Exploiting structure and pseudoconvexity in iterative parallel optimization algorithms for real time and large scale applications

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Problem formulation: differentiable problems

- Consider the following optimization problem:

$$(P): \quad \begin{array}{ll} \underset{\mathbf{x}}{\text{minimize}} & f(\mathbf{x}) \\ \text{subject to} & \mathbf{x} \in \mathcal{X}. \end{array}$$

  Assume that

  - $f(\mathbf{x})$ is differentiable and $\nabla f(\mathbf{x})$ is continuous;
  - $\mathcal{X}$ is closed and convex;
  - a solution of (P) exists.

- Objective: Iterative algorithms that can solve (P) efficiently.
- Application: mutual information maximization of the MIMO BC

$$\begin{array}{ll} \underset{(\mathbf{Q}_k)_{k=1}^K}{\text{maximize}} & \log\left|\mathbf{I} + \sum_{k=1}^K \mathbf{H}_k^H \mathbf{Q}_k \mathbf{H}_k\right| \\ \text{subject to} & \mathbf{Q}_k \succeq \mathbf{0},\ k = 1, \ldots, K, \sum_{k=1}^K \text{tr}(\mathbf{Q}_k) \leq P. \end{array}$$

