fitting

The eigendecomposition of the finite sample covariance matrix $\hat{\boldsymbol{R}}$ is given by:

$$\hat{\boldsymbol{R}} = \hat{\boldsymbol{U}}\hat{\boldsymbol{\Lambda}}\hat{\boldsymbol{U}}^{\mathsf{H}} = \hat{\boldsymbol{U}}_{\mathrm{s}}\hat{\boldsymbol{\Lambda}}_{\mathrm{s}}\hat{\boldsymbol{U}}_{\mathrm{s}}^{\mathsf{H}} + \hat{\boldsymbol{U}}_{\mathrm{n}}\boldsymbol{\Lambda}_{\mathrm{n}}\hat{\boldsymbol{U}}_{\mathrm{n}}^{\mathsf{H}}$$

where we define for $\hat{\lambda}_1 \geq \hat{\lambda}_2 \geq \ldots \geq \hat{\lambda}_M$

$\hat{\boldsymbol{U}}_{\mathrm{s}} = [\hat{\boldsymbol{u}}_1, \ldots, \hat{\boldsymbol{u}}_N] \in \mathbb{C}^{M\times N}$ sample signal eigenvector matrix

$\hat{\boldsymbol{U}}_{\mathrm{n}} = [\hat{\boldsymbol{u}}_{N+1}, \ldots, \hat{\boldsymbol{u}}_M] \in \mathbb{C}^{M\times (M-N)}$ sample noise eigenvector matrix

$\hat{\boldsymbol{\Lambda}}_{\mathrm{s}} = \mathrm{diag}(\hat{\lambda}_1, \ldots, \hat{\lambda}_N) \in \mathbb{S}_+^{N\times N}$ sample signal eigenvalues

$\hat{\boldsymbol{\Lambda}}_{\mathrm{n}} = \mathrm{diag}(\hat{\lambda}_{N+1}, \ldots, \hat{\lambda}_M) \in \mathbb{S}_+^{(M-N)\times (M-N)}$ sample noise eigenvalues

and

$\hat{\boldsymbol{U}} = \left[\hat{\boldsymbol{U}}_{\mathrm{s}}, \hat{\boldsymbol{U}}_{\mathrm{n}}\right] \in \mathbb{C}^{M\times M}$ unitary matrix of eigenvectors

$\hat{\boldsymbol{\Lambda}} = \mathrm{blkdiag}\left(\hat{\boldsymbol{\Lambda}}_{\mathrm{s}}, \hat{\boldsymbol{\Lambda}}_{\mathrm{n}}\right) \in \mathbb{S}_+^{M\times M}$ diagonal matrix of eigenvalues.

Optimal Parametric Methods
Weighted Subspace Fitting

The DML cost function

$$f_{\mathrm{DML}}(\boldsymbol{\theta}) = \mathrm{Tr}\left(\boldsymbol{\Pi}^{\perp}_{\boldsymbol{A}(\boldsymbol{\theta})}\hat{\boldsymbol{R}}\right)$$

is equivalently obtained from minimizing the Least-Squares fitting problem w.r.t. to the fitting matrix $\boldsymbol{S}$:

$$f_{\mathrm{LS}}(\boldsymbol{\theta}, \boldsymbol{S}) = \|\boldsymbol{X} - \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{S}\|_{\mathsf{F}}^2.$$

The minimization yields the LS estimate

$$\hat{\boldsymbol{S}}_{\mathrm{LS}} = \left(\boldsymbol{A}^{\mathsf{H}}(\boldsymbol{\theta})\boldsymbol{A}(\boldsymbol{\theta})\right)^{-1}\boldsymbol{A}^{\mathsf{H}}(\boldsymbol{\theta})\boldsymbol{X} = \boldsymbol{A}^{\dagger}(\boldsymbol{\theta})\boldsymbol{X}$$

which, if substituted back in the LS function yields the DML function above.

Optimal Parametric Methods
Weighted Subspace Fitting

The LS fitting problem can be generalized. A general data matrix $\boldsymbol{M}$ (as some transformation of the data $\boldsymbol{X}$) can be used instead of $\boldsymbol{X}$.

Examples are $\boldsymbol{M} = \hat{\boldsymbol{U}}_{\text{s}}$ and $\boldsymbol{M} = \hat{\boldsymbol{U}}_{\text{s}} \hat{\boldsymbol{\Lambda}}_{\text{s}}^{\frac{1}{2}}$ or most generally

$$\boldsymbol{M} = \hat{\boldsymbol{U}}_{\text{s}} \boldsymbol{W}^{\frac{1}{2}}$$

for arbitrary weighting matrix $\boldsymbol{W}$.
The corresponding weighted subspace fitting (WSF) problem becomes
[Viberg'91],[Ottersten'90],[Stoica'90]

$$f_{\text{WSF}}(\boldsymbol{\theta}, \boldsymbol{F}) = \|\boldsymbol{M} - \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{F}\|_{\mathsf{F}}^{2}$$

or after concentration w.r.t. $\boldsymbol{F}$ with $\hat{\boldsymbol{F}}_{\text{WSF}} = \boldsymbol{A}^{\dagger}(\boldsymbol{\theta})\boldsymbol{M}$

$$f_{\text{WSF}}(\boldsymbol{\theta}) = \text{Tr}\left(\boldsymbol{\Pi}^{\perp}_{\boldsymbol{A}(\boldsymbol{\theta})} \hat{\boldsymbol{U}}_{\text{s}} \boldsymbol{W} \hat{\boldsymbol{U}}_{\text{s}}^{\mathsf{H}}\right).$$

Optimal Parametric Methods
Weighted Subspace Fitting

The WSF estimates for the DOAs $\boldsymbol{\theta}$ are obtained as

$$\hat{\boldsymbol{\theta}}_{\mathrm{WSF}} = \arg\min_{\boldsymbol{\theta}} \mathrm{Tr}\left(\boldsymbol{\Pi}^{\perp}_{\boldsymbol{A}(\boldsymbol{\theta})} \hat{\boldsymbol{U}}_{\mathrm{s}} \boldsymbol{W} \hat{\boldsymbol{U}}_{\mathrm{s}}^{\mathsf{H}}\right).$$

- The minimization of the WSF cost function cannot be carried out in closed-form and generally requires multi-dimensional search.
- Similarly to the multi-dimensional ML methods, the complexity associated with the minimization becomes prohibitive if the number of source $N > 3$.
- The choice of the weighting matrix as

$$\boldsymbol{W}_{\mathrm{ao}} = \left(\hat{\boldsymbol{\Lambda}}_{\mathrm{s}} - \hat{\nu}_{\mathrm{w}} \boldsymbol{I}_N\right)^2 \hat{\boldsymbol{\Lambda}}_{\mathrm{s}}^{-1} \text{ for } \hat{\nu}_{\mathrm{w}} = \frac{1}{M-N} \mathrm{Tr}\left(\hat{\boldsymbol{\Lambda}}_{\mathrm{n}}\right)$$

  is asymptotically (for large T) optimal in terms of the Mean-Squared-Error (MSE) of DOA estimates which achieves the CRB under the stochastic model.

Recall the Covariance Matrix $\boldsymbol{R}$

$$\boldsymbol{R} = \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{P}\boldsymbol{A}^{\mathsf{H}}(\boldsymbol{\theta}) + \nu\boldsymbol{I}$$

Formulation of Covariance Matching Estimation Techniques (COMET)

[Ottersten'98]

$$\hat{\boldsymbol{A}}_{\text{COMET}} = \underset{\boldsymbol{A}(\boldsymbol{\theta})\in\mathcal{A}_N}{\arg\min} \min_{\boldsymbol{P}\succeq 0, \nu\geq 0} \left|\left|\boldsymbol{W} \operatorname{vec}\left(\hat{\boldsymbol{R}} - \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{P}\boldsymbol{A}^{\mathsf{H}}(\boldsymbol{\theta}) - \nu\boldsymbol{I}\right)\right|\right|_{\text{F}}^2$$

where $\boldsymbol{W} \in \mathbb{C}^{M^2\times M^2}$ is a proper weighting matrix, e.g., $\boldsymbol{W} = \boldsymbol{I}$.

Asymptotically Optimal Weighting Matrix

The MSE of COMET is asymptotically equal to the Stochastic Cramér-Rao bound if the weighting matrix $\boldsymbol{W}$ is chosen as

$$\boldsymbol{W} = \hat{\boldsymbol{W}}_{\text{asymp}} = \left(\hat{\boldsymbol{R}}^{\mathsf{T}} \otimes \hat{\boldsymbol{R}}\right)^{-1/2}.$$

Optimal Parametric Methods
Covariance Matching Estimation Techniques

Observation

$$\begin{aligned}\operatorname{vec}(\boldsymbol{R}) &= \operatorname{vec}\left(\boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{P}\boldsymbol{A}^{\mathsf{H}}(\boldsymbol{\theta}) + \nu\boldsymbol{I}\right)\\ &= \boldsymbol{\Phi}(\boldsymbol{\theta})\boldsymbol{\gamma}\end{aligned}$$

- $\boldsymbol{\Phi} \in \mathbb{C}^{M^2\times(N^2+1)}$ is full-rank matrix depending on the steering matrix $\boldsymbol{A}(\boldsymbol{\theta})$.
- $\boldsymbol{\gamma} \in \mathbb{R}^{(N^2+1)\times 1}$ contains the noise power ν and real-valued entries which characterize the elements on the source covariance matrix $\boldsymbol{P}$.

Relaxed Formulation of COMET

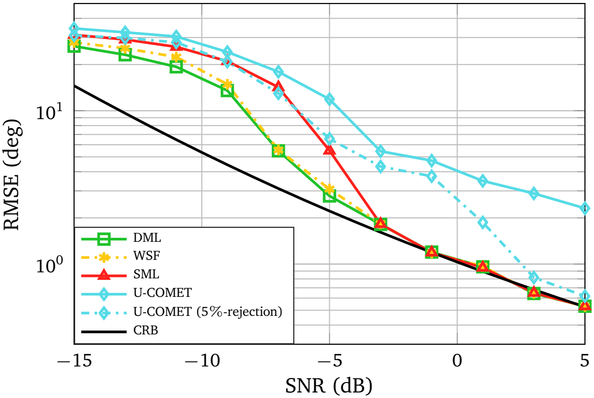
$$\begin{aligned}\hat{\boldsymbol{\theta}}_{\text{COMET}} &= \arg\min_{\boldsymbol{\theta}\in\Theta^N} \min_{\boldsymbol{\gamma}\in\mathbb{C}^{(N^2+1)\times 1}} \left\|\boldsymbol{W}\operatorname{vec}\left(\hat{\boldsymbol{R}}\right) - \boldsymbol{W}\boldsymbol{\Phi}\left(\boldsymbol{\theta}\right)\boldsymbol{\gamma}\right\|_{\mathrm{F}}^2\\ &= \arg\min_{\boldsymbol{\theta}\in\Theta^N} \operatorname{vec}\left(\hat{\boldsymbol{R}}\right)^{\mathsf{H}} \boldsymbol{W}^{\mathsf{H}}\, \boldsymbol{\Pi}^{\perp}_{\boldsymbol{W}\boldsymbol{\Phi}(\boldsymbol{\theta})}\, \boldsymbol{W}\operatorname{vec}\left(\hat{\boldsymbol{R}}\right)\end{aligned}$$

Optimal Parametric Methods

Simulation Results

Uncorrelated Source Signals

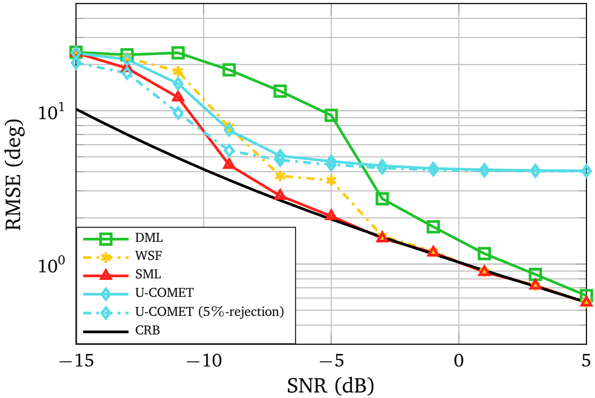
$$M = 5,\ \boldsymbol{\theta} = [90^\circ, 100^\circ]^\mathsf{T},\ T = 200,\ \rho = 0$$



Optimal Parametric Methods
Simulation Results

Correlated Source Signals

$$M = 5,\ \boldsymbol{\theta} = [90^\circ, 100^\circ]^\mathsf{T},\ T = 200,\ \rho = 0.99$$



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Approximation/Relaxation Concept
Motivation

General Formulation of Parametric DOA Estimation

$$\boldsymbol{A}\left(\hat{\boldsymbol{\theta}}\right) = \underset{\boldsymbol{A}(\boldsymbol{\theta}) \in \mathcal{A}_N}{\arg\min}\, f\left(\boldsymbol{A}\left(\boldsymbol{\theta}\right)\right)$$

- Different choices on the cost function $f(\cdot)$ leads to different estimators.
- Prohibitively expensive computational cost to obtain the global minimum.

Adoption of Approximation/Relaxation Techniques required!

- Relaxation/Restriction of the feasible set
- Successive approximation of the cost function
- ...

Approximation/Relaxation Concept
Motivation

Potential Approaches

Original Problem

$$\hat{\boldsymbol{A}} = \min_{\boldsymbol{A} \in \mathcal{A}_N} f(\boldsymbol{A})$$

Relaxation

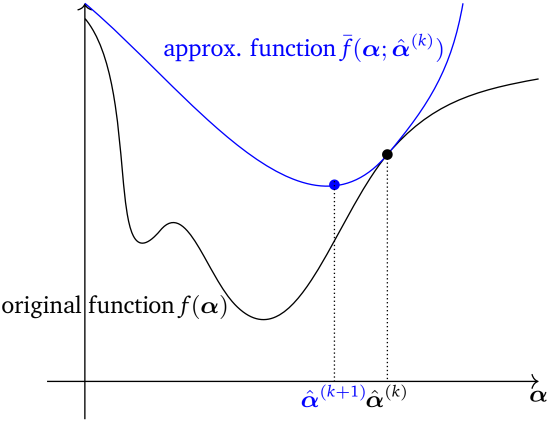
$$\hat{\boldsymbol{A}}_{\text{relaxed}} = \min_{\boldsymbol{A} \in \bar{\mathcal{A}}_N \supset \mathcal{A}_N} f(\boldsymbol{A})$$

Approximation

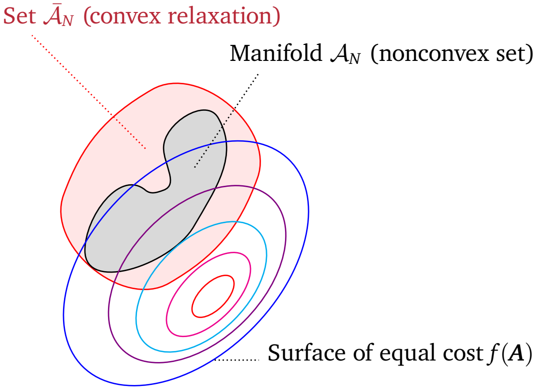
$$\hat{\boldsymbol{A}}_{\text{approx}} = \min_{\boldsymbol{A} \in \mathcal{A}_N} \bar{f}(\boldsymbol{A})$$

- Back-projection is generally required after the relaxation step.
- Possible combination of both relaxation and approximation.

Approximation/Relaxation Concept
Approximation



Approximation/Relaxation Concept
Relaxation



Approximation/Relaxation Concept
Relaxation

Concept of Relaxation-and-Projection Method

1. Replace the original array manifold $\mathcal{A}_N$ by a relaxed manifold $\bar{\mathcal{A}}_N \supset \mathcal{A}_N$

$$\hat{\boldsymbol{A}} = \underset{\boldsymbol{A} \in \mathcal{A}_N}{\arg\min}\, f(\boldsymbol{A}) \quad \longrightarrow \quad \hat{\boldsymbol{A}}_{\text{relaxed}} = \underset{\boldsymbol{A} \in \bar{\mathcal{A}}_N}{\arg\min}\, f(\boldsymbol{A}).$$

2. Project the relaxed estimate $\hat{\boldsymbol{A}}_{\text{relaxed}}$ back to the original array manifold $\mathcal{A}_N$.

Remarks

- The choice on the relaxed array manifold $\bar{\mathcal{A}}_N$ generally depends on the underlying structure of the sensor array.
- Relaxation-and-Projection may, in particular cases, preserve optimality, e.g., in the Extended Invariance Principle (EXIP) [Stoica'89-2].

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Single-source Approximation Techniques
Concept

Suboptimal solutions of the DOA estimation problem can be obtained by adopting the Single-source Approximation.

Recall the General DOA Estimation Problem

$$\boldsymbol{A}\left(\hat{\boldsymbol{\theta}}\right) = \underset{\boldsymbol{A}\left(\boldsymbol{\theta}\right) \in \mathcal{A}_N}{\arg\min}\ f\left(\boldsymbol{A}\left(\boldsymbol{\theta}\right)\right)$$

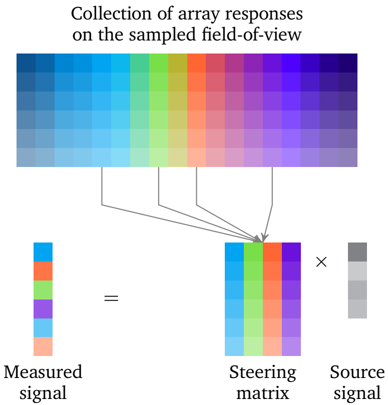
Single-source Approximation

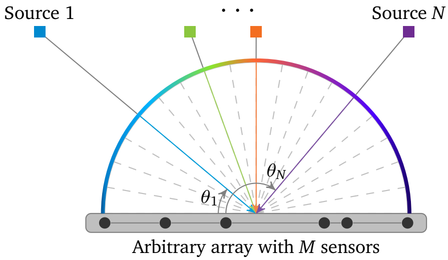
Spectral sweep to find the N deepest local minima $\hat{\boldsymbol{\theta}} = \left[\hat{\theta}_1, \ldots, \hat{\theta}_N\right]^{\mathsf{T}}$ of $f\left(\boldsymbol{a}\left(\theta\right)\right)$

$$\boldsymbol{A}\left(\hat{\boldsymbol{\theta}}\right) = {}^{N}\underset{\boldsymbol{a}(\theta) \in \mathcal{A}_1}{\arg\min} f\left(\boldsymbol{a}(\theta)\right).$$

Interpretation: The cost function measures the goodness-of-fit under the assumption of only one source signal located at the candidate DOA $\theta \in \Theta$.

Single-source Approximation Techniques
Concept





Single-source Approximation Techniques
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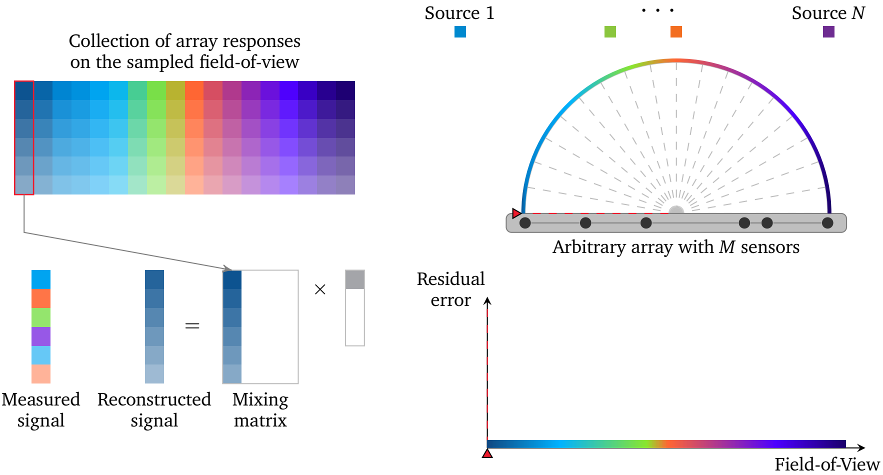

Single-source Approximation Techniques
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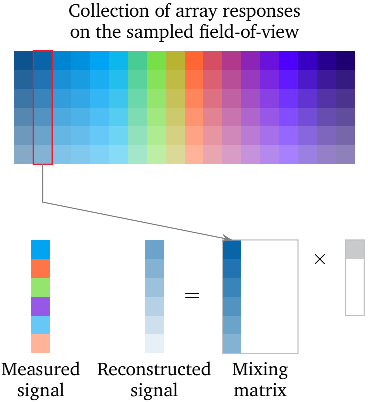

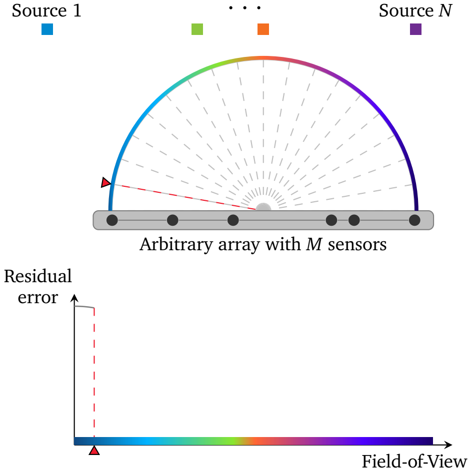

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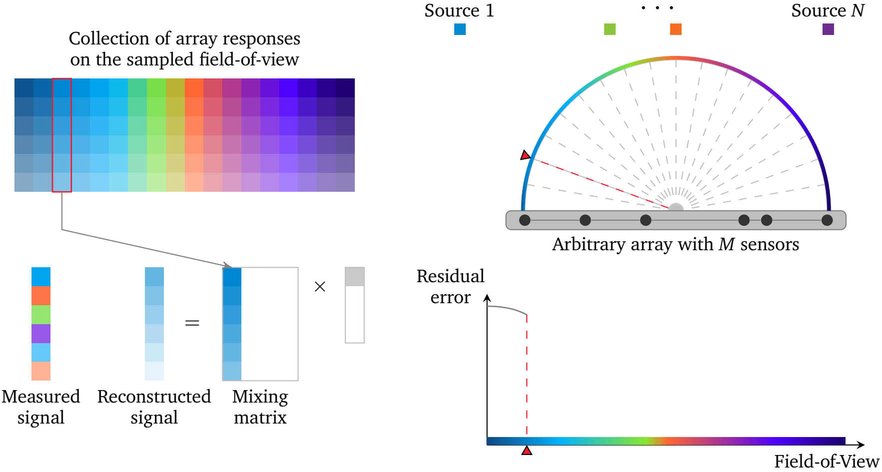
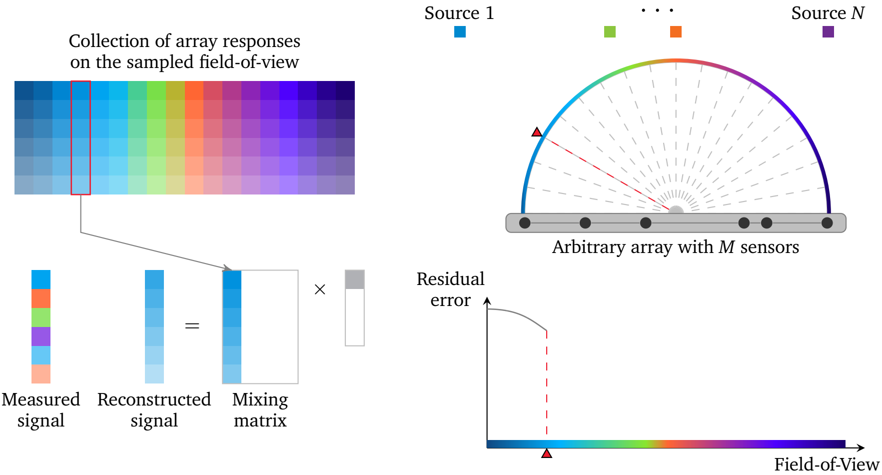
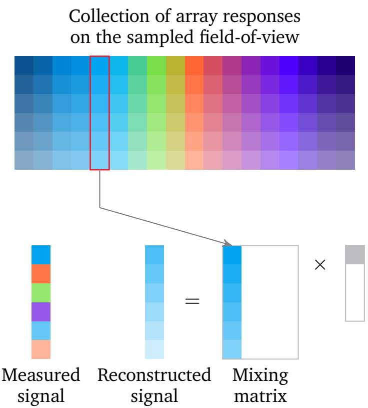


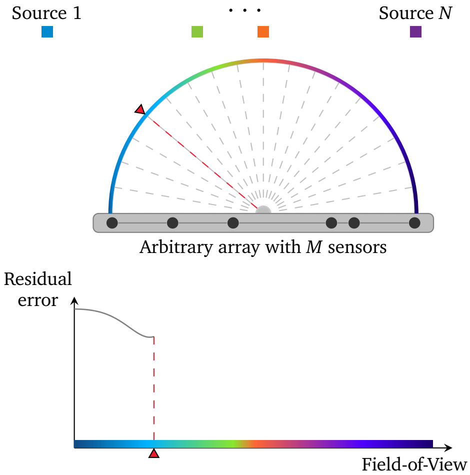
Single-source Approximation Techniques
Concept



Single-source Approximation Techniques

Concept





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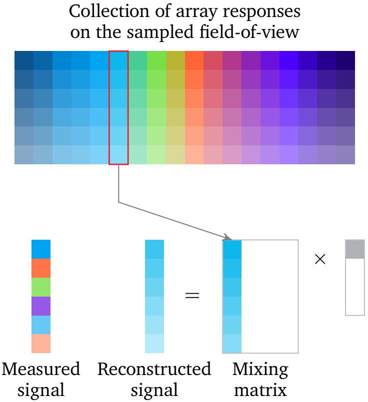

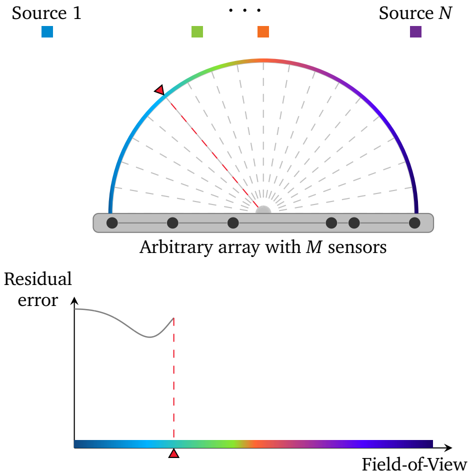

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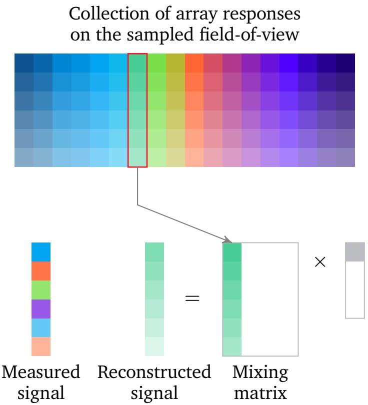

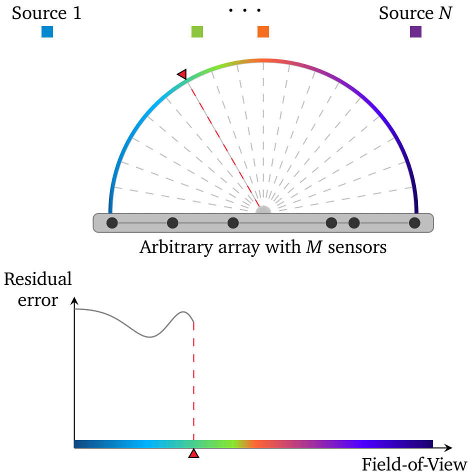

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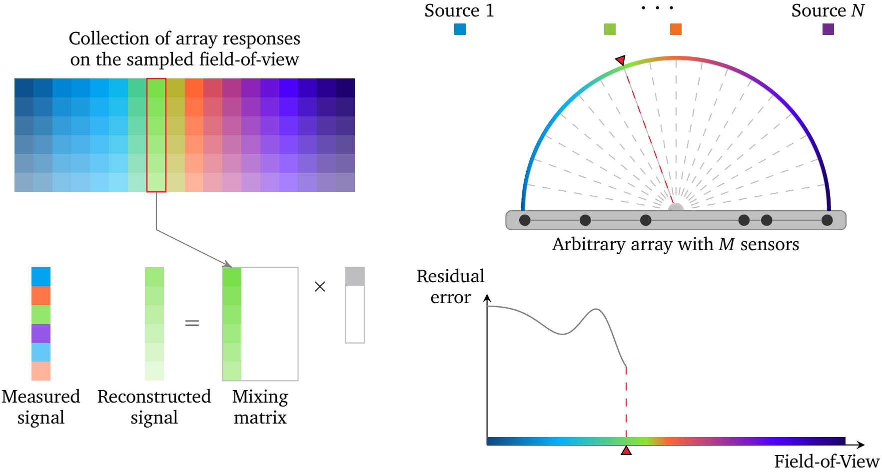

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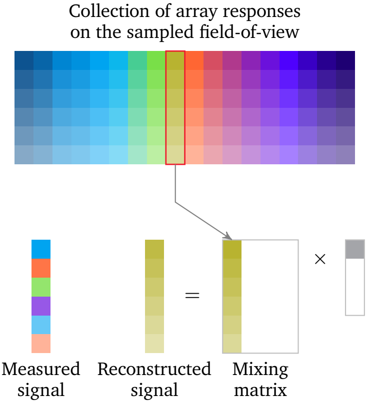

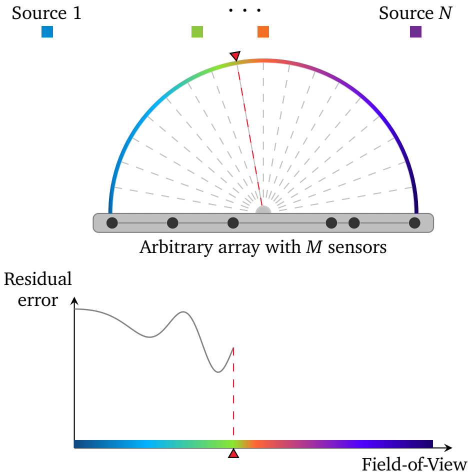

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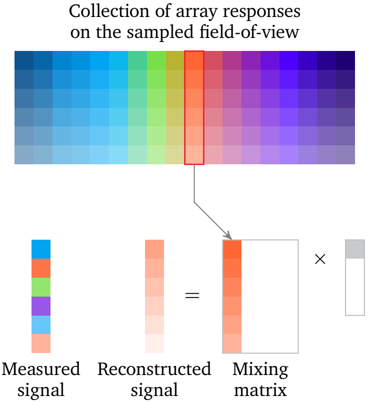

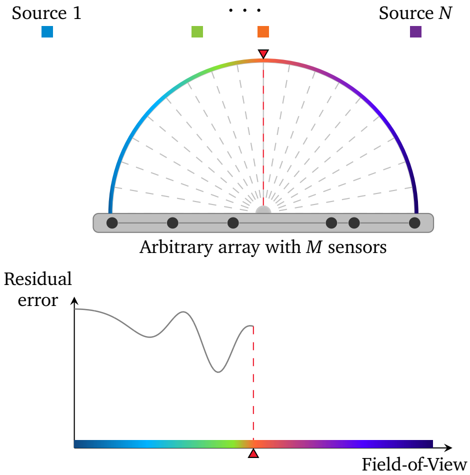

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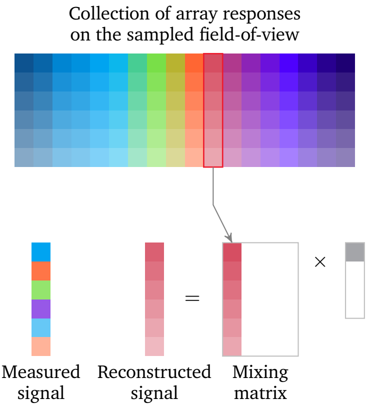

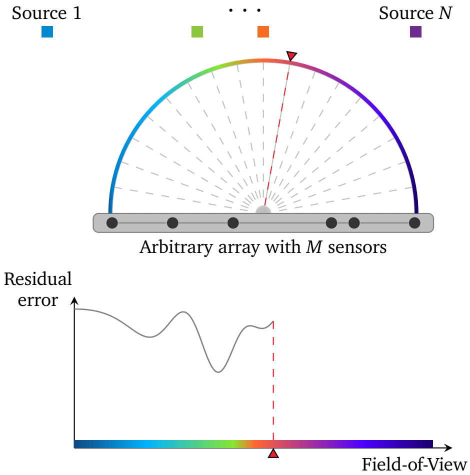

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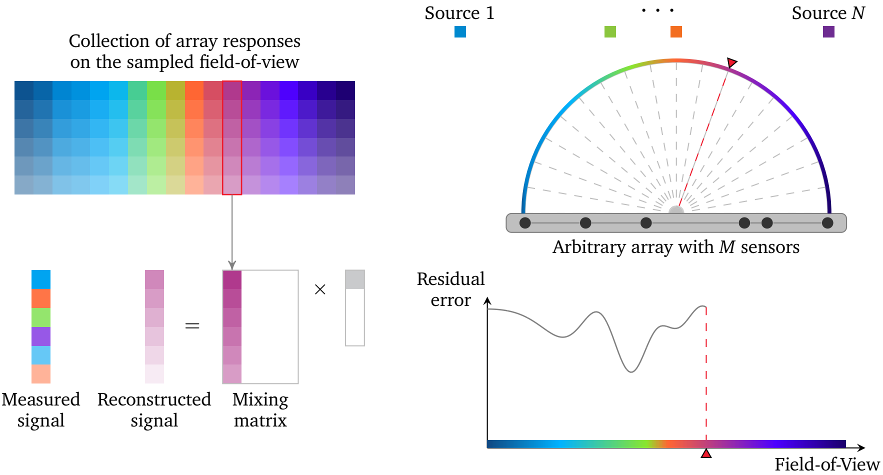

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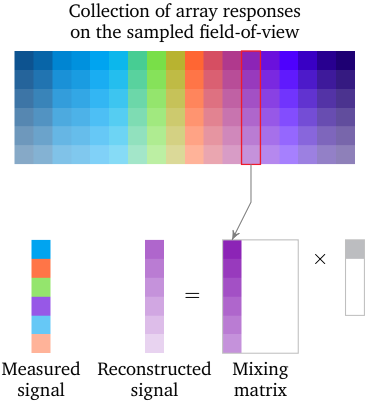

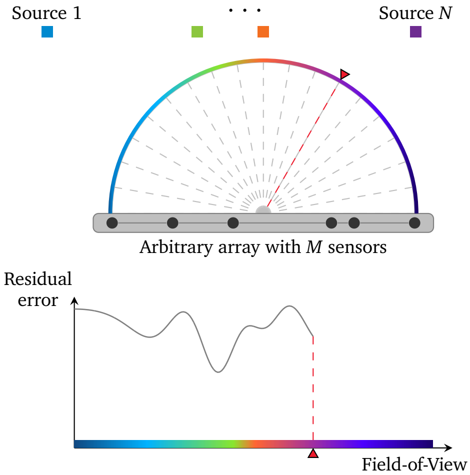

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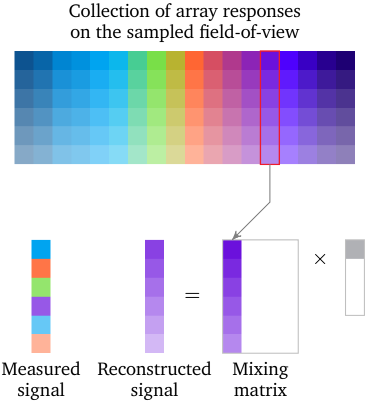

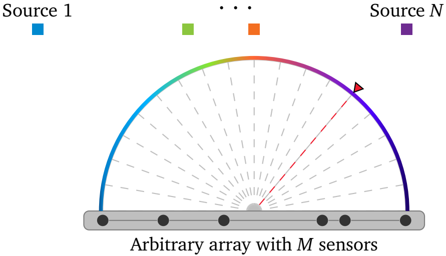

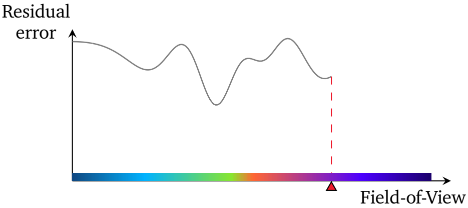

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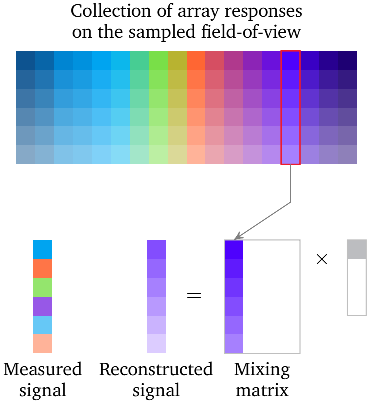
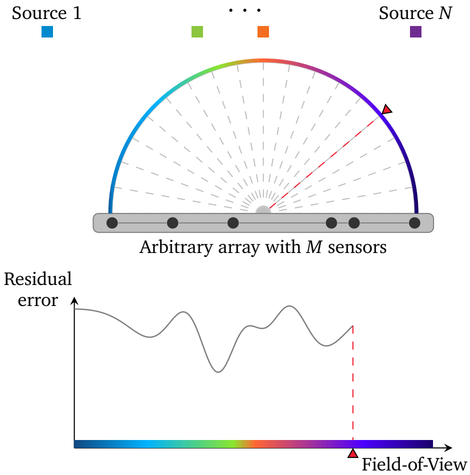
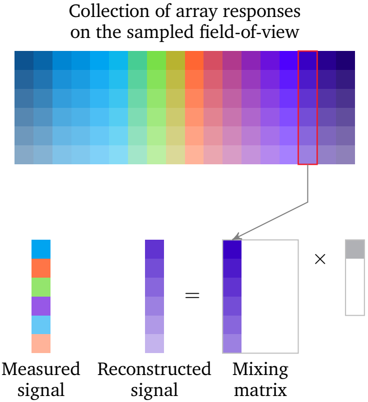
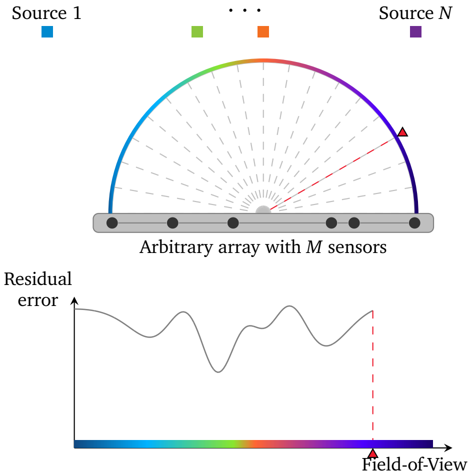
Concept



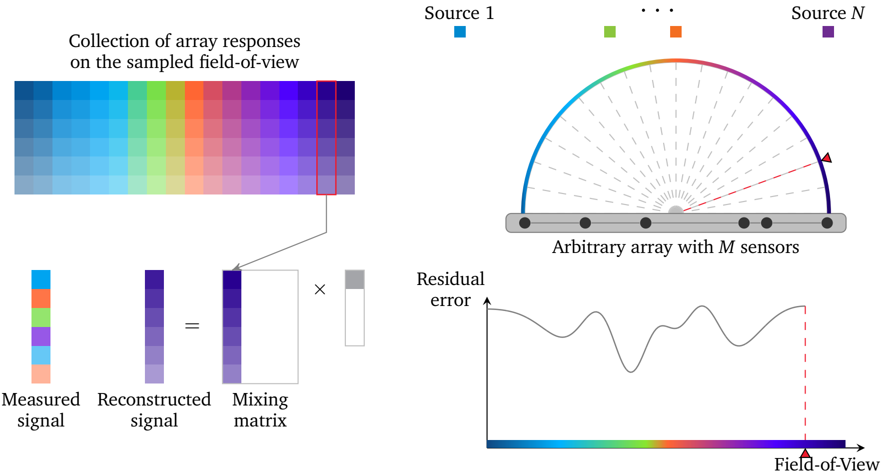


Single-source Approximation Techniques
Concept

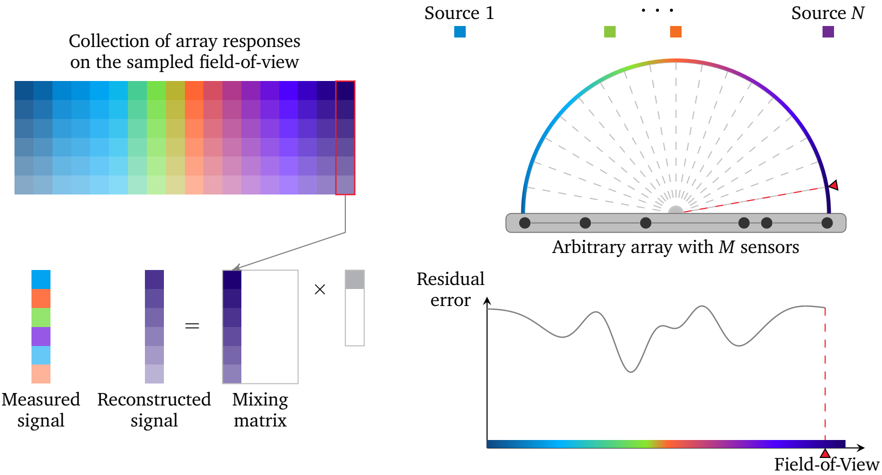




Single-source Approximation Techniques
Concept

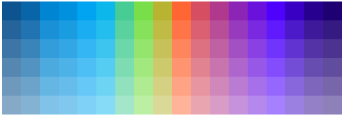


Single-source Approximation Techniques
Concept

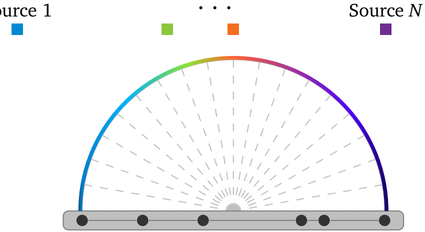


Single-source Approximation Techniques
Concept

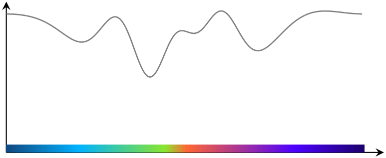
Collection of array responses
on the sampled field-of-view



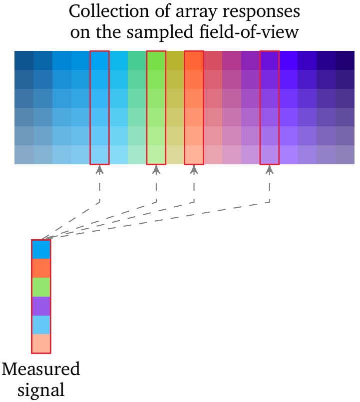
Measured
signal

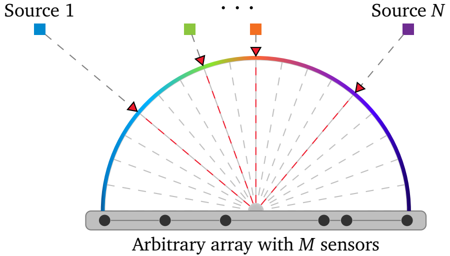


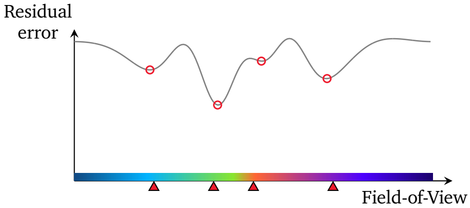
Arbitrary array with M sensors



Single-source Approximation Techniques
Concept







Single-source Approximation Techniques
Conventional Beamformer

Original Derivation

- Output power of the receive signal $\boldsymbol{x}(t)$ after spatial filtering with the beamforming vector $\boldsymbol{w}(\theta)$

$$P(\theta) = \mathbb{E}\left\{\left|\boldsymbol{w}^{\mathsf{H}}(\theta)\boldsymbol{x}(t)\right|^2\right\}$$
$$= \boldsymbol{w}^{\mathsf{H}}(\theta)\boldsymbol{R}\boldsymbol{w}(\theta).$$

- In practice, the true covariance matrix $\boldsymbol{R}$ of the receive signal $\boldsymbol{x}(t)$ is not available and therefore replaced by the sample covariance matrix $\hat{\boldsymbol{R}}$

$$\hat{P}(\theta) = \frac{1}{T}\sum_{t=1}^{T}\left|\boldsymbol{w}^{\mathsf{H}}(\theta)\boldsymbol{x}(t)\right|^2$$
$$= \boldsymbol{w}^{\mathsf{H}}(\theta)\hat{\boldsymbol{R}}\boldsymbol{w}(\theta).$$

Single-source Approximation Techniques
Conventional Beamformer

Beamformer Vector

$$\boldsymbol{w}_{\mathrm{CBF}}(\theta) = \frac{\boldsymbol{a}(\theta)}{||\boldsymbol{a}(\theta)||}$$

Conventional Beamforming Estimator [Bartlett'48]

Find the N highest local maxima of the beamformer spectrum

$$\hat{P}_{\mathrm{CBF}}(\theta) = \frac{\boldsymbol{a}^{\mathsf{H}}(\theta)\hat{\boldsymbol{R}}\boldsymbol{a}(\theta)}{||\boldsymbol{a}(\theta)||^2}.$$

Interpretation

- $\boldsymbol{w}_{\mathrm{CBF}}(\theta)$ can be considered as a spatially matched filter that maximizes the power impinging on the sensor array from the direction θ.

Single-source Approximation Techniques
Conventional Beamformer

Alternative Derivation: Starting from the Covariance Matrix $\boldsymbol{R}$

$$\boldsymbol{R} = \boldsymbol{A}\boldsymbol{P}\boldsymbol{A}^{\mathsf{H}} + \nu \boldsymbol{I}$$

Single-source approximation of Covariance Fitting Problem

$$\begin{aligned}\hat{\sigma}_{\mathrm{s}}^2 &= \arg\min_{\sigma_{\mathrm{s}}^2} \left|\left|\hat{\boldsymbol{R}} - \sigma_{\mathrm{s}}^2 \boldsymbol{a}\boldsymbol{a}^{\mathsf{H}}\right|\right|_{\mathrm{F}}^2 \\ &= \frac{\boldsymbol{a}^{\mathsf{H}}\hat{\boldsymbol{R}}\boldsymbol{a}}{\left(\boldsymbol{a}^{\mathsf{H}}\boldsymbol{a}\right)^2}\end{aligned}$$

- Conventional beamformer spectrum measures the power impinging at the sensor array from the direction $\boldsymbol{a} = \boldsymbol{a}(\theta)$.
- Disadvantage: limited angular resolution.

Single-source Approximation Techniques
Capon Beamformer

Design of the Capon beamformer

For each direction $\boldsymbol{a} = \boldsymbol{a}(\theta)$, find the beamformer vector $\boldsymbol{w} = \boldsymbol{w}(\theta)$ such that

- the power from the direction $\boldsymbol{a}$ is maintained
- the power from remaining directions is suppressed as much as possible.

Optimization Problem

$$\min_{\boldsymbol{w}} \boldsymbol{w}^{\mathsf{H}} \hat{\boldsymbol{R}} \boldsymbol{w}$$

$$\text{subject to } \boldsymbol{w}^{\mathsf{H}} \boldsymbol{a} = 1$$

- Also known as Minimum Variance Distortionless Response beamformer.
- Optimal beamformer vector $\boldsymbol{w}_{\text{Capon}} = \dfrac{\hat{\boldsymbol{R}}^{-1}\boldsymbol{a}}{\boldsymbol{a}^{\mathsf{H}}\hat{\boldsymbol{R}}^{-1}\boldsymbol{a}}$.

Single-source Approximation Techniques
Capon Beamformer

Capon spectrum [Capon'66]

$$\hat{P}_{\text{Capon}}(\theta) = \boldsymbol{w}^{\mathsf{H}}_{\text{Capon}}(\theta)\hat{\boldsymbol{R}}\boldsymbol{w}_{\text{Capon}}(\theta)$$
$$= \frac{1}{\boldsymbol{a}^{\mathsf{H}}(\theta)\hat{\boldsymbol{R}}^{-1}\boldsymbol{a}(\theta)}$$

- Estimate the DOAs $\hat{\boldsymbol{\theta}}$ from the N highest peaks of $\hat{P}_{\text{Capon}}(\theta)$.
- Higher resolution capability than the conventional beamformer.
- Applicable if the sample covariance matrix $\hat{\boldsymbol{R}}$ is full rank.
- Values of Capon peaks are roughly proportional to the signal power of the sources.

Single-source Approximation Techniques
Capon Beamformer

Recall the Conventional Beamfomer

$$\hat{\sigma}_s^2 = \arg\min_{\sigma_s^2} \left\| \hat{\boldsymbol{R}} - \sigma_s^2 \boldsymbol{a}\boldsymbol{a}^{\mathsf{H}} \right\|_{\mathrm{F}}^2$$

Alternative Formulation of the Capon Spectrum

$$\hat{\sigma}_s^2 = \arg\min_{\sigma_s^2} \left\| \hat{\boldsymbol{R}} - \sigma_s^2 \boldsymbol{a}\boldsymbol{a}^{\mathsf{H}} \right\|_{\mathrm{F}}^2$$
$$\text{subject to } \hat{\boldsymbol{R}} - \sigma_s^2 \boldsymbol{a}\boldsymbol{a}^{\mathsf{H}} \succeq \mathbf{0}$$

Remarks

- Both formulations are based on covariance fitting criteria under single-source approximation.
- Constraint in the Capon formulation prevents the residual matrix to be indefinite.

Single-source Approximation Techniques
MUSIC

Recall the Eigendecomposition of the Covariance Matrix $\boldsymbol{R}$

$$\boldsymbol{R} = \boldsymbol{A}\boldsymbol{P}\boldsymbol{A}^{\mathsf{H}} + \nu\boldsymbol{I} = \boldsymbol{U}_{\mathrm{s}}\boldsymbol{\Lambda}_{\mathrm{s}}\boldsymbol{U}_{\mathrm{s}}^{\mathsf{H}} + \nu\boldsymbol{U}_{\mathrm{n}}\boldsymbol{U}_{\mathrm{n}}^{\mathsf{H}}$$

- Assumption: Non-coherent source signals.
- Key observation: $\boldsymbol{U}_{\mathrm{n}}^{\mathsf{H}}\boldsymbol{a}(\theta) = \boldsymbol{0}$ iff θ coincides with one of the true DOAs $\boldsymbol{\theta}$.

MUSIC Pseudo-spectrum [Schmidt'79]

$$\hat{P}_{\mathrm{MUSIC}}(\theta) = \frac{1}{\left\|\hat{\boldsymbol{U}}_{\mathrm{n}}^{\mathsf{H}}\boldsymbol{a}(\theta)\right\|_2^2} = \frac{1}{\boldsymbol{a}^{\mathsf{H}}(\theta)\hat{\boldsymbol{U}}_{\mathrm{n}}\hat{\boldsymbol{U}}_{\mathrm{n}}^{\mathsf{H}}\boldsymbol{a}(\theta)}$$

- MUSIC pseudo-spectrum is inversely proportional to the distance between the steering vector $\boldsymbol{a}(\theta)$ and the sample noise subspace $\mathrm{span}(\boldsymbol{U}_{\mathrm{n}})$.

Single-source Approximation Techniques
MUSIC

Recall the WSF Estimator [Viberg'91]

$$\hat{\boldsymbol{A}} = \arg\min_{\boldsymbol{A}\in\mathcal{A}_N} \min_{\boldsymbol{F}} \left|\left|\hat{\boldsymbol{U}}_{\mathrm{s}} - \boldsymbol{A}\boldsymbol{F}\right|\right|_{\mathrm{F}}^2$$

MUSIC Null-spectrum

$$f_{\mathrm{MUSIC}}(\theta) = \boldsymbol{a}^{\mathsf{H}}(\theta)\hat{\boldsymbol{U}}_{\mathrm{n}}\hat{\boldsymbol{U}}_{\mathrm{n}}^{\mathsf{H}}\boldsymbol{a}(\theta)$$

Alternative Interpretation

$$f_{\mathrm{MUSIC}}(\theta) \propto \min_{\boldsymbol{f}} \left|\left|\hat{\boldsymbol{U}}_{\mathrm{s}} - \boldsymbol{a}(\theta)\boldsymbol{f}^{\mathsf{T}}\right|\right|_{\mathrm{F}}^2$$

- MUSIC can be considered as a single-source approximation of WSF with identity weighting.

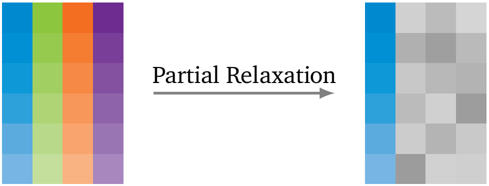
Partial Relaxation Techniques
General Concept

Formulation of the Multi-dimensional Search

$$\left\{\hat{\boldsymbol{A}}\right\} = \underset{\boldsymbol{A}\in\mathcal{A}_N}{\arg\min} \, f\left(\boldsymbol{A}\right)$$

Relaxed Array Manifold

$$\bar{\mathcal{A}}_N = \left\{\boldsymbol{A} \in \mathbb{C}^{M\times N} \;\middle|\; \boldsymbol{A} = \left[\boldsymbol{a}(\theta), \boldsymbol{B}\right], \boldsymbol{B} \in \mathbb{C}^{M\times(N-1)} \text{ and } \operatorname{rank}\left(\boldsymbol{A}\right) = N\right\}$$



$\boldsymbol{A} \in \mathcal{A}_N$ $\qquad$ $\boldsymbol{A} \in \bar{\mathcal{A}}_N$

Partial Relaxation Techniques
General Concept

Formulation of the Multi-dimensional Search

$$\left\{\hat{\boldsymbol{A}}\right\} = \underset{\boldsymbol{A}\in\mathcal{A}_N}{\arg\min} f\left(\boldsymbol{A}\right)$$

Relaxed Array Manifold

$$\bar{\mathcal{A}}_N = \left\{\boldsymbol{A}\in\mathbb{C}^{M\times N} \mid \boldsymbol{A} = \left[\boldsymbol{a}(\theta), \boldsymbol{B}\right], \boldsymbol{B}\in\mathbb{C}^{M\times(N-1)} \text{ and } \operatorname{rank}\left(\boldsymbol{A}\right) = N\right\}$$

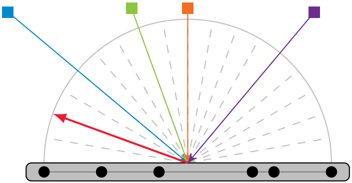
Formulation of Partial Relaxation (PR) Framework [Trinh-Hoang'18]

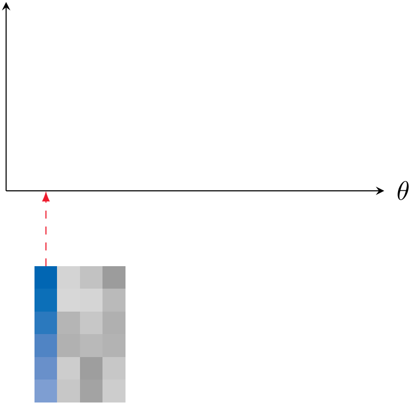
$$\{\hat{\boldsymbol{a}}_{\mathrm{PR}}\} = {}^{N}\underset{\boldsymbol{a}\in\mathcal{A}_1}{\arg\min}\ \min_{\boldsymbol{B}\in\mathbb{C}^{M\times(N-1)}} f\left(\left[\boldsymbol{a}, \boldsymbol{B}\right]\right)$$

- Compute the null-spectrum $f_{\mathrm{PR}}(\theta) = \min_{\boldsymbol{B}\in\mathbb{C}^{M\times(N-1)}} f\left(\left[\boldsymbol{a}(\theta), \boldsymbol{B}\right]\right)$.
- N-deepest local minimizers of $f_{\mathrm{PR}}(\theta)$ are the DOA estimates.

Partial Relaxation Techniques
General Concept

$$\min_{\boldsymbol{B}\in\mathbb{C}^{M\times(N-1)}} f\left(\left[\boldsymbol{a}(\theta),\boldsymbol{B}\right]\right)$$

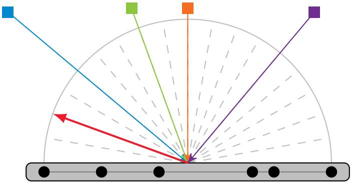


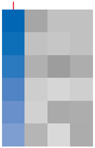


$$\left[\boldsymbol{a}(\theta),\boldsymbol{B}\right]$$

Partial Relaxation Techniques
General Concept

$$\min_{\boldsymbol{B} \in \mathbb{C}^{M \times (N-1)}} f\left(\left[\boldsymbol{a}(\theta), \boldsymbol{B}\right]\right)$$

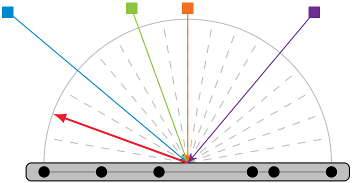


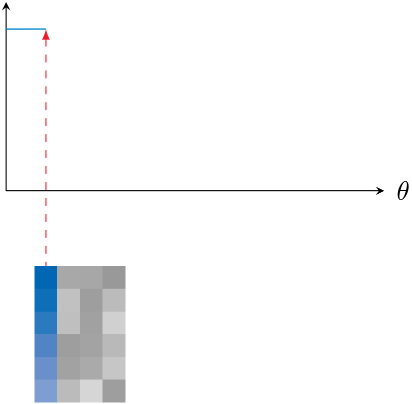


$\left[\boldsymbol{a}(\theta), \boldsymbol{B}\right]$

Partial Relaxation Techniques
General Concept

$$\min_{\boldsymbol{B}\in\mathbb{C}^{M\times(N-1)}} f\left(\left[\boldsymbol{a}(\theta),\boldsymbol{B}\right]\right)$$

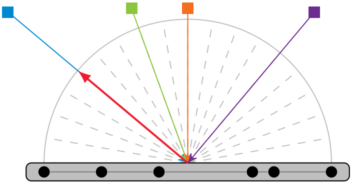


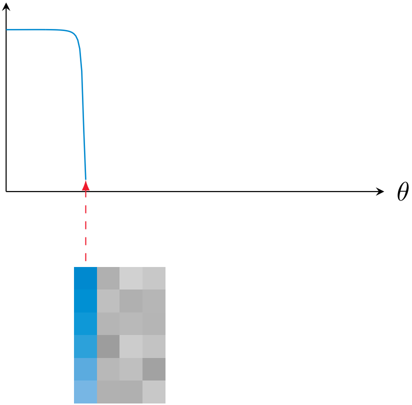


$\left[\boldsymbol{a}(\theta),\boldsymbol{B}\right]$

Partial Relaxation Techniques
General Concept

$$\min_{\boldsymbol{B}\in\mathbb{C}^{M\times(N-1)}} f\left(\left[\boldsymbol{a}(\theta),\boldsymbol{B}\right]\right)$$

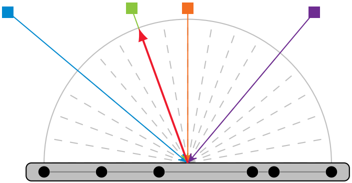


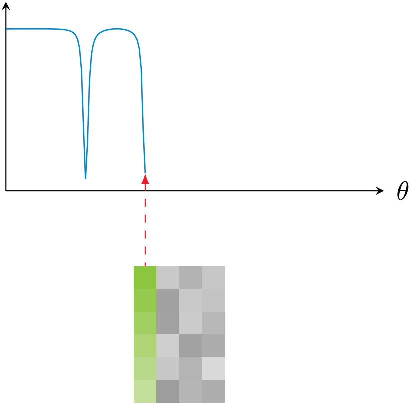


$$\left[\boldsymbol{a}(\theta),\boldsymbol{B}\right]$$

Partial Relaxation Techniques
General Concept

$$\min_{\boldsymbol{B}\in\mathbb{C}^{M\times(N-1)}} f\left(\left[\boldsymbol{a}(\theta),\boldsymbol{B}\right]\right)$$

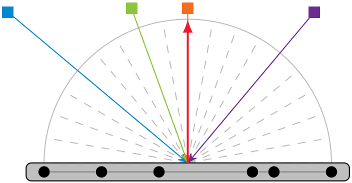


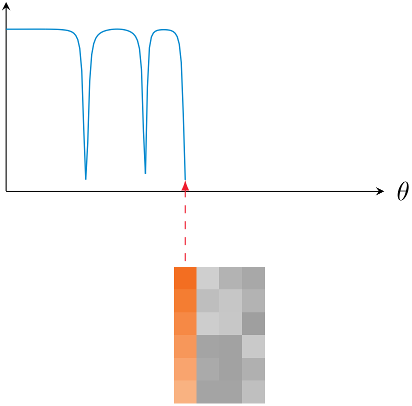


$\left[\boldsymbol{a}(\theta),\boldsymbol{B}\right]$

Partial Relaxation Techniques
General Concept

$$\min_{\boldsymbol{B}\in\mathbb{C}^{M\times(N-1)}} f\left(\left[\boldsymbol{a}(\theta),\boldsymbol{B}\right]\right)$$

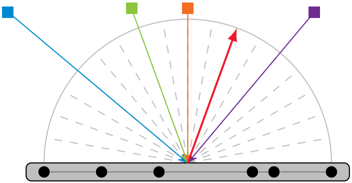


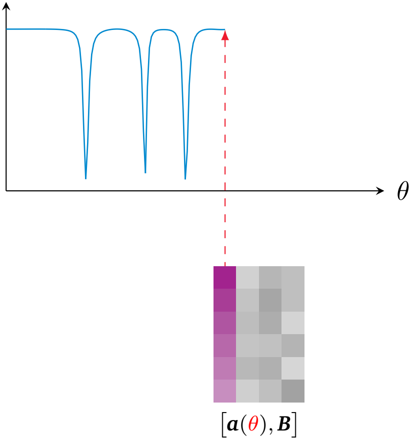


$\left[\boldsymbol{a}({\color{red}\theta}),\boldsymbol{B}\right]$

Partial Relaxation Techniques
General Concept

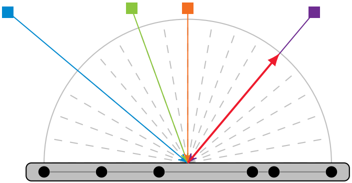
$$\min_{\boldsymbol{B}\in\mathbb{C}^{M\times(N-1)}} f\left(\left[\boldsymbol{a}(\theta),\boldsymbol{B}\right]\right)$$

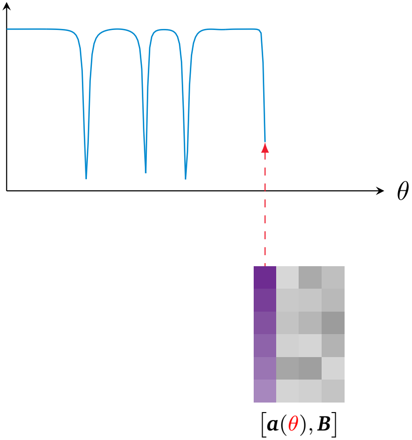




Partial Relaxation Techniques
General Concept

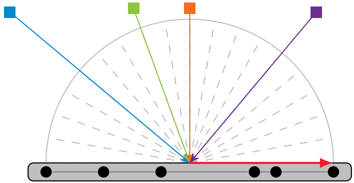
$$\min_{\boldsymbol{B}\in\mathbb{C}^{M\times(N-1)}} f\left(\left[\boldsymbol{a}(\theta),\boldsymbol{B}\right]\right)$$

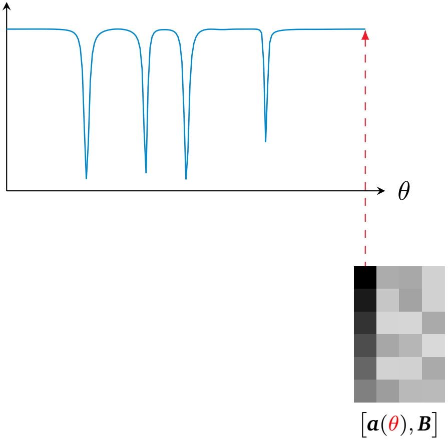




Partial Relaxation Techniques
General Concept

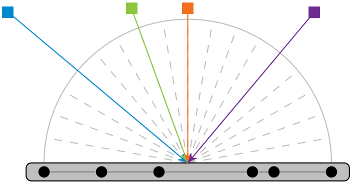
$$\min_{\boldsymbol{B}\in\mathbb{C}^{M\times(N-1)}} f\left(\left[\boldsymbol{a}(\theta),\boldsymbol{B}\right]\right)$$

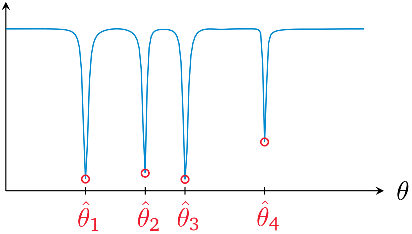




Partial Relaxation Techniques
General Concept

$$\min_{\boldsymbol{B}\in\mathbb{C}^{M\times(N-1)}} f\left(\left[\boldsymbol{a}(\theta),\boldsymbol{B}\right]\right)$$





- Relax the manifold structure of the signals from “interfering” directions.
- Generally lower complexity than multi-dimensional search.

Partial Relaxation Techniques
PR Deterministic Maximum Likelihood

Recall the DML estimator

$$\left\{\hat{\boldsymbol{A}}_{\text{DML}}\right\} = \underset{\boldsymbol{A}\in\mathcal{A}_N}{\arg\min}\ \text{Tr}\left(\boldsymbol{\Pi}_{\boldsymbol{A}}^{\perp}\hat{\boldsymbol{R}}\right)$$

Partially-relaxed (PR) Formulation

$$\begin{aligned}\{\hat{\boldsymbol{a}}_{\text{PR-DML}}\} &= {}^{N}\underset{\boldsymbol{a}\in\mathcal{A}_1}{\arg\min}\ \min_{\boldsymbol{B}\in\mathbb{C}^{M\times(N-1)}} \text{Tr}\left(\boldsymbol{\Pi}_{[\boldsymbol{a},\boldsymbol{B}]}^{\perp}\hat{\boldsymbol{R}}\right)\\ &= {}^{N}\underset{\boldsymbol{a}\in\mathcal{A}_1}{\arg\min}\ \min_{\boldsymbol{B}\in\mathbb{C}^{M\times(N-1)}} \text{Tr}\left(\boldsymbol{\Pi}_{\boldsymbol{a}}^{\perp}\hat{\boldsymbol{R}}\right) - \text{Tr}\left(\boldsymbol{\Pi}_{\boldsymbol{\Pi}_{\boldsymbol{a}}^{\perp}\boldsymbol{B}}\hat{\boldsymbol{R}}\right)\end{aligned}$$

Null-spectrum of the PR-DML Estimator with $\boldsymbol{a} = \boldsymbol{a}(\theta)$

$$f_{\text{PR-DML}}(\theta) = \text{Tr}\left(\boldsymbol{\Pi}_{\boldsymbol{a}}^{\perp}\hat{\boldsymbol{R}}\right) - \max_{\boldsymbol{B}\in\mathbb{C}^{M\times(N-1)}} \text{Tr}\left(\boldsymbol{\Pi}_{\boldsymbol{\Pi}_{\boldsymbol{a}}^{\perp}\boldsymbol{B}}\hat{\boldsymbol{R}}\right)$$

Partial Relaxation Techniques
PR Deterministic Maximum Likelihood

New Optimization Problem

$$\max_{\boldsymbol{B} \in \mathbb{C}^{M \times (N-1)}} \operatorname{Tr}\left(\boldsymbol{\Pi}_{\boldsymbol{\Pi}_{\boldsymbol{a}}^{\perp}\boldsymbol{B}} \hat{\boldsymbol{R}}\right)$$

Eigenvalue Decomposition of $\boldsymbol{\Pi}_{\boldsymbol{\Pi}_{\boldsymbol{a}}^{\perp}\boldsymbol{B}}$

$$\boldsymbol{\Pi}_{\boldsymbol{\Pi}_{\boldsymbol{a}}^{\perp}\boldsymbol{B}} = \boldsymbol{Z}\boldsymbol{Z}^{\mathsf{H}} \text{ with } \boldsymbol{Z} \in \mathbb{C}^{M \times K}$$

- $\operatorname{rank}\left(\boldsymbol{\Pi}_{\boldsymbol{\Pi}_{\boldsymbol{a}}^{\perp}\boldsymbol{B}}\right) = K \leq N - 1$
- $\boldsymbol{Z}^{\mathsf{H}}\boldsymbol{a} = \boldsymbol{0}$

Equivalent Reformulation

$$\max_{\boldsymbol{Z} \in \mathbb{C}^{M \times K}} \operatorname{Tr}\left(\boldsymbol{Z}^{\mathsf{H}}\boldsymbol{\Pi}_{\boldsymbol{a}}^{\perp}\hat{\boldsymbol{R}}\boldsymbol{\Pi}_{\boldsymbol{a}}^{\perp}\boldsymbol{Z}\right) = \sum_{k=1}^{N-1} \lambda_k(\boldsymbol{\Pi}_{\boldsymbol{a}}^{\perp}\hat{\boldsymbol{R}}\boldsymbol{\Pi}_{\boldsymbol{a}}^{\perp}) = \sum_{k=1}^{N-1} \lambda_k(\boldsymbol{\Pi}_{\boldsymbol{a}}^{\perp}\hat{\boldsymbol{R}})$$

$$\text{subject to } \boldsymbol{Z}^{\mathsf{H}}\boldsymbol{a} = \boldsymbol{0}, \quad \boldsymbol{Z}^{\mathsf{H}}\boldsymbol{Z} = \boldsymbol{I}$$

Partial Relaxation Techniques
PR Deterministic Maximum Likelihood

Null-spectrum of the PR-DML Estimator

$$f_{\text{PR-DML}}(\theta) = \mathrm{Tr}\left(\boldsymbol{\Pi}^{\perp}_{\boldsymbol{a}(\theta)}\hat{\boldsymbol{R}}\right) - \max_{\boldsymbol{B}\in\mathbb{C}^{M\times(N-1)}} \mathrm{Tr}\left(\boldsymbol{\Pi}_{\boldsymbol{\Pi}^{\perp}_{\boldsymbol{a}(\theta)}\boldsymbol{B}}\hat{\boldsymbol{R}}\right)$$

$$= \sum_{k=N}^{M} \lambda_k(\boldsymbol{\Pi}^{\perp}_{\boldsymbol{a}(\theta)}\hat{\boldsymbol{R}})$$

$$= \sum_{k=N}^{M} \lambda_k\left(\hat{\boldsymbol{R}} - \frac{1}{||\boldsymbol{a}(\theta)||^2}\hat{\boldsymbol{R}}^{1/2}\boldsymbol{a}(\theta)\boldsymbol{a}^{\mathsf{H}}(\theta)\hat{\boldsymbol{R}}^{1/2}\right)$$

Remarks

- Multiple minimizers for $\boldsymbol{B}$.
- Closed-form expressions for the null-spectrum.
- $(M-N+1)$- smallest eigenvalues are required.

Partial Relaxation Techniques
PR Deterministic Maximum Likelihood

Alternative Derivation of Null-spectrum of PR-DML

$$f_{\text{PR-DML}}(\theta) = \min_{\boldsymbol{B}\in\mathbb{C}^{M\times(N-1)}} \operatorname{Tr}\left(\boldsymbol{\Pi}^{\perp}_{[\boldsymbol{a}(\theta),\boldsymbol{B}]}\hat{\boldsymbol{R}}\right)$$

$$= \min_{\boldsymbol{B}\in\mathbb{C}^{M\times(N-1)}} \min_{\boldsymbol{s}\in\mathbb{C}^{T\times 1},\boldsymbol{H}\in\mathbb{C}^{(N-1)\times T}} \frac{1}{T}\left|\left|\boldsymbol{X}-\boldsymbol{a}(\theta)\boldsymbol{s}^{\mathsf{T}}-\boldsymbol{B}\boldsymbol{H}\right|\right|_{\mathrm{F}}^{2}$$

Substitute $\boldsymbol{E}=\boldsymbol{B}\boldsymbol{H}$ and Concentrate with Respect to $\boldsymbol{s}$

$$f_{\text{PR-DML}}(\theta) = \min_{\operatorname{rank}(\boldsymbol{E})\leq N-1} \frac{1}{T}\left|\left|\boldsymbol{\Pi}^{\perp}_{\boldsymbol{a}(\theta)}\boldsymbol{X}-\boldsymbol{\Pi}^{\perp}_{\boldsymbol{a}(\theta)}\boldsymbol{E}\right|\right|_{\mathrm{F}}^{2}$$

$$= \frac{1}{T}\sum_{k=N}^{M}\sigma_k^2\left(\boldsymbol{\Pi}^{\perp}_{\boldsymbol{a}(\theta)}\boldsymbol{X}\right)$$

$$= \sum_{k=N}^{M}\lambda_k\left(\boldsymbol{\Pi}^{\perp}_{\boldsymbol{a}(\theta)}\hat{\boldsymbol{R}}\right)$$

Partial Relaxation Techniques
PR Weighted Subspace Fitting

Recall the WSF estimator

$$\left\{\hat{\boldsymbol{A}}_{\text{WSF}}\right\} = \underset{\boldsymbol{A}\in\mathcal{A}_N}{\arg\min}\ \text{Tr}\left(\boldsymbol{\Pi}_{\boldsymbol{A}}^{\perp}\hat{\boldsymbol{U}}_{\text{s}}\boldsymbol{W}\hat{\boldsymbol{U}}_{\text{s}}^{\mathsf{H}}\right)$$

Partially-relaxed (PR) Formulation

$$\{\hat{\boldsymbol{a}}_{\text{PR-WSF}}\} = {}^{N}\underset{\boldsymbol{a}\in\mathcal{A}_1}{\arg\min}\ \min_{\boldsymbol{B}\in\mathbb{C}^{M\times(N-1)}} \text{Tr}\left(\boldsymbol{\Pi}_{[\boldsymbol{a},\boldsymbol{B}]}^{\perp}\hat{\boldsymbol{U}}_{\text{s}}\boldsymbol{W}\hat{\boldsymbol{U}}_{\text{s}}^{\mathsf{H}}\right)$$

Null-spectrum of the PR-WSF Estimator

$$f_{\text{PR-WSF}}(\theta) = \lambda_N\left(\boldsymbol{\Pi}_{\boldsymbol{a}(\theta)}^{\perp}\hat{\boldsymbol{U}}_{\text{s}}\boldsymbol{W}\hat{\boldsymbol{U}}_{\text{s}}^{\mathsf{H}}\right)$$

- Only one eigenvalue required.
- PR-WSF with $\boldsymbol{W} = \boldsymbol{I}$ is equivalent to MUSIC estimator.

Partial Relaxation Techniques
PR Constrained Covariance Fitting

Recall the Covariance Matrix $\boldsymbol{R}$

$$\begin{aligned}\boldsymbol{R} &= \boldsymbol{A}\boldsymbol{P}\boldsymbol{A}^{\mathsf{H}} + \nu\boldsymbol{I} \\ &= \begin{bmatrix}\boldsymbol{a} & \boldsymbol{B}\end{bmatrix}\begin{bmatrix}\sigma_s^2 & \boldsymbol{\rho}^{\mathsf{H}} \\ \boldsymbol{\rho} & \boldsymbol{Q}\end{bmatrix}\begin{bmatrix}\boldsymbol{a}^{\mathsf{H}} \\ \boldsymbol{B}^{\mathsf{H}}\end{bmatrix} + \nu\boldsymbol{I}\end{aligned}$$

Formulation of PR-Constrained Covariance Fitting (PR-CCF)

$$\{\hat{\boldsymbol{a}}_{\text{PR-CCF}}\} = {}^{N}\underset{\boldsymbol{a}\in\mathcal{A}_1}{\arg\min}\ \min_{\boldsymbol{B},\sigma_s^2\geq 0,\boldsymbol{Q}\succeq\boldsymbol{0}} \left\|\hat{\boldsymbol{R}} - \sigma_s^2\boldsymbol{a}\boldsymbol{a}^{\mathsf{H}} - \boldsymbol{B}\boldsymbol{Q}\boldsymbol{B}^{\mathsf{H}}\right\|_{\mathrm{F}}^2$$

$$\text{subject to } \hat{\boldsymbol{R}} - \sigma_s^2\boldsymbol{a}\boldsymbol{a}^{\mathsf{H}} - \boldsymbol{B}\boldsymbol{Q}\boldsymbol{B}^{\mathsf{H}} \succeq \boldsymbol{0}$$

- Neglect the correlation between source signals.
- Replace the noise component with the positive-semidefinite constraint.

Partial Relaxation Techniques
PR Constrained Covariance Fitting

Equivalent formulation of the inner optimization

$$\min_{\sigma_s^2 \geq 0} \sum_{k=N}^{M} \lambda_k^2 \left(\hat{\boldsymbol{R}} - \sigma_s^2 \boldsymbol{a}\boldsymbol{a}^{\mathsf{H}}\right)$$

$$\text{subject to } \hat{\boldsymbol{R}} - \sigma_s^2 \boldsymbol{a}\boldsymbol{a}^{\mathsf{H}} \succeq \boldsymbol{0}$$

Closed-form solution for the minimizer $\hat{\sigma}^2_{\mathrm{s,\,C}}$

$$\hat{\sigma}^2_{\mathrm{s,\,C}} = \frac{1}{\boldsymbol{a}^{\mathsf{H}} \hat{\boldsymbol{R}}^{-1} \boldsymbol{a}}$$

Null-spectrum of the PR-CCF Estimator

$$f_{\text{PR-CCF}}(\theta) = \sum_{k=N}^{M} \lambda_k^2 \left(\hat{\boldsymbol{R}} - \frac{1}{\boldsymbol{a}^{\mathsf{H}}(\theta) \hat{\boldsymbol{R}}^{-1} \boldsymbol{a}(\theta)} \boldsymbol{a}(\theta) \boldsymbol{a}^{\mathsf{H}}(\theta)\right)$$

Partial Relaxation Techniques
PR Unconstrained Covariance Fitting

Formulation of PR-Unconstrained Covariance Fitting (PR-UCF)

$$\{\hat{\boldsymbol{a}}_{\text{PR-UCF}}\} =^{N} \underset{\boldsymbol{a}\in\mathcal{A}_1}{\arg\min} \min_{\boldsymbol{B},\sigma_{\text{s}}^2\geq 0,\boldsymbol{Q}\succeq\boldsymbol{0}} \left|\left|\hat{\boldsymbol{R}} - \sigma_{\text{s}}^2\boldsymbol{a}\boldsymbol{a}^{\mathsf{H}} - \boldsymbol{B}\boldsymbol{Q}\boldsymbol{B}^{\mathsf{H}}\right|\right|_{\text{F}}^2$$

Null-spectrum of the PR-UCF Estimator with $\boldsymbol{a} = \boldsymbol{a}(\theta)$

$$f_{\text{PR-UCF}}(\theta) = \min_{\sigma_{\text{s}}^2\geq 0} \sum_{k=N}^{M} \lambda_k^2\left(\hat{\boldsymbol{R}} - \sigma_{\text{s}}^2\boldsymbol{a}\boldsymbol{a}^{\mathsf{H}}\right)$$

- No closed-form solution for the minimizer $\hat{\sigma}_{s,\text{U}}^2$.
- $\bar{\lambda}_k(\sigma_{\text{s}}^2) = \lambda_k\left(\hat{\boldsymbol{R}} - \sigma_{\text{s}}^2\boldsymbol{a}\boldsymbol{a}^{\mathsf{H}}\right)$ is continuously differentiable with respect to σ_s^2

$$\frac{\mathrm{d}\bar{\lambda}_k\left(\sigma_{\text{s}}^2\right)}{\mathrm{d}\sigma_{\text{s}}^2} = -\frac{1}{\sigma_{\text{s}}^4\boldsymbol{a}^{\mathsf{H}}\left(\hat{\boldsymbol{R}} - \bar{\lambda}_k(\sigma_{\text{s}}^2)\boldsymbol{I}_M\right)^{-2}\boldsymbol{a}}.$$

Partial Relaxation Techniques
PR Unconstrained Covariance Fitting

Define

$$g(\sigma_\mathrm{s}^2) = \sum_{k=N}^{M} \lambda_k^2 \left(\hat{\boldsymbol{R}} - \sigma_\mathrm{s}^2 \boldsymbol{a}\boldsymbol{a}^\mathsf{H} \right)$$

Objective: Find $\hat{\sigma}_{s,\mathrm{U}}^2$ where the derivative $g'(\sigma_\mathrm{s}^2)$ vanishes

$$g'(\sigma_\mathrm{s}^2) = -\sum_{k=N}^{M} \frac{2\bar{\lambda}_k(\sigma_\mathrm{s}^2)}{\sigma_\mathrm{s}^4 \boldsymbol{a}^\mathsf{H} \left(\hat{\boldsymbol{R}} - \bar{\lambda}_k(\sigma_\mathrm{s}^2)\boldsymbol{I}_M \right)^{-2} \boldsymbol{a}}$$

- If $\sigma_\mathrm{s}^2 \to 0 \implies g'(\sigma_\mathrm{s}^2) < 0$
- If $\sigma_\mathrm{s}^2 \to \infty \implies g(\sigma_\mathrm{s}^2) \approx \sigma_\mathrm{s}^4 \, ||\boldsymbol{a}||_2^4 \implies g'(\sigma_\mathrm{s}^2) > 0$

Solution: Find an interval where $g'(\sigma_\mathrm{s}^2)$ changes sign and perform bisection search

Partial Relaxation Techniques
PR Full Covariance Fitting

Formulation of PR-Full Covariance Fitting (PR-FCF)

$$\{\hat{\boldsymbol{a}}_{\text{PR-UCF}}\} =^{N} \underset{\boldsymbol{a}\in\mathcal{A}_1}{\arg\min} \min_{\boldsymbol{B},\sigma_s^2\geq 0,\boldsymbol{Q}\succeq\boldsymbol{0},\nu\geq 0} \left|\left|\hat{\boldsymbol{R}} - \sigma_s^2 \boldsymbol{a}\boldsymbol{a}^{\mathsf{H}} - \boldsymbol{B}\boldsymbol{Q}\boldsymbol{B} - \nu\boldsymbol{I}\right|\right|_{\text{F}}^2$$

Null-spectrum of the PR-FCF Estimator with $\boldsymbol{a} = \boldsymbol{a}(\theta)$

$$f_{\text{PR-FCF}}(\theta) = \min_{\sigma_s^2\geq 0} \sum_{k=N}^{M} \lambda_k^2\left(\hat{\boldsymbol{R}} - \sigma_s^2\boldsymbol{a}\boldsymbol{a}^{\mathsf{H}}\right) - \frac{\left(\sum_{k=N}^{M}\lambda_k\left(\hat{\boldsymbol{R}} - \sigma_s^2\boldsymbol{a}\boldsymbol{a}^{\mathsf{H}}\right)\right)^2}{M-N+1}$$

- No closed-form solution for the minimizer $\hat{\sigma}_{s,\,\text{F}}^2$.
- Numerical suboptimal solution obtained from Newton's method.

Partial Relaxation Techniques
Insights and Relation

Methods	Multi-dimensional Search	Partial Relaxation	Single-source Approximation
Signal Fitting	DML	PR-DML	Conv. Beamformer
Subspace Fitting	WSF	PR-WSF	Weighted MUSIC
Covariance Fitting	Unweighted COMET	PR-CCF PR-UCF PR-FCF	Capon Beamformer Conv. Beamformer

- Degraded performance of PR methods in the case of correlated signals.
- Null-spectra of PR methods require the computation of eigenvalues.

Explanation of Performance Degradation of PR Methods

Case study: Two fully coherent source signals without sensor noise

$$\begin{aligned} \boldsymbol{X} &= \boldsymbol{a}(\theta_1)\boldsymbol{s}^{\mathsf{T}} + \boldsymbol{a}(\theta_2)\boldsymbol{s}^{\mathsf{T}} \\ &= \Big(\boldsymbol{a}(\theta_1) + \boldsymbol{a}(\theta_2)\Big)\boldsymbol{s}^{\mathsf{T}}. \end{aligned}$$

Null-spectrum of the PR-DML estimator for $N = 2$ source signals

$$f_{\text{PR-DML}}(\theta) = \min_{\boldsymbol{b}\in\mathbb{C}^{M\times 1}} \min_{\boldsymbol{s}\in\mathbb{C}^{T\times 1}, \boldsymbol{h}\in\mathbb{C}^{T\times 1}} \frac{1}{T} \left|\left| \boldsymbol{X} - \boldsymbol{a}(\theta)\boldsymbol{s}^{\mathsf{T}} - \boldsymbol{b}\boldsymbol{h}^{\mathsf{T}} \right|\right|_{\text{F}}^2$$

- Cost function is non-negative.
- Perfect match is achieved if $\boldsymbol{b} = \boldsymbol{a}(\theta_1) + \boldsymbol{a}(\theta_2)$ regardless of θ.
- Flat null-spectrum for all look-direction $\theta \implies$ no reliable DOA estimation.

Null-spectrum of the PR-DML Estimator

$$f_{\text{PR-DML}}(\theta) = \sum_{k=N}^{M} \lambda_k \left(\hat{\boldsymbol{R}} - \frac{1}{||\boldsymbol{a}||^2} \hat{\boldsymbol{R}}^{1/2} \boldsymbol{a}\boldsymbol{a}^{\mathsf{H}} \hat{\boldsymbol{R}}^{1/2} \right) \text{ with } \boldsymbol{a} = \boldsymbol{a}(\theta)$$

Partial Relaxation Techniques
Efficient Implementation

Null-spectrum of the PR-CCF Estimator

$$f_{\text{PR-CCF}}(\theta) = \sum_{k=N}^{M} \lambda_k^2 \left(\hat{\boldsymbol{R}} - \frac{1}{\boldsymbol{a}^{\mathsf{H}} \hat{\boldsymbol{R}}^{-1} \boldsymbol{a}} \boldsymbol{a}\boldsymbol{a}^{\mathsf{H}} \right) \text{ with } \boldsymbol{a} = \boldsymbol{a}(\theta)$$

- Dependent on eigenvalues but not on eigenvectors.
- Similar structure of the matrix argument.

Partial Relaxation Techniques
Efficient Implementation

Null-spectrum of the PR-CCF Estimator

$$f_{\text{PR-CCF}}(\theta) = \sum_{k=N}^{M} \lambda_k^2 \left(\hat{\boldsymbol{R}} - \frac{1}{\boldsymbol{a}^{\mathsf{H}} \hat{\boldsymbol{R}}^{-1} \boldsymbol{a}} \boldsymbol{a}\boldsymbol{a}^{\mathsf{H}} \right) \text{ with } \boldsymbol{a} = \boldsymbol{a}(\theta)$$

- Dependent on eigenvalues but not on eigenvectors.
- Similar structure of the matrix argument.

Core Numerical Problem: Efficient Computation of Eigenvalues

$$\bar{d}_k = \lambda_k \left(\boldsymbol{D} - \bar{\rho} \boldsymbol{z}\boldsymbol{z}^{\mathsf{H}} \right) \text{ with } \rho > 0$$

- $\boldsymbol{D} = \text{diag}(d_1, \ldots, d_K) \in \mathbb{R}^{K \times K}$ with $d_1 > \ldots > d_K$.
- $\boldsymbol{z} = [z_1, \ldots, z_K]^{\mathsf{T}} \in \mathbb{C}^{K \times 1}$ has no zero entry.

Partial Relaxation Techniques
Efficient Implementation

Remarks

- Corresponding to the routine `dlaed4()` in LAPACK [Anderson'99].
- Applicable to PR estimators using orthogonal transformation.
- Adaptive initialization using previous eigenvalues.
- Reduction in execution time using alternative expressions.

Example: PR-DML Estimator

$$\{\hat{\boldsymbol{a}}_{\text{PR-DML}}\} =^{N} \underset{\boldsymbol{a}\in\mathcal{A}_1}{\arg\min} \sum_{k=N}^{M} \lambda_k \left(\hat{\boldsymbol{R}} - \frac{1}{||\boldsymbol{a}||^2} \hat{\boldsymbol{R}}^{1/2} \boldsymbol{a}\boldsymbol{a}^{\mathsf{H}} \hat{\boldsymbol{R}}^{1/2} \right)$$

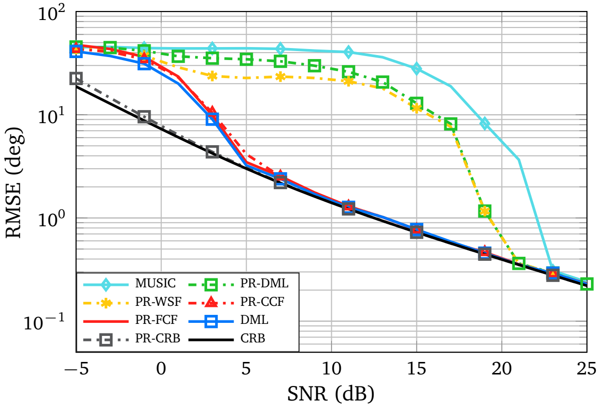
$$=^{N} \underset{\boldsymbol{a}\in\mathcal{A}_1}{\arg\min} \operatorname{Tr}\left(\hat{\boldsymbol{R}}\right) - \frac{\boldsymbol{a}^{\mathsf{H}}\hat{\boldsymbol{R}}\boldsymbol{a}}{\boldsymbol{a}^{\mathsf{H}}\boldsymbol{a}} - \sum_{k=1}^{N-1} \lambda_k \left(\hat{\boldsymbol{\Lambda}} - \frac{1}{||\boldsymbol{a}||_2^2} \hat{\boldsymbol{\Lambda}}^{1/2} \hat{\boldsymbol{U}}^{\mathsf{H}} \boldsymbol{a}\boldsymbol{a}^{\mathsf{H}} \hat{\boldsymbol{U}} \hat{\boldsymbol{\Lambda}}^{1/2} \right)$$

Partial Relaxation Techniques

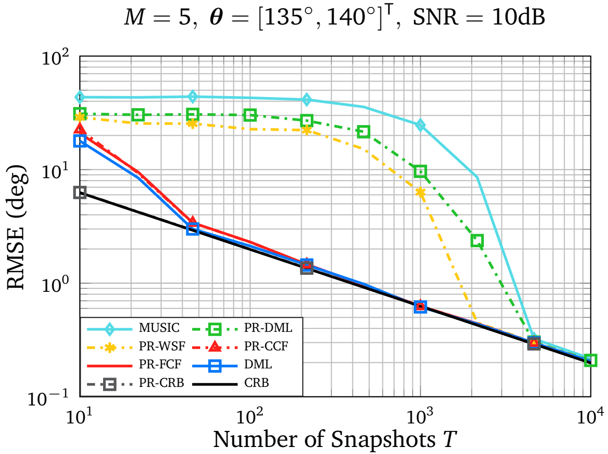
Simulation Results

Uncorrelated Source Signals

$$M = 5,\ \boldsymbol{\theta} = [135^\circ, 140^\circ]^\mathsf{T},\ T = 150$$



Uncorrelated Source Signals



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Shift-Invariant Array

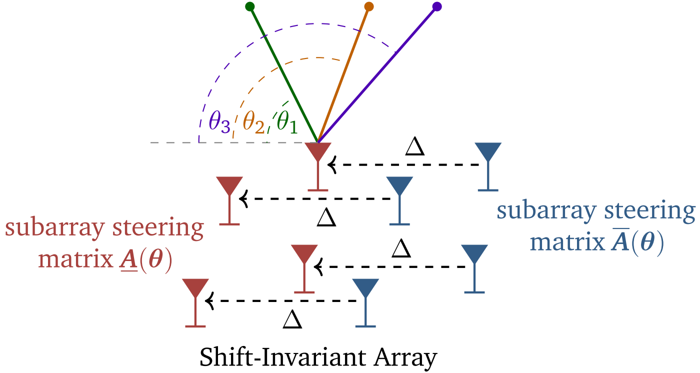


Figure: Antenna array composed of two identical subarrays (subarray 1 in red color) and (subarray 2 in blue color) shifted by baseline Δ.

Relaxation Based on Geometry Exploitation
ESPRIT

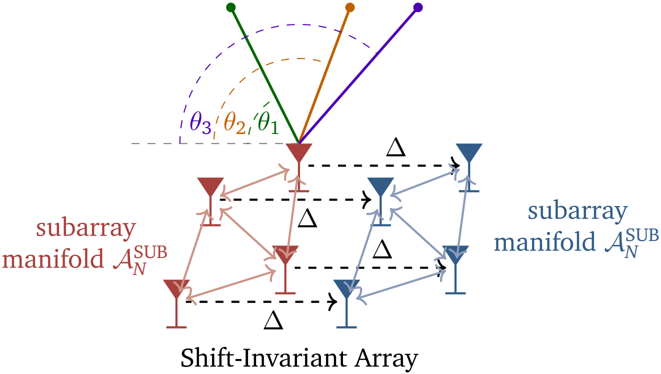


Figure: The subarray displacement (shift) Δ must be known. $\mathcal{A}_N^{\text{SUB}}$ is the manifold of each subarray.

Relaxation Based on Geometry Exploitation
ESPRIT

We assume $\frac{M}{2} \geq N$. Given the steering matrix $\underline{\boldsymbol{A}}(\boldsymbol{\theta}) \in \mathcal{A}_N^{\mathrm{SUB}}$ of the first subarray, the steering matrix $\overline{\boldsymbol{A}}(\boldsymbol{\theta}) \in \mathcal{A}_N^{\mathrm{SUB}}$ of the second subarray can be expressed as

$$\overline{\boldsymbol{A}}(\boldsymbol{\theta}) = \underline{\boldsymbol{A}}(\boldsymbol{\theta})\boldsymbol{D}(\boldsymbol{\theta}), \quad \boldsymbol{D}(\boldsymbol{\theta}) = \operatorname{diag}\left(e^{-\mathrm{j}\frac{2\pi}{\lambda}\Delta\cos(\theta_1)}, e^{-\mathrm{j}\frac{2\pi}{\lambda}\Delta\cos(\theta_2)}, \cdots, e^{-\mathrm{j}\frac{2\pi}{\lambda}\Delta\cos(\theta_N)}\right)$$

The array steering matrix can be decomposed in subarray responses as

$$\boldsymbol{A}(\boldsymbol{\theta}) = \begin{bmatrix} \underline{\boldsymbol{A}}(\boldsymbol{\theta}) \\ \overline{\boldsymbol{A}}(\boldsymbol{\theta}) \end{bmatrix} = \begin{bmatrix} \underline{\boldsymbol{A}}(\boldsymbol{\theta}) \\ \underline{\boldsymbol{A}}(\boldsymbol{\theta})\boldsymbol{D}(\boldsymbol{\theta}) \end{bmatrix}$$

Similarly, let $\boldsymbol{U}_{\mathrm{s}}$ be partitioned as

$$\boldsymbol{U}_{\mathrm{s}} = \begin{bmatrix} \underline{\boldsymbol{U}}_{\mathrm{s}} \\ \overline{\boldsymbol{U}}_{\mathrm{s}} \end{bmatrix}$$

Relaxation Based on Geometry Exploitation
ESPRIT

From an optimization perspective ESPRIT and TLS-ESPRIT can be understood as a subspace matching approach with manifold relaxation.

We consider TLS-ESPRIT: Recall that $\boldsymbol{A}(\boldsymbol{\theta})$ and $\boldsymbol{U}_{\mathrm{s}}$ span the same space and consider the subspace fitting problem

$$f_{\mathrm{WSF}}(\boldsymbol{\theta}) = \min_{\boldsymbol{A}(\boldsymbol{\theta}) \in \mathcal{A}_N} \min_{\boldsymbol{K} \in \mathbb{C}^{N \times N}} \|\hat{\boldsymbol{U}}_{\mathrm{s}} - \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{K}\|_{\mathsf{F}}^2$$

which involves a multi-dimensional multi-modal optimization over the manifold $\mathcal{A}_N$:

$$\mathcal{A}_N = \left\{ \boldsymbol{A} \in \mathbb{C}^{M \times N} \middle| \begin{bmatrix} \underline{\boldsymbol{A}}(\boldsymbol{\vartheta}) \\ \underline{\boldsymbol{A}}(\boldsymbol{\vartheta})\boldsymbol{D}(\boldsymbol{\vartheta}) \end{bmatrix}, \underline{\boldsymbol{A}}(\boldsymbol{\vartheta}) \in \mathcal{A}_N^{\mathrm{SUB}}, \boldsymbol{\vartheta} \in \Omega^N \right\}$$

Relaxation Based on Geometry Exploitation
ESPRIT

To make the problem tractable the original array manifold $\mathcal{A}_N$ is replaced by the relaxed manifold $\mathcal{A}_N^{\text{ESPRIT}}$

$$\mathcal{A}_N^{\text{ESPRIT}} = \left\{ \boldsymbol{A} \in \mathbb{C}^{M \times N} \middle| \, \boldsymbol{A} = \begin{bmatrix} \underline{\boldsymbol{A}} \\ \underline{\boldsymbol{A}}\boldsymbol{D} \end{bmatrix}, \, \underline{\boldsymbol{A}} \in \mathbb{C}^{\frac{M}{2} \times N}, \boldsymbol{D} \in \mathbb{D}^{N \times N} \right\}$$

where $\underline{\boldsymbol{A}} \in \mathbb{C}^{\frac{M}{2} \times N}$ is an arbitrary complex matrix and $\boldsymbol{D}$ an arbitrary diagonal matrix parameterized as

$$\boldsymbol{D}(\boldsymbol{\vartheta}, \boldsymbol{r}) = \operatorname{diag}\left(r_1 e^{-\mathrm{j}\frac{2\pi}{\lambda}\Delta\cos(\vartheta_1)}, r_2 e^{-\mathrm{j}\frac{2\pi}{\lambda}\Delta\cos(\vartheta_2)}, \cdots, r_N e^{-\mathrm{j}\frac{2\pi}{\lambda}\Delta\cos(\vartheta_N)} \right)$$

with $\boldsymbol{r} = [r_1, r_2, \ldots, r_N]^{\mathsf{T}} \in \mathbb{R}_+^N$.

Relaxation Based on Geometry Exploitation
ESPRIT

The subspace fitting problem over manifold $\mathcal{A}_N^{\text{ESPRIT}}$ can also be written as the Total Least Squares (TLS) ESPRIT problem:

$$
\begin{aligned}
&\min_{\boldsymbol{A}\in\mathcal{A}_N^{\text{ESPRIT}}} \min_{\boldsymbol{K}\in\mathbb{C}^{N\times N}} \left\|\hat{\boldsymbol{U}}_{\text{s}} - \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{K}\right\|_{\mathsf{F}}^2 \\
&= \min_{\boldsymbol{D}\in\mathbb{D}^{N\times N}} \min_{\boldsymbol{K}\in\mathbb{C}^{N\times N}} \min_{\underline{\boldsymbol{A}}\in\mathbb{C}^{(M/2)\times N}} \left(\left\| \left[\underline{\hat{\boldsymbol{U}}}_{\text{s}}, \bar{\hat{\boldsymbol{U}}}_{\text{s}}\right] - \underline{\boldsymbol{A}}\left[\boldsymbol{K}, \boldsymbol{D}\boldsymbol{K}\right] \right\|_{\mathsf{F}}^2\right) \\
&= \min_{\boldsymbol{D}\in\mathbb{D}^{N\times N}} \min_{\boldsymbol{K}\in\mathbb{C}^{N\times N}} \min_{\underline{\boldsymbol{A}}\in\mathbb{C}^{(M/2)\times N}} \left(\left\| \left[\underline{\hat{\boldsymbol{U}}}_{\text{s}}, \bar{\hat{\boldsymbol{U}}}_{\text{s}}\right] - \left[\underline{\breve{\boldsymbol{U}}}_{\text{s}}, \bar{\breve{\boldsymbol{U}}}_{\text{s}}\right] \right\|_{\mathsf{F}}^2\right) \\
&\qquad\qquad\text{subject to} \quad \underline{\breve{\boldsymbol{U}}}_{\text{s}} = \underline{\boldsymbol{A}}\boldsymbol{K} \\
&\qquad\qquad\qquad\qquad\quad\; \bar{\breve{\boldsymbol{U}}}_{\text{s}} = \underline{\boldsymbol{A}}\boldsymbol{D}\boldsymbol{K}
\end{aligned}
$$

Relaxation Based on Geometry Exploitation
ESPRIT

If the source signals are not coherent, i.e., $\boldsymbol{K}$ is an invertible matrix, we can rewrite the previous optimization problem as follows:

$$\min_{\boldsymbol{D}\in\mathbb{D}^{N\times N}} \min_{\boldsymbol{K}\in\mathbb{C}^{N\times N}} \min_{\left[\underline{\breve{\boldsymbol{U}}}_{\mathrm{s}},\overline{\breve{\boldsymbol{U}}}_{\mathrm{s}}\right]\in\mathbb{C}^{\frac{M}{2}\times 2N}} \left(\left\| \left[\underline{\hat{\boldsymbol{U}}}_{\mathrm{s}},\overline{\hat{\boldsymbol{U}}}_{\mathrm{s}}\right] - \left[\underline{\breve{\boldsymbol{U}}}_{\mathrm{s}},\overline{\breve{\boldsymbol{U}}}_{\mathrm{s}}\right] \right\|_{\mathsf{F}}^{2}\right)$$

$$\text{subject to} \quad \overline{\breve{\boldsymbol{U}}}_{\mathrm{s}} = \underline{\breve{\boldsymbol{U}}}_{\mathrm{s}}\boldsymbol{K}^{-1}\boldsymbol{D}\boldsymbol{K}$$

$$= \min_{\boldsymbol{D}\in\mathbb{D}^{N\times N}} \min_{\boldsymbol{K}\in\mathbb{C}^{N\times N}} \min_{\left[\underline{\breve{\boldsymbol{U}}}_{\mathrm{s}},\overline{\breve{\boldsymbol{U}}}_{\mathrm{s}}\right]\in\mathbb{C}^{\frac{M}{2}\times 2N}} \left(\left\| \left[\underline{\hat{\boldsymbol{U}}}_{\mathrm{s}},\overline{\hat{\boldsymbol{U}}}_{\mathrm{s}}\right] - \left[\underline{\breve{\boldsymbol{U}}}_{\mathrm{s}},\overline{\breve{\boldsymbol{U}}}_{\mathrm{s}}\right] \right\|_{\mathsf{F}}^{2}\right)$$

$$\text{subject to} \quad \left[\underline{\breve{\boldsymbol{U}}}_{\mathrm{s}},\overline{\breve{\boldsymbol{U}}}_{\mathrm{s}}\right]\begin{bmatrix}\boldsymbol{K}^{-1}\boldsymbol{D}\boldsymbol{K}\\ -\boldsymbol{I}_N\end{bmatrix} = \boldsymbol{0}_{\frac{M}{2}\times N}$$

It follows from the constraint that the solution $\left[\underline{\breve{\boldsymbol{U}}}_{\mathrm{s}}^{\star},\overline{\breve{\boldsymbol{U}}}_{\mathrm{s}}^{\star}\right]$ of the inner optimization problem satisfies

$$\operatorname{rank}\left(\left[\underline{\breve{\boldsymbol{U}}}_{\mathrm{s}}^{\star},\overline{\breve{\boldsymbol{U}}}_{\mathrm{s}}^{\star}\right]\right) \leq N$$

Consequence

The minimizer $\left[\underline{\breve{\boldsymbol{U}}}_{\mathrm{s}}^{\star}, \overline{\breve{\boldsymbol{U}}}_{\mathrm{s}}^{\star}\right]$ is the best rank-N approximation of $\left[\underline{\hat{\boldsymbol{U}}}_{\mathrm{s}}, \overline{\hat{\boldsymbol{U}}}_{\mathrm{s}}\right]$

- Defining the Singular Value Decomposition

$$\left[\underline{\hat{\boldsymbol{U}}}_{\mathrm{s}}, \overline{\hat{\boldsymbol{U}}}_{\mathrm{s}}\right] = \sum_{k=1}^{2N} \sigma_k \boldsymbol{g}_k \boldsymbol{h}_k^{\mathsf{H}}$$

  with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_{2N}$, the minimizer $\left[\underline{\breve{\boldsymbol{U}}}_{\mathrm{s}}^{\star}, \overline{\breve{\boldsymbol{U}}}_{\mathrm{s}}^{\star}\right]$ is given by

$$\left[\underline{\breve{\boldsymbol{U}}}_{\mathrm{s}}^{\star}, \overline{\breve{\boldsymbol{U}}}_{\mathrm{s}}^{\star}\right] = \sum_{k=1}^{N} \sigma_k \boldsymbol{g}_k \boldsymbol{h}_k^{\mathsf{H}}.$$

- From the constraint

$$\left[\underline{\breve{\boldsymbol{U}}}_{\mathrm{s}}^{\star}, \overline{\breve{\boldsymbol{U}}}_{\mathrm{s}}^{\star}\right] \begin{bmatrix} \boldsymbol{K}^{-1}\boldsymbol{D}\boldsymbol{K} \\ -\boldsymbol{I}_N \end{bmatrix} = \boldsymbol{0}_{\frac{M}{2} \times N} \quad \Rightarrow \quad \hat{\boldsymbol{\Psi}} = \boldsymbol{K}^{-1}\boldsymbol{D}\boldsymbol{K} = \left(\underline{\breve{\boldsymbol{U}}}_{\mathrm{s}}^{\star\mathsf{H}} \underline{\breve{\boldsymbol{U}}}_{\mathrm{s}}^{\star}\right)^{-1} \underline{\breve{\boldsymbol{U}}}_{\mathrm{s}}^{\star\mathsf{H}} \overline{\breve{\boldsymbol{U}}}_{\mathrm{s}}^{\star}$$

- The eigenvalues of $\boldsymbol{\Psi}$ form the diagonal element of $\hat{\boldsymbol{D}}_{\mathrm{TLS-ESPRIT}}$.

To summarize, the TLS-ESPRIT algorithm is carried out in the following steps:

Step 1: Compute the eigendecomposition of the sample covariance matrix $\hat{\boldsymbol{R}}$ and obtain the sample signal-subspace $\hat{\boldsymbol{U}}_{\mathrm{s}}$ and form the partitions $\overline{\hat{\boldsymbol{U}}}_{\mathrm{s}}$ and $\underline{\hat{\boldsymbol{U}}}_{\mathrm{s}}$.

Step 2: Compute the best rank-N approximation $\left[\underline{\breve{\boldsymbol{U}}}_{\mathrm{s}}^{\star}, \overline{\breve{\boldsymbol{U}}}_{\mathrm{s}}^{\star}\right]$.

Step 3: Compute

$$\hat{\boldsymbol{\Psi}} = (\underline{\breve{\boldsymbol{U}}}_{\mathrm{s}}^{\star\mathsf{H}} \underline{\breve{\boldsymbol{U}}}_{\mathrm{s}}^{\star})^{-1} \underline{\breve{\boldsymbol{U}}}_{\mathrm{s}}^{\star\mathsf{H}} \overline{\breve{\boldsymbol{U}}}_{\mathrm{s}}$$

Step 4: Find the eigenvalues $\lambda_n(\hat{\boldsymbol{\Psi}})$ of $\hat{\boldsymbol{\Psi}}$ and determine DOA estimates as $\hat{\theta}_{n,\mathrm{ESPRIT}} = \arccos\left(-\frac{\lambda}{2\pi\Delta} \arg\left(\lambda_n(\hat{\boldsymbol{\Psi}})\right)\right)$, for $n = 1, \ldots, N$.

Relaxation Based on Geometry Exploitation
ESPRIT

Recall the Formulation of TLS-ESPRIT

$$\hat{\boldsymbol{\theta}}_{\text{TLS}-\text{ESPRIT}} = \underset{\boldsymbol{A}(\boldsymbol{\theta}) \in \mathcal{A}_N^{\text{ESPRIT}}}{\arg\min} \min_{\boldsymbol{K} \in \mathbb{C}^{N \times N}} \left\| \hat{\boldsymbol{U}}_{\mathsf{s}} - \boldsymbol{A}(\boldsymbol{\theta}) \boldsymbol{K} \right\|_{\mathsf{F}}^2$$

Formulation of (conventional) Least Squares (LS-)ESPRIT

$$\hat{\boldsymbol{\theta}}_{\text{ESPRIT}} = \underset{\boldsymbol{A}(\boldsymbol{\theta}) \in \mathcal{A}_N^{\text{ESPRIT}}}{\arg\min} \min_{\boldsymbol{K} \in \mathbb{C}^{N \times N}} \left\| \hat{\boldsymbol{U}}_{\mathsf{s}} \boldsymbol{K}^{-1} - \boldsymbol{A}(\boldsymbol{\theta}) \right\|_{\mathsf{F}}^2$$

- Both LS-ESPRIT and TLS-ESPRIT technique are search-free approaches.
- The subarray manifold must not be known.

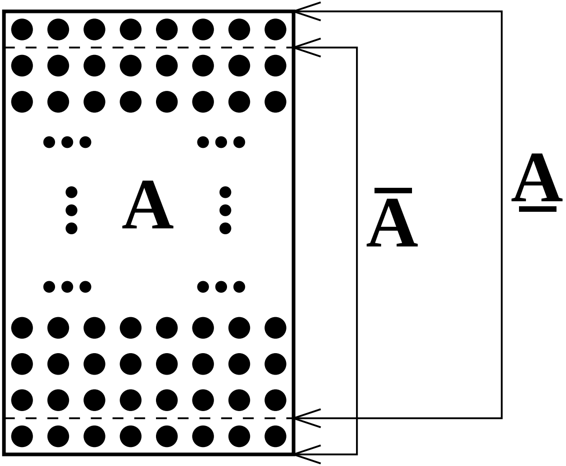
Relaxation Based on Geometry Exploitation
ESPRIT

In the ESPRIT algorithm the subarrays can also overlap, such as in the case of ULA:

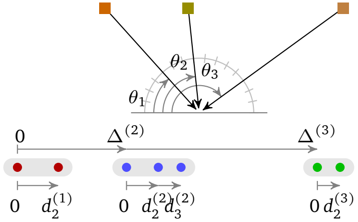
$$\boldsymbol{A}(\boldsymbol{\theta}) = \begin{bmatrix} 1 & 1 & \dots & 1 \\ e^{-\mathrm{j}\frac{2\pi}{\lambda}d\cos(\theta_1)} & e^{-\mathrm{j}\frac{2\pi}{\lambda}d\cos(\theta_2)} & \dots & e^{-\mathrm{j}\frac{2\pi}{\lambda}d\cos(\theta_N)} \\ \vdots & \vdots & & \vdots \\ e^{-\mathrm{j}\frac{2\pi}{\lambda}(M-1)d\cos(\theta_1)} & e^{-\mathrm{j}\frac{2\pi}{\lambda}(M-1)d\cos(\theta_2)} & \dots & e^{-\mathrm{j}\frac{2\pi}{\lambda}(M-1)d\cos(\theta_N)} \end{bmatrix}$$

with partition $\overline{\boldsymbol{A}}(\boldsymbol{\theta})$ and $\underline{\boldsymbol{A}}(\boldsymbol{\theta})$ denoting the matrices with eliminated first and last row, respectively.

Relaxation Based on Geometry Exploitation
ESPRIT



Relaxation Based on Geometry Exploitation
Partly Calibrated Array



- Partition array into P subarrays, with sensor positions

$$d_{\sum_{l=1}^{p-1} M_l + m} = d_m^{(p)} + \Delta^{(p)}$$

- Reverse setup as in ESPRIT:
  - known intra-subarray sensor positions $d_m^{(p)}$ and
  - unknown inter-subarray displacements $\Delta^{(p)}$
  - $\underline{\boldsymbol{d}} = [\boldsymbol{d}_1^\mathsf{T}, \boldsymbol{d}_2^\mathsf{T}, \ldots, \boldsymbol{d}_P^\mathsf{T}]^\mathsf{T}$ with $\boldsymbol{d}_p = [d_1^{(p)}, \ldots, d_{M_p}^{(p)}]^\mathsf{T}$ where M_p is number of sensors in p-th subarray.

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The array response of the p-th subarray for a source at DOA θ can be characterized as

$$\boldsymbol{a}_p(\theta) = [1, e^{-\mathrm{j}\frac{2\pi}{\lambda} d_p^{(2)} \cos(\theta)}, \ldots, e^{-\mathrm{j}\frac{2\pi}{\lambda} d_p^{(P)} \cos(\theta)}]^{\mathsf{T}}.$$

Let

$$\mathcal{A}_N^{(p)} = \left\{ \boldsymbol{A}_p \in \mathbb{C}^{M_p \times N} \middle| \, \boldsymbol{A}_p = [\boldsymbol{a}_p(\vartheta_1), \ldots, \boldsymbol{a}_p(\vartheta_N)] \text{ with } \vartheta_1 < \ldots < \vartheta_N \in \Theta \right\}$$

denote the array manifold corresponding to the p-th subarray.

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The overall array response is then characterized as

$$\begin{aligned}\boldsymbol{a}(\theta) &= [\boldsymbol{a}_1^{\mathsf{T}}(\theta), e^{-\mathrm{j}\frac{2\pi}{\lambda}\Delta^{(2)}\cos(\theta)}\boldsymbol{a}_2^{\mathsf{T}}(\theta), \ldots, e^{-\mathrm{j}\frac{2\pi}{\lambda}\Delta^{(P)}\cos(\theta)}\boldsymbol{a}_P^{\mathsf{T}}(\theta)]^{\mathsf{T}} \\ &= \underbrace{\begin{bmatrix} \boldsymbol{a}_1(\theta) & \mathbf{0}_{M_1\times 1} & \cdots & \mathbf{0}_{M_1\times 1} \\ \mathbf{0}_{M_2\times 1} & \boldsymbol{a}_2(\theta) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0}_{M_{P-1}\times 1} \\ \mathbf{0}_{M_P\times 1} & \cdots & \mathbf{0}_{M_P\times 1} & \boldsymbol{a}_P(\theta) \end{bmatrix}}_{\boldsymbol{T}(\theta)} \underbrace{\begin{bmatrix} 1 \\ e^{-\mathrm{j}\frac{2\pi}{\lambda}\Delta^{(2)}\cos(\theta)} \\ \vdots \\ e^{-\mathrm{j}\frac{2\pi}{\lambda}\Delta^{(P)}\cos(\theta)} \end{bmatrix}}_{\boldsymbol{h}(\theta,\boldsymbol{\Delta})}\end{aligned}$$

where $\boldsymbol{\Delta} = [\Delta^{(2)}, \ldots, \Delta^{(P)}]^{\mathsf{T}} \in \mathbb{R}^{(P-1)\times 1}$.

Relaxation Based on Geometry Exploitation

Partly Calibrated Array

Defining the block-diagonal subarray responses matrix

$$\boldsymbol{T}(\theta) = \begin{bmatrix} \boldsymbol{a}_1(\theta) & \mathbf{0}_{M_1 \times 1} & \cdots & \mathbf{0}_{M_1 \times 1} \\ \mathbf{0}_{M_2 \times 1} & \boldsymbol{a}_2(\theta) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0}_{M_{P-1} \times 1} \\ \mathbf{0}_{M_P \times 1} & \cdots & \mathbf{0}_{M_P \times 1} & \boldsymbol{a}_P(\theta) \end{bmatrix}$$

and the reference sensor steering vector

$$\boldsymbol{h}(\theta) = \left[1, e^{-\mathrm{j}\frac{2\pi}{\lambda}\Delta^{(2)}\cos(\theta)}, \cdots, e^{-\mathrm{j}\frac{2\pi}{\lambda}\Delta^{(P)}\cos(\theta)}\right]^{\mathsf{T}}$$

we can factorize the array response vector as

$$\boldsymbol{a}(\theta) = \boldsymbol{T}(\theta)\boldsymbol{h}(\theta, \boldsymbol{\Delta}).$$

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The overall array manifold depends on the subarray displacements vector $\boldsymbol{\Delta}$:

$$\mathcal{A}_N = \Big\{\boldsymbol{A} = [\boldsymbol{T}_1\boldsymbol{h}_1, \ldots, \boldsymbol{T}_N\boldsymbol{h}_N] \in \mathbb{C}^{M\times N} \Big| \; \boldsymbol{T}_n = \boldsymbol{T}(\vartheta_n) \in \mathcal{T}_1, \boldsymbol{h}_n = \boldsymbol{h}(\vartheta_n, \boldsymbol{\Delta}) \in \mathcal{H}_1 \text{ with } \vartheta_1 < \ldots < \vartheta_N \in \Theta\Big\}$$

where

$$\mathcal{T}_1 = \Big\{\boldsymbol{T} \in \mathbb{C}^{M\times P} \big| \; \boldsymbol{T} = \boldsymbol{T}(\vartheta) \text{ with } \vartheta \in \Theta\Big\}$$

$$\mathcal{H}_1 = \Big\{\boldsymbol{h} \in \mathbb{C}^{P\times 1} \big| \; \boldsymbol{h} = \boldsymbol{h}(\vartheta, \boldsymbol{\Delta}) \text{ with } \vartheta \in \Theta; \; \boldsymbol{\Delta} \in \mathbb{R}^{(P-1)\times 1}\Big\}.$$

Relaxation Based on Geometry Exploitation

Rank Reduction Algorithm

- Consider first the case of a fully calibrated array, hence the subarray displacements $\boldsymbol{\Delta}$ are known.
- In this case the spectral MUSIC estimator introduced above can be applied, hence

$$\{\hat{\boldsymbol{a}}\} =^{N} \underset{\boldsymbol{a}\in\mathcal{A}_1}{\arg\min} f_{\text{MUSIC}}(\boldsymbol{a}) =^{N} \underset{\boldsymbol{T}\in\mathcal{T}_1,\boldsymbol{h}\in\mathcal{H}_1}{\arg\min} f_{\text{MUSIC}}(\boldsymbol{T},\boldsymbol{h})$$

with

$$f_{\text{MUSIC}}(\boldsymbol{a}) = \boldsymbol{a}^{\mathsf{H}}\hat{\boldsymbol{U}}_{\text{n}}\hat{\boldsymbol{U}}_{\text{n}}^{\mathsf{H}}\boldsymbol{a}$$

$$f_{\text{MUSIC}}(\boldsymbol{T},\boldsymbol{h}) = \boldsymbol{h}^{\mathsf{H}}\boldsymbol{T}^{\mathsf{H}}\hat{\boldsymbol{U}}_{\text{n}}\hat{\boldsymbol{U}}_{\text{n}}^{\mathsf{H}}\boldsymbol{T}\boldsymbol{h},$$

Relaxation Based on Geometry Exploitation
Rank Reduction Algorithm

- In the partly calibrated array case the subarray displacements $\boldsymbol{\Delta} \in \mathbb{R}^{(P-1)\times 1}$ are unknown.
- Hence, the reference sensor steering vector $\boldsymbol{h}(\theta, \boldsymbol{\Delta}) \in \mathcal{H}_1$ depends on the unknown displacements $\boldsymbol{\Delta}$ that must be estimated along with the DOAs $\theta_1, \ldots, \theta_N$.
- This requires a prohibitive P dimensional parameter search (with ambiguities).
- However, the subarray responses matrix $\boldsymbol{T}(\theta) \in \mathcal{T}_1$ is independent of the displacements $\boldsymbol{\Delta}$.

Relaxation Based on Geometry Exploitation
Rank Reduction Algorithm

Relaxation Approach

- Relax the manifold structure of the reference sensor steering vector $\Rightarrow$ Replace $\boldsymbol{h}(\theta, \boldsymbol{\Delta}) \in \mathcal{H}_1$ by an unstructured vector $\boldsymbol{c} \in \mathbb{C}^{P \times 1}$ with $||\boldsymbol{c}||_2^2 = ||\boldsymbol{h}||_2^2 = P$
- Maintain the manifold structure of the subarray responses matrix $\boldsymbol{T}(\theta) \in \mathcal{T}_1$

Relaxed Array Manifold for Partly Calibrated Array

$$\bar{\mathcal{A}}_N = \left\{ \boldsymbol{A} = [\boldsymbol{T}_1 \boldsymbol{c}_1, \ldots, \boldsymbol{T}_N \boldsymbol{c}_N] \,\middle|\, \boldsymbol{T}_n \in \mathcal{T}_1, ||\boldsymbol{c}_n||_2^2 = P \text{ with } \vartheta_1 < \ldots < \vartheta_N \in \Theta \right\}$$

with $\mathcal{T}_1 = \left\{ \boldsymbol{T} \in \mathbb{C}^{M \times P} \middle| \, \boldsymbol{T} = \boldsymbol{T}(\vartheta) \text{ with } \vartheta \in \Theta \right\}$.

MUSIC Estimator on Relaxed Array Manifold

$$\{\hat{\boldsymbol{a}}\} = {}^N \underset{\boldsymbol{a} \in \bar{\mathcal{A}}_1}{\arg\min} \, f_{\text{MUSIC}}(\boldsymbol{a}) = {}^N \underset{\boldsymbol{T} \in \mathcal{T}_1, \boldsymbol{c}}{\arg\min} \, f_{\text{MUSIC}}(\boldsymbol{T}, \boldsymbol{c})$$

with

$$f_{\text{MUSIC}}(\boldsymbol{a}) = \boldsymbol{a}^{\mathsf{H}} \hat{\boldsymbol{U}}_{\text{n}} \hat{\boldsymbol{U}}_{\text{n}}^{\mathsf{H}} \boldsymbol{a} = \boldsymbol{c}^{\mathsf{H}} \boldsymbol{T}^{\mathsf{H}} \hat{\boldsymbol{U}}_{\text{n}} \hat{\boldsymbol{U}}_{\text{n}}^{\mathsf{H}} \boldsymbol{T} \boldsymbol{c}.$$

Relaxation Based on Geometry Exploitation

Rank Reduction Algorithm

MUSIC Estimator on Relaxed Array Manifold

$$\{\hat{\boldsymbol{a}}\} =^N \arg\min_{\boldsymbol{a}\in\bar{\mathcal{A}}_1} f_{\text{MUSIC}}(\boldsymbol{a}) =^N \arg\min_{\boldsymbol{T}\in\mathcal{T}_1} \min_{\boldsymbol{c}} f_{\text{MUSIC}}(\boldsymbol{T}, \boldsymbol{c})$$

with

$$f_{\text{MUSIC}}(\boldsymbol{a}) = \boldsymbol{a}^{\mathsf{H}}\hat{\boldsymbol{U}}_{\text{n}}\hat{\boldsymbol{U}}_{\text{n}}^{\mathsf{H}}\boldsymbol{a} = \boldsymbol{c}^{\mathsf{H}}\boldsymbol{T}^{\mathsf{H}}\hat{\boldsymbol{U}}_{\text{n}}\hat{\boldsymbol{U}}_{\text{n}}^{\mathsf{H}}\boldsymbol{T}\boldsymbol{c}.$$

With the relaxation of the reference sensor steering vector manifold the inner optimization problem exhibits a simple solution.

The solution vector $\boldsymbol{c}^{\star}$ corresponds to a minor eigenvector of the matrix

$$\boldsymbol{M}_{\text{RARE}}^{(P)}(\theta) = \boldsymbol{T}^{\mathsf{H}}(\theta)\hat{\boldsymbol{U}}_{\text{n}}\hat{\boldsymbol{U}}_{\text{n}}^{\mathsf{H}}\boldsymbol{T}(\theta).$$

Relaxation Based on Geometry Exploitation
Rank Reduction Algorithm

Hence the RARE estimator corresponds to

$$\left\{\hat{\theta}\right\} = {}^{N}\underset{\theta\in\Theta}{\arg\min}\, f_{\text{RARE}}(\theta)$$

where the RARE null-spectrum is defined as

$$\begin{aligned} f_{\text{RARE}}(\theta) &= \lambda_P\big(\boldsymbol{M}_{\text{RARE}}^{(P)}(\theta)\big) \\ &= \lambda_P\left(\boldsymbol{T}^{\mathsf{H}}(\theta)\hat{\boldsymbol{U}}_{\text{n}}\hat{\boldsymbol{U}}_{\text{n}}^{\mathsf{H}}\boldsymbol{T}(\theta)\right), \end{aligned}$$

and $\lambda_P\big(\boldsymbol{M}_{\text{RARE}}^{(P)}(\theta)\big)$ denotes the minor eigenvalue of the $P \times P$ matrix $\boldsymbol{M}_{\text{RARE}}^{(P)}(\theta)$.

To simplify the evaluation the RARE null-spectrum is often defined as

$$\begin{aligned} f_{\text{RARE}}(\theta) &= \det\big(\boldsymbol{M}_{\text{RARE}}^{(P)}(\theta)\big) \\ &= \det\left(\boldsymbol{T}^{\mathsf{H}}(\theta)\hat{\boldsymbol{U}}_{\text{n}}\hat{\boldsymbol{U}}_{\text{n}}^{\mathsf{H}}\boldsymbol{T}(\theta)\right). \end{aligned}$$

Relaxation Based on Geometry Exploitation

Rank Reduction Algorithm

For $P > N$ it follows from Schur complement that the RARE matrix $\boldsymbol{M}_{\text{RARE}}^{(P)}(\theta)$ can be alternatively expressed as

$$\boldsymbol{M}_{\text{RARE}}^{(N)}(\theta) = \boldsymbol{I}_P - \hat{\boldsymbol{U}}_{\text{s}}^{\mathsf{H}} \boldsymbol{T}(\theta) \boldsymbol{\Omega} \boldsymbol{T}^{\mathsf{H}}(\theta) \hat{\boldsymbol{U}}_{\text{s}},$$

for $\boldsymbol{\Omega}$ denoting a constant diagonal matrix defined as $\boldsymbol{\Omega} = \left(\boldsymbol{T}^{\mathsf{H}}(\theta)\boldsymbol{T}(\theta)\right)^{-1}$.

In this case the RARE null-spectrum is written as

$$f_{\text{RARE}}(\theta) = \left(\boldsymbol{M}_{\text{RARE}}^{(N)}(\theta)\right) = \lambda_N \left(\boldsymbol{I}_N - \hat{\boldsymbol{U}}_{\text{s}}^{\mathsf{H}} \boldsymbol{T}(\theta) \boldsymbol{\Omega} \boldsymbol{T}^{\mathsf{H}}(\theta) \hat{\boldsymbol{U}}_{\text{s}}\right),$$

or

$$f_{\text{RARE}}(\theta) = \det\left(\boldsymbol{M}_{\text{RARE}}^{(N)}(\theta)\right) = \det\left(\boldsymbol{I}_N - \hat{\boldsymbol{U}}_{\text{s}}^{\mathsf{H}} \boldsymbol{T}(\theta) \boldsymbol{\Omega} \boldsymbol{T}^{\mathsf{H}}(\theta) \hat{\boldsymbol{U}}_{\text{s}}\right),$$

respectively.

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Sparse Relaxation Techniques
ℓ_1-relaxation Techniques

To avoid the difficulty of the multi-dimensional multimodal optimization over a nonconvex manifold $\mathcal{A}_N$ the compressed sensing (CS) approach is to sample the field of view Ω on a fine grid of DOAs

$$\tilde{\boldsymbol{\theta}} = [\tilde{\theta}_1, \tilde{\theta}_2, \ldots, \tilde{\theta}_K]^\mathsf{T} \in \Theta^K$$

with $K \gg N$ constructing an fixed overcomplete (fat) dictionary (sensing) matrix

$$\tilde{\boldsymbol{A}} = \boldsymbol{A}(\tilde{\boldsymbol{\theta}}) \in \mathcal{A}_K.$$

In the following we assume for simplicity that the true source DoAs in vector $\boldsymbol{\theta}$ lie on the grid, hence

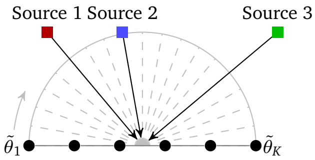
$$\theta_n \in \tilde{\Theta} = \{\tilde{\theta}_1, \ldots, \tilde{\theta}_K\} \text{ for } n = 1, \ldots, N.$$

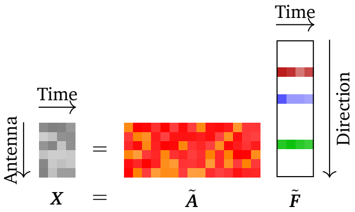
- Observe T snapshots of N source signals impinging on array of M sensors
- Sparse representation of $M \times T$ measurement matrix

$$\boldsymbol{X} = \tilde{\boldsymbol{A}}\tilde{\boldsymbol{F}} + \boldsymbol{N}$$

with

- $M \times K$ sensing matrix $\tilde{\boldsymbol{A}} = \left[\boldsymbol{a}(\tilde{\theta}_1), \ldots, \boldsymbol{a}(\tilde{\theta}_K)\right]$
- $K \times T$ joint sparse signal matrix $\tilde{\boldsymbol{F}} = [\tilde{\boldsymbol{f}}(1), \ldots, \tilde{\boldsymbol{f}}(T)]$
- $M \times T$ sensor noise matrix $\boldsymbol{N} = [\boldsymbol{n}(1), \ldots, \boldsymbol{n}(T)]$.



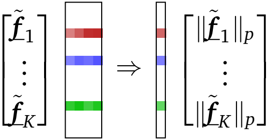


Sparse Relaxation Techniques

ℓ_1-relaxation Techniques

- $\ell_{p,q}$ mixed-norm of matrix $\tilde{\boldsymbol{F}} = \left[\tilde{\boldsymbol{f}}_1, \ldots, \tilde{\boldsymbol{f}}_K\right]^{\mathsf{T}}$:

$$\|\tilde{\boldsymbol{F}}\|_{p,q} = \left(\sum_{k=1}^{K} \|\tilde{\boldsymbol{f}}_k\|_p^q \right)^{\frac{1}{q}} .$$



- Nonlinear coupling of elements in row vectors $\tilde{\boldsymbol{f}}_k$ by ℓ_p-norm.
- Ideal for sparse reconstruction: $\ell_{p,0}$-norm with $p \geq 2$.

With dictionary $\tilde{\boldsymbol{A}}$ the LS fitting problem can be equivalently reformulated as

$$\begin{aligned} \min_{\tilde{\boldsymbol{F}} \in \mathbb{C}^{K \times T}} \quad & \|\boldsymbol{X} - \tilde{\boldsymbol{A}}\tilde{\boldsymbol{F}}\|_\mathsf{F}^2 \\ \text{subject to} \quad & \|\tilde{\boldsymbol{F}}\|_{p,0} = N. \end{aligned}$$

- Note, that the sensing matrix $\tilde{\boldsymbol{A}}$ is fat, hence the equation $\boldsymbol{X} = \tilde{\boldsymbol{A}}\tilde{\boldsymbol{F}}$ has infinitely many exact solutions.
- Hence, in the $\ell_{p,0}$-constrained problem we search for an N-row sparse solution that minimizes the fitting error.
- Dictionary $\tilde{\boldsymbol{A}}$ is constant, hence the optimization over manifold $\mathcal{A}_N$ has been avoided in the problem reformulation.
- However, the $\ell_{p,0}$-constraint is still nonconvex and combinatorial.

To solve the problem Lagrangian relaxation can be applied. The corresponding dual function is

$$d(\lambda) = \min_{\tilde{\boldsymbol{F}} \in \mathbb{C}^{K \times T}} \frac{1}{2} \|\boldsymbol{X} - \tilde{\boldsymbol{A}} \tilde{\boldsymbol{F}}\|_\mathsf{F}^2 + \lambda \|\tilde{\boldsymbol{F}}\|_{p,0} - \lambda N$$

for $\lambda \geq 0$.

- The Lagrange multiplier λ marks the cost associated with the violation of the $\ell_{p,0}$ constraint.
- The Lagrangian minimization problem provides a lower bound for the objective function value of the $\ell_{p,0}$ constrained LS matching problem above.
- We will later discuss a practical procedure for finding a suitable λ.
- The relaxed problem is still nonconvex due to the nonconvexity of the $\ell_{p,0}$ mixed-norm, hence convex approximation techniques can be applied.

Sparse Relaxation Techniques
ℓ_1-relaxation Techniques

- A common convex approximation of the $\ell_{p,0}$-pseudo-norm that is known to promote sparse solutions is the $\ell_{p,1}$-norm. This approximation is commonly termed ℓ_1-norm relaxation,...
- ... even though depending on the choice of λ it may not necessarily represent a relaxation of the the ℓ_0 constrained LS matching problem above in the optimization relaxation sense (the lower bound property is not necessarily satisfied).
- Further, for fixed λ dropping constant terms we obtain the ℓ_1 regularized LS problem also known as LASSO [Yang'18].

$$\hat{\tilde{\boldsymbol{F}}}_\lambda = \min_{\tilde{\boldsymbol{F}} \in \mathbb{C}^{K \times T}} \frac{1}{2} \|\boldsymbol{X} - \tilde{\boldsymbol{A}} \tilde{\boldsymbol{F}}\|_\mathsf{F}^2 + \lambda \|\tilde{\boldsymbol{F}}\|_{p,1}$$

where $\lambda \geq 0$.

Multiple Snapshot Problem – Mixed-Norm Regularization

- $\ell_{2,1}$ Mixed-norm minimization [Malioutov'05], [Yuan'05]

$$\min_{\tilde{\boldsymbol{F}}} \frac{1}{2} \left\| \boldsymbol{X} - \tilde{\boldsymbol{A}}\tilde{\boldsymbol{F}} \right\|_{\mathsf{F}}^2 + \lambda \left\| \tilde{\boldsymbol{F}} \right\|_{2,1} .$$

- Problem: For large number of snapshots T or large number of candidate frequencies K the problem becomes computationally intractable.
- Heuristic approach: Reduction of the dimension of measurement matrix $\boldsymbol{X}$ by ℓ_1-SVD and adaptive grid refinement,

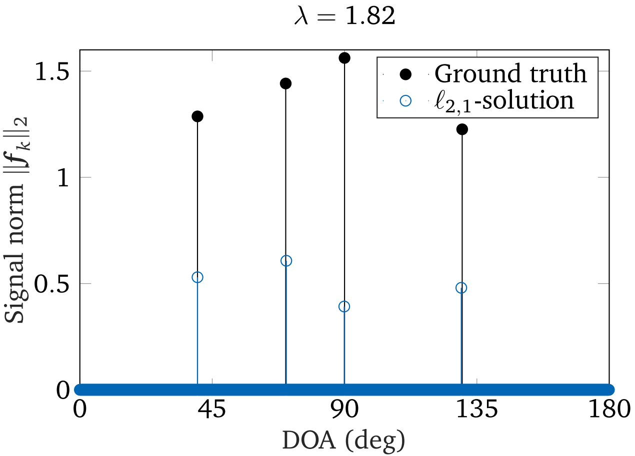
Choice of regularization parameter λ

- It can be proven that with the choice

$$\lambda \geq \lambda_{\max} = \max_{k=1,\ldots,K} \|\tilde{\boldsymbol{a}}_k^{\mathsf{H}} \boldsymbol{X}\|_2$$

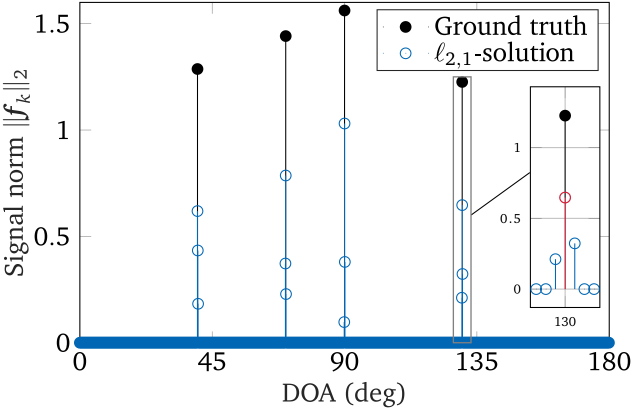
  the all zero matrix $\hat{\tilde{\boldsymbol{F}}}_{\lambda} = \hat{\tilde{\boldsymbol{F}}}_{\lambda_{\max}} = \mathbf{0}_{K\times T}$ is always the optimal solution of the $\ell_{2,1}$ mixed-norm problem.
- Hence $\lambda_{\max}$ provides an upper bound for the choice of λ.
- The bisection algorithm can be used to find the smallest value of $\lambda_{N,\min}$ for which an N-row-sparse solution matrix $\hat{\tilde{\boldsymbol{F}}}_{\lambda_{N,\min}}$ is obtained, i.e., $\|\hat{\tilde{\boldsymbol{F}}}_{\lambda_{N,\min}}\|_{2,0} = N$.

Sparse Relaxation Techniques
ℓ_1-relaxation Techniques



Sparse Relaxation Techniques
ℓ_1-relaxation Techniques

$\lambda = 0.88$



- If the solution is not N-row sparse, choose the N-largest local maxima.

Sparse Relaxation Techniques
Equivalent Formulation

SPARROW Formulation [Steffen'16]

The $\ell_{2,1}$ mixed-norm minimization problem

$$\min_{\tilde{\boldsymbol{F}} \in \mathbb{C}^{K \times T}} \frac{1}{2} \left\| \boldsymbol{X} - \tilde{\boldsymbol{A}}\tilde{\boldsymbol{F}} \right\|_{\mathsf{F}}^{2} + \lambda\sqrt{T} \left\| \tilde{\boldsymbol{F}} \right\|_{2,1}$$

is equivalent to SPARse ROW-norm reconstruction (SPARROW)

$$\min_{\boldsymbol{G} \in \mathbb{D}_{+}^{K}} \operatorname{Tr}\big((\tilde{\boldsymbol{A}}\boldsymbol{G}\tilde{\boldsymbol{A}}^{\mathsf{H}} + \lambda\boldsymbol{I})^{-1} \hat{\boldsymbol{R}} \big) + \operatorname{Tr}(\boldsymbol{G}),$$

with $\hat{\boldsymbol{R}} = \boldsymbol{X}\boldsymbol{X}^{\mathsf{H}}/T$ and minimizers $\hat{\tilde{\boldsymbol{F}}} = [\hat{\tilde{\boldsymbol{f}}}_1 \ldots, \hat{\tilde{\boldsymbol{f}}}_K]^{\mathsf{T}}$ and $\hat{\boldsymbol{G}} = \operatorname{diag}(\hat{g}_1, \ldots, \hat{g}_K)$ as

$$\hat{\tilde{\boldsymbol{F}}} = \hat{\boldsymbol{G}}\tilde{\boldsymbol{A}}^{\mathsf{H}}(\tilde{\boldsymbol{A}}\hat{\boldsymbol{G}}\tilde{\boldsymbol{A}}^{\mathsf{H}} + \lambda\boldsymbol{I})^{-1}\boldsymbol{X} \quad \text{and} \quad \hat{g}_k = \|\hat{\tilde{\boldsymbol{f}}}_k\|_2/\sqrt{T} \quad \text{for } k = 1, \ldots, K.$$

Sparse Relaxation Techniques
Equivalent Formulation

- SPARROW formulation

$$\min_{\boldsymbol{G}\in\mathbb{D}_+^K} \operatorname{Tr}\Big(\big(\tilde{\boldsymbol{A}}\boldsymbol{G}\tilde{\boldsymbol{A}}^{\mathsf{H}} + \lambda\boldsymbol{I}\big)^{-1}\hat{\boldsymbol{R}}\Big) + \operatorname{Tr}(\boldsymbol{G}).$$

- SDP implementation for oversampled case $T > M$

$$\min_{\boldsymbol{G}\in\mathbb{D}_+^K, \boldsymbol{U}_M} \operatorname{Tr}(\boldsymbol{U}_M\hat{\boldsymbol{R}}) + \operatorname{Tr}(\boldsymbol{G})$$

$$\text{subject to} \quad \begin{bmatrix} \boldsymbol{U}_M & \boldsymbol{I}_M \\ \boldsymbol{I}_M & \tilde{\boldsymbol{A}}\boldsymbol{G}\tilde{\boldsymbol{A}}^{\mathsf{H}} + \lambda\boldsymbol{I}_M \end{bmatrix} \succeq \boldsymbol{0} \quad \Leftrightarrow \quad \boldsymbol{U}_M \succeq \left(\tilde{\boldsymbol{A}}\boldsymbol{G}\tilde{\boldsymbol{A}}^{\mathsf{H}} + \lambda\boldsymbol{I}_M\right)^{-1}.$$

- SDP implementation for undersampled case $T \leq M$

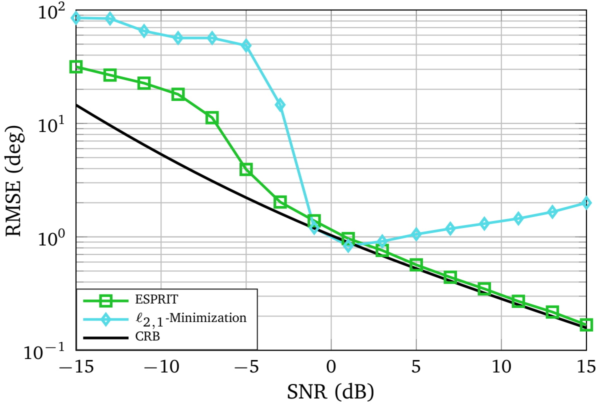
$$\min_{\boldsymbol{G}\in\mathbb{D}_+^K, \boldsymbol{U}_T} \frac{1}{T}\operatorname{Tr}(\boldsymbol{U}_T) + \operatorname{Tr}(\boldsymbol{G})$$

$$\text{subject to} \quad \begin{bmatrix} \boldsymbol{U}_T & \boldsymbol{X}^{\mathsf{H}} \\ \boldsymbol{X} & \tilde{\boldsymbol{A}}\boldsymbol{G}\tilde{\boldsymbol{A}}^{\mathsf{H}} + \lambda\boldsymbol{I}_M \end{bmatrix} \succeq \boldsymbol{0} \quad \Leftrightarrow \quad \boldsymbol{U}_T \succeq \boldsymbol{X}^{\mathsf{H}}\left(\tilde{\boldsymbol{A}}\boldsymbol{G}\tilde{\boldsymbol{A}}^{\mathsf{H}} + \lambda\boldsymbol{I}_M\right)^{-1}\boldsymbol{X}.$$

Sparse Relaxation Techniques
Simulation Results

Uncorrelated Source Signals

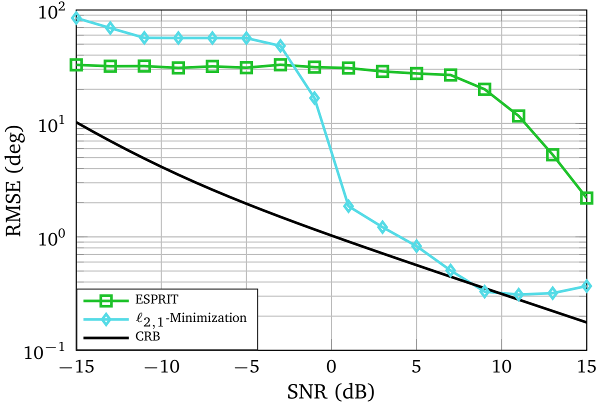
$$M = 5,\ \boldsymbol{\theta} = [90^\circ, 100^\circ]^\mathsf{T},\ T = 200,\ \rho = 0,\ \lambda = \sqrt{\nu M T \log M}$$



Sparse Relaxation Techniques
Simulation Results

Correlated Source Signals

$$M = 5,\ \boldsymbol{\theta} = [90^\circ, 100^\circ]^\mathsf{T},\ T = 200,\ \rho = 0.99,\ \lambda = \sqrt{\nu M T \log M}$$



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Majorization-Minimization Techniques
Expectation-Maximization

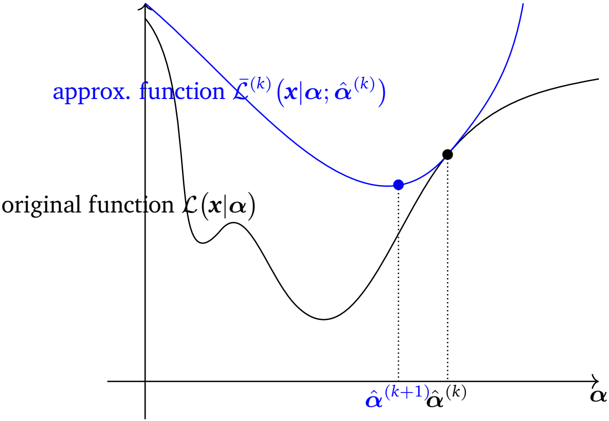
Properties of Multi-source Criteria

- Excellent threshold and asymptotic estimation performance.
- Full N-dimensional search required.
- Prohibitive complexity for scenarios where $N > 3$.

Solution: Approximation Methods

- Approximation techniques such as Alternating Projection, Block Coordinate Descent, viable options for local convergence.
- Majorization-minimization (MM) approach is an iterative optimization technique.
- Original optimization problem approximated by a sequence of upper bound problems.
- The approximate problems much easier to solve than the original problem (e.g. closed form).

Majorization-Minimization Techniques
Expectation-Maximization



Majorization-Minimization Techniques
Expectation-Maximization

ML problem:

$$\hat{\boldsymbol{\alpha}}_{\text{ML}} = \arg\min_{\boldsymbol{\alpha}} \mathcal{L}\big(\boldsymbol{x}|\boldsymbol{\alpha}\big).$$

Approximate problem at point $\hat{\boldsymbol{\alpha}}^{(k)}$ in iteration k:

$$\hat{\boldsymbol{\alpha}}^{(k+1)} = \arg\min_{\boldsymbol{\alpha}} \bar{\mathcal{L}}^{(k)}\big(\boldsymbol{x}|\boldsymbol{\alpha};\hat{\boldsymbol{\alpha}}^{(k)}\big)$$

where the approximate function $\bar{\mathcal{L}}^{(k)}\big(\boldsymbol{x}|\boldsymbol{\alpha};\hat{\boldsymbol{\alpha}}^{(k)}\big)$ is chosen such that it satisfies

- upper bound property:

$$\bar{\mathcal{L}}^{(k)}\big(\boldsymbol{x}|\boldsymbol{\alpha};\hat{\boldsymbol{\alpha}}^{(k)}\big) \geq \mathcal{L}\big(\boldsymbol{x}|\boldsymbol{\alpha}\big), \quad \forall \boldsymbol{\alpha}$$

- tightness at $\hat{\boldsymbol{\alpha}}^{(k)}$:

$$\bar{\mathcal{L}}^{(k)}\big(\boldsymbol{x}|\hat{\boldsymbol{\alpha}}^{(k)};\hat{\boldsymbol{\alpha}}^{(k)}\big) = \mathcal{L}\big(\boldsymbol{x}|\hat{\boldsymbol{\alpha}}^{(k)}\big).$$

Majorization-Minimization Techniques
Expectation-Maximization

- Expectation-maximization (EM) algorithm [Miller'90] [Dempster'77] is a special case of the MM algorithm [Hunter'04], [Luo'16].
- Unobserved data $\boldsymbol{y}$ only available through mapping $\boldsymbol{x} = \mathcal{T}(\boldsymbol{y})$, hence given $\boldsymbol{y}$ the observed data $\boldsymbol{x}$ is fully determined.
- $f(\boldsymbol{x}|\boldsymbol{y}, \boldsymbol{\alpha})$ is conditional pdf of observations $\boldsymbol{x}$ given unobserved data $\boldsymbol{y}$ with parameterization $\boldsymbol{\alpha}$.
- $f(\boldsymbol{y}|\boldsymbol{\alpha})$ is pdf of unobserved data $\boldsymbol{y}$ with parameterization $\boldsymbol{\alpha}$.
- In the EM algorithm the negative likelihood is approximated by Jensen's inequality

$$\begin{aligned}\mathcal{L}(\boldsymbol{x}|\boldsymbol{\alpha}) &= -\ln \mathrm{E}_{\boldsymbol{y}|\boldsymbol{x},\hat{\boldsymbol{\alpha}}^{(k)}}\left(\frac{f(\boldsymbol{x},\boldsymbol{y}|\boldsymbol{\alpha})}{f(\boldsymbol{y}|\boldsymbol{x},\hat{\boldsymbol{\alpha}}^{(k)})}\right)\\ &\leq -\mathrm{E}_{\boldsymbol{y}|\boldsymbol{x},\hat{\boldsymbol{\alpha}}^{(k)}}\Big(\ln\big(f(\boldsymbol{y}|\boldsymbol{\alpha})\big)\Big) + \text{constant} \triangleq \bar{\mathcal{L}}^{(k)}\big(\boldsymbol{x}|\boldsymbol{\alpha};\hat{\boldsymbol{\alpha}}^{(k)}\big).\end{aligned}$$

Majorization-Minimization Techniques
Expectation-Maximization

- Consider example of DML signal model with known noise variance ν

$$\boldsymbol{x}(t) = \sum_{n=1}^{N} \boldsymbol{a}(\theta_n) s_n(t) + \boldsymbol{n}(t)$$

where $\boldsymbol{A} = [\boldsymbol{a}(\theta_1), \ldots, \boldsymbol{a}(\theta_N)] \in \mathcal{A}_N$ and $\boldsymbol{n}(t) \sim \mathcal{N}_{\mathrm{C}}\big(\mathbf{0}_M, \nu \boldsymbol{I}_M\big)$.

- Define unobserved data $\boldsymbol{y}^{\mathsf{T}}(t) = [\boldsymbol{y}_1^{\mathsf{T}}(t), \ldots, \boldsymbol{y}_N^{\mathsf{T}}(t)]$ as individual source contributions

$$\boldsymbol{y}_n(t) = \boldsymbol{a}(\theta_n) s_n(t) + \boldsymbol{n}_n(t), \quad n = 1, \ldots, N$$

with i.i.d. $\boldsymbol{n}_n(t) \sim \mathcal{N}_{\mathrm{C}}\big(\mathbf{0}_{M\times 1}, \nu_n \boldsymbol{I}_M\big)$ and $\sum_{n=1}^{N} \nu_n = \nu$.

- Then

$$\boldsymbol{x}(t) = \sum_{n=1}^{N} \boldsymbol{y}_n(t) = \sum_{n=1}^{N} \boldsymbol{a}(\theta_n) s_n(t) + \boldsymbol{n}(t), \quad \text{where} \quad \boldsymbol{n}(t) = \sum_{n=1}^{N} \boldsymbol{n}_n(t).$$

Majorization-Minimization Techniques
Expectation-Maximization

Expectation Step

At point $\hat{\boldsymbol{\alpha}}^{(k)} = [\hat{\boldsymbol{\theta}}^{(k)\mathsf{T}}, \hat{\boldsymbol{s}}^{(k)\mathsf{T}}]^{\mathsf{T}}$ in iteration k, the approximate upper bound function can be characterized as

$$\bar{\mathcal{L}}^{(k)}(\boldsymbol{x}, \boldsymbol{\theta}, \boldsymbol{s} | \hat{\boldsymbol{\theta}}^{(k)}, \hat{\boldsymbol{s}}^{(k)}) \propto \sum_{n=1}^{N} \mathrm{E}_{\boldsymbol{y}_n | \boldsymbol{x}, \hat{\boldsymbol{\alpha}}^{(k)}} \Big(\ln \big(f(\boldsymbol{y}_n | \boldsymbol{\alpha}) \big) \Big)$$

$$\propto - \sum_{n=1}^{N} \Big\| \underbrace{\boldsymbol{a}(\hat{\theta}_n^{(k)}) \hat{s}_n^{(k)} - \frac{1}{N} \Big(\boldsymbol{x} - \boldsymbol{A}(\hat{\boldsymbol{\theta}}^{(k)}) \hat{\boldsymbol{s}}^{(k)} \Big)}_{\hat{\boldsymbol{y}}_n^{(k)}(t)} - \boldsymbol{a}(\theta_n) s_n \Big\|^2$$

where we omitted constant terms.

Maximization Step

$$\Big(\hat{\theta}_n^{(k+1)}, \hat{s}_n^{(k+1)} \Big) = \underset{\theta_n, s_n(1), \ldots, s_n(T)}{\arg\min} \sum_{t=1}^{T} \Big\| \boldsymbol{a}(\theta_n) s_n(t) - \hat{\boldsymbol{y}}_n^{(k)}(t) \Big\|^2, \quad \text{for } n = 1, \ldots, N.$$

Solved in parallel or sequentially. Each subproblem is simple to solve.

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Asymptotic Performance Bound
Review of Crámer-Rao Bound

Parametric Model

- Random stationary process x.
- Observations over time $\boldsymbol{x}(t) \in \mathcal{X}$ for $t = 1, \ldots, T$ of the random process x.
- Non-redundant deterministic parameter vector $\boldsymbol{\alpha} = [\alpha_1, \ldots, \alpha_I]^\mathsf{T} \in \mathbb{R}^{I \times 1}$.
- Probability density function for a given parameter $f_{\mathsf{x}}(\boldsymbol{x}|\boldsymbol{\alpha})$.

Objective of Parametric Estimation

- Assumption: Independent observations over time drawn from the same probability density function with the true parameter $\boldsymbol{\alpha}_{\text{true}}$.
- Given the observations $\{\boldsymbol{x}(1), \ldots, \boldsymbol{x}(T)\}$ and the family of the probability density functions $f_{\mathsf{x}}(\boldsymbol{x}|\boldsymbol{\alpha})$.
- Estimate $\boldsymbol{\alpha}_{\text{true}}$ by an estimator $\hat{\boldsymbol{\alpha}}$.

Asymptotic Performance Bound
Review of Crámer-Rao Bound

For a given estimator $\hat{\alpha} = T\big(\mathbf{x}(1), \ldots, \mathbf{x}(T)\big)$

- Bias $\boldsymbol{\mu} = \mathbb{E}\{\hat{\boldsymbol{\alpha}}\}$.
- Covariance $\boldsymbol{\Sigma} = \mathbb{E}\Big\{(\hat{\boldsymbol{\alpha}} - \boldsymbol{\mu})(\hat{\boldsymbol{\alpha}} - \boldsymbol{\mu})^{\mathsf{H}}\Big\}$.

Fisher Information Matrix

Under some regularity conditions, the Fisher Information Matrix (FIM) is defined as

$$\boldsymbol{\mathcal{I}}(\boldsymbol{\alpha}) = -\mathbb{E}\left\{\nabla^2_{\boldsymbol{\alpha}}\big(\log f_{\mathbf{x}}(\boldsymbol{x}|\boldsymbol{\alpha})\big)\right\}.$$

Crámer-Rao Inequality

For any unbiased estimator $\hat{\alpha}$ with the covariance matrix $\boldsymbol{\Sigma}$, we have

$$\boldsymbol{\Sigma} \succeq \boldsymbol{C}(\boldsymbol{\alpha}_{\text{true}}) = \Big[\boldsymbol{\mathcal{I}}(\boldsymbol{\alpha}_{\text{true}})\Big]^{-1}.$$

Asymptotic Performance Bound
Review of Crámer-Rao Bound

Special Case: Gaussian case

- Parameter vector: $\boldsymbol{\alpha} = [\alpha_1, \ldots, \alpha_I]^{\mathsf{T}}$.
- Circularly-symmetric complex Gaussian observation: $\mathbf{x} \sim \mathcal{N}_{\mathsf{C}}\big(\boldsymbol{m}(\boldsymbol{\alpha}), \boldsymbol{K}(\boldsymbol{\alpha})\big)$.

Slepian-Bangs Formula

The *ij*-th element of the FIM matrix is given by

$$\Big[\boldsymbol{\mathcal{I}}(\boldsymbol{\alpha})\Big]_{ij} = \mathrm{Tr}\left(\boldsymbol{K}(\boldsymbol{\alpha})^{-1}\frac{\partial \boldsymbol{K}(\boldsymbol{\alpha})}{\partial \alpha_i}\boldsymbol{K}(\boldsymbol{\alpha})^{-1}\frac{\partial \boldsymbol{K}(\boldsymbol{\alpha})}{\partial \alpha_j}\right) + 2\mathrm{Re}\left\{\frac{\partial \boldsymbol{m}(\boldsymbol{\alpha})^{\mathsf{H}}}{\partial \alpha_i}\boldsymbol{K}(\boldsymbol{\alpha})^{-1}\frac{\partial \boldsymbol{m}^{\mathsf{H}}(\boldsymbol{\alpha})}{\partial \alpha_j}\right\}.$$

Necessary condition for the invertibility of the FIM matrix

- The parameter vector must be locally identifiable.
- Consequence: the parameters must be non-redundant.

Asymptotic Performance Bound
Review of Crámer-Rao Bound

Partition the FIM matrix

$$\boldsymbol{\mathcal{I}}(\boldsymbol{\alpha}) = \begin{bmatrix} \boldsymbol{\mathcal{I}}_{\boldsymbol{\theta}\boldsymbol{\theta}} & \boldsymbol{\mathcal{I}}_{\boldsymbol{\theta}\boldsymbol{\beta}} \\ \boldsymbol{\mathcal{I}}_{\boldsymbol{\beta}\boldsymbol{\theta}} & \boldsymbol{\mathcal{I}}_{\boldsymbol{\beta}\boldsymbol{\beta}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{C}_{\boldsymbol{\theta}\boldsymbol{\theta}} & \boldsymbol{C}_{\boldsymbol{\theta}\boldsymbol{\beta}} \\ \boldsymbol{C}_{\boldsymbol{\beta}\boldsymbol{\theta}} & \boldsymbol{C}_{\boldsymbol{\beta}\boldsymbol{\beta}} \end{bmatrix}^{-1} \text{ with } \boldsymbol{\alpha} = \left[\boldsymbol{\theta}^{\mathsf{T}}, \boldsymbol{\beta}^{\mathsf{T}}\right]^{\mathsf{T}}$$

- $\boldsymbol{\theta}$ contains desired parameters.
- $\boldsymbol{\beta}$ contains nuisance parameters.

Crámer-Rao bound of the desired parameters $\boldsymbol{\theta}$

$$\boldsymbol{C}_{\boldsymbol{\theta}\boldsymbol{\theta}} = \left(\boldsymbol{\mathcal{I}}_{\boldsymbol{\theta}\boldsymbol{\theta}} - \boldsymbol{\mathcal{I}}_{\boldsymbol{\theta}\boldsymbol{\beta}} \boldsymbol{\mathcal{I}}_{\boldsymbol{\beta}\boldsymbol{\beta}}^{-1} \boldsymbol{\mathcal{I}}_{\boldsymbol{\beta}\boldsymbol{\theta}}\right)^{-1}$$

Asymptotic Performance Bound
Review of Crámer-Rao Bound

Recall the Deterministic Signal Model

$$\mathbf{x}(t) \sim \mathcal{N}_{\mathsf{C}}\big(\boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{s}(t), \nu\boldsymbol{I}\big) \text{ for all } t = 1, \ldots, T.$$

Deterministic Crámer-Rao Bound

$$\boldsymbol{C}_{\text{det}}(\boldsymbol{\theta}) = \boldsymbol{C}_{\boldsymbol{\theta}\boldsymbol{\theta}} = \frac{\nu}{2T}\text{Re}\left\{\hat{\boldsymbol{P}}^{\mathsf{T}} \odot \left(\boldsymbol{D}^{\mathsf{H}}\boldsymbol{\Pi}_{\boldsymbol{A}}^{\perp}\boldsymbol{D}\right)\right\}^{-1}$$

- $\hat{\boldsymbol{P}} = \frac{1}{T}\sum_{t=1}^{T}\boldsymbol{s}(t)\boldsymbol{s}^{\mathsf{H}}(t) = \frac{1}{T}\boldsymbol{S}\boldsymbol{S}^{\mathsf{H}}$
- $\boldsymbol{D} = \left[\frac{\mathrm{d}\boldsymbol{a}(\theta_1)}{\mathrm{d}\theta}, \ldots, \frac{\mathrm{d}\boldsymbol{a}(\theta_N)}{\mathrm{d}\theta}\right]$

Asymptotic Performance Bound
Review of Crámer-Rao Bound

Recall the Stochastic Signal Model

$$\mathbf{x}(t) \sim \mathcal{N}_{\mathsf{C}}\big(\mathbf{0}, \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{P}\boldsymbol{A}^{\mathsf{H}}(\boldsymbol{\theta}) + \nu \boldsymbol{I}\big) \text{ for all } t = 1, \ldots, T$$

Stochastic Crámer-Rao Bound

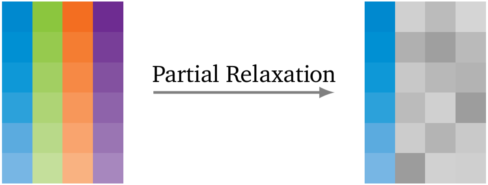
$$\boldsymbol{C}_{\mathrm{sto}}(\boldsymbol{\theta}) = \boldsymbol{C}_{\boldsymbol{\theta}\boldsymbol{\theta}} = \frac{\nu}{2T}\mathrm{Re}\left\{\boldsymbol{M}^{\mathsf{T}} \odot \left(\boldsymbol{D}^{\mathsf{H}}\boldsymbol{\Pi}_{\boldsymbol{A}}^{\perp}\boldsymbol{D}\right)\right\}^{-1}$$

- $\boldsymbol{M} = \boldsymbol{P}\boldsymbol{A}^{\mathsf{H}}\boldsymbol{R}^{-1}\boldsymbol{A}\boldsymbol{P}$
- $\boldsymbol{D} = \left[\frac{\mathrm{d}\boldsymbol{a}(\theta_1)}{\mathrm{d}\theta}, \ldots, \frac{\mathrm{d}\boldsymbol{a}(\theta_N)}{\mathrm{d}\theta}\right]$

Asymptotic Performance Bound
Crámer-Rao Bound for Partial Relaxation Model

Relaxed Array Manifold

$$\bar{\mathcal{A}}_N = \left\{ \boldsymbol{A} | \boldsymbol{A} = [\boldsymbol{a}(\theta), \boldsymbol{B}], \boldsymbol{a}(\theta) \in \mathcal{A}_1, \boldsymbol{B} \in \mathbb{C}^{M \times (N-1)} \text{ and rank}(\boldsymbol{A}) = N \right\}$$



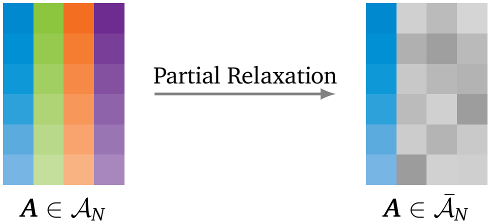
$\boldsymbol{A} \in \mathcal{A}_N$ $\quad$ $\boldsymbol{A} \in \bar{\mathcal{A}}_N$

Partial Relaxation Model for Time Instant t

$$\boldsymbol{x}(t) = \boldsymbol{A}\boldsymbol{s}(t) + \boldsymbol{n}(t) \text{ with } \boldsymbol{A} \in \bar{\mathcal{A}}_N.$$

Relaxed Array Manifold

$$\bar{\mathcal{A}}_N = \left\{ \boldsymbol{A} | \boldsymbol{A} = \left[\boldsymbol{a}(\theta), \boldsymbol{B} \right], \boldsymbol{a}(\theta) \in \mathcal{A}_1, \boldsymbol{B} \in \mathbb{C}^{M \times (N-1)} \text{ and rank} \left(\boldsymbol{A} \right) = N \right\}$$



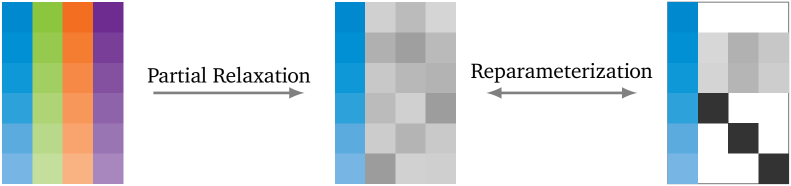
How does the array manifold relaxation affect the DOA estimation?

Asymptotic Performance Bound

Crámer-Rao Bound for Partial Relaxation Model

Reparameterization for Redundancy Elimination [Trinh-Hoang'20-2]

$$\boldsymbol{A}(\boldsymbol{\theta}) \in \mathcal{A}_N \qquad \boldsymbol{A} = \begin{bmatrix} a_1(\vartheta) & \boldsymbol{b}_1^{\mathrm{T}} \\ \boldsymbol{a}_2(\vartheta) & \boldsymbol{B}_2 \\ \boldsymbol{a}_3(\vartheta) & \boldsymbol{B}_3 \end{bmatrix} \in \bar{\mathcal{A}}_N \qquad \bar{\boldsymbol{A}} = \boldsymbol{A}\boldsymbol{T} = \begin{bmatrix} a_1(\vartheta) & \boldsymbol{0}^{\mathrm{T}} \\ \boldsymbol{a}_2(\vartheta) & \bar{\boldsymbol{B}} \\ \boldsymbol{a}_3(\vartheta) & \boldsymbol{I}_{N-1} \end{bmatrix}$$



$$\boldsymbol{R} = \boldsymbol{A}\boldsymbol{P}\boldsymbol{A}^{\mathrm{H}} + \nu \boldsymbol{I}_M \qquad \boldsymbol{R} = \boldsymbol{A}\boldsymbol{P}\boldsymbol{A}^{\mathrm{H}} + \nu \boldsymbol{I}_M \qquad \boldsymbol{R} = \bar{\boldsymbol{A}}\bar{\boldsymbol{P}}\bar{\boldsymbol{A}}^{\mathrm{H}} + \nu \boldsymbol{I}_M$$

- Structure of the desired direction is unaltered.
- Non-redundancy of the parameterization is ensured.

Asymptotic Performance Bound
Expression of the PR-CRB

Recall the conventional Crámer-Rao Bound

$$\boldsymbol{C}_{\text{sto}}(\boldsymbol{\theta}) = \frac{\nu}{2T}\text{Re}\left\{\boldsymbol{M} \odot \left(\boldsymbol{D}^{\mathsf{H}}\boldsymbol{\Pi}_{\boldsymbol{A}}^{\perp}\boldsymbol{D}\right)\right\}^{-1}$$

- $\boldsymbol{M} = \left(\boldsymbol{P}\boldsymbol{A}^{\mathsf{H}}\boldsymbol{R}^{-1}\boldsymbol{A}\boldsymbol{P}\right)^{\mathsf{T}} = \begin{bmatrix} M_{11} & \boldsymbol{M}_{21}^{\mathsf{H}} \\ \boldsymbol{M}_{21} & \boldsymbol{M}_{22} \end{bmatrix}$
- $\boldsymbol{D} = \left[\frac{\mathrm{d}\boldsymbol{a}(\theta_1)}{\mathrm{d}\theta}, \dots, \frac{\mathrm{d}\boldsymbol{a}(\theta_N)}{\mathrm{d}\theta}\right] = [\boldsymbol{d}, \boldsymbol{D}_2]$

Crámer-Rao Bound for $\vartheta = \theta_1$ under the PR model

$$C_{\text{PR-CRB}}(\vartheta) = \frac{\nu}{2T}\left(\left(M_{11} - \boldsymbol{M}_{21}^{\mathsf{H}}\boldsymbol{M}_{22}^{-1}\boldsymbol{M}_{21}\right)\boldsymbol{d}^{\mathsf{H}}\boldsymbol{\Pi}_{\boldsymbol{A}}^{\perp}\boldsymbol{d}\right)^{-1}.$$

Asymptotic Performance Bound
Expression of the PR-CRB - Implications

Crámer-Rao Bounds

$$\boldsymbol{C}_{\text{sto}}(\boldsymbol{\theta}) = \frac{\nu}{2T}\text{Re}\left\{\boldsymbol{M} \odot \left(\boldsymbol{D}^{\mathsf{H}}\boldsymbol{\Pi}_{\boldsymbol{A}}^{\perp}\boldsymbol{D}\right)\right\}^{-1}$$

$$C_{\text{PR-CRB}}(\vartheta) = \frac{\nu}{2T}\left(\left(M_{11} - \boldsymbol{M}_{21}^{\mathsf{H}}\boldsymbol{M}_{22}^{-1}\boldsymbol{M}_{21}\right)\boldsymbol{d}^{\mathsf{H}}\boldsymbol{\Pi}_{\boldsymbol{A}}^{\perp}\boldsymbol{d}\right)^{-1}$$

- PR-CRB is always lower-bounded by the conventional CRB, i.e.

$$C_{\text{PR-CRB}}(\vartheta_n) \geq \left[\boldsymbol{C}_{\text{sto}}(\boldsymbol{\theta})\right]_{nn}, \qquad \text{for } n = 1, \ldots, N.$$

- In the case of high SNR and uncorrelated source signals, the two bounds are approximately equal.

Asymptotic Performance Bound
Expression of the PR-CRB - Implications

Recall the null-spectrum of PR-DML and PR-WSF estimator

$$f_{\text{PR-DML}}(\boldsymbol{a}) = \sum_{k=N}^{M} \lambda_k \left(\boldsymbol{\Pi}_{\boldsymbol{a}}^{\perp} \hat{\boldsymbol{R}} \right)$$

$$f_{\text{PR-WSF}}(\boldsymbol{a}) = \lambda_N \left(\boldsymbol{\Pi}_{\boldsymbol{a}}^{\perp} \hat{\boldsymbol{U}}_{\text{s}} \boldsymbol{W} \hat{\boldsymbol{U}}_{\text{s}}^{\mathsf{H}} \right)$$

Asymptotically as $T \to \infty$,

- The mean-square error of PR-WSF achieves PR-CRB for all positive definite weighting matrix $\boldsymbol{W}$.

- The mean-square error of PR-WSF, PR-DML and MUSIC are identical.

Concluding Remarks

Problem relaxation

Deliberately ignoring part of the prior knowledge is a powerful approach to make complicated estimation problems computationally tractable (without sacrificing much performance).

- Partial array geometry relaxation.
- Relaxation of interference structure.

Extensions?

- Revisit established algorithms for more advanced measurement models and design your own relaxation algorithms!!!

Use PR models in the performance analysis:

- Understand which model information is relaxed in a particular algorithm.

MATLAB Code is available at

`https://git.rwth-aachen.de/minh.trinh_hoang/eusipco-2020-tutorial-source-code.git`

Thank you for your attention!

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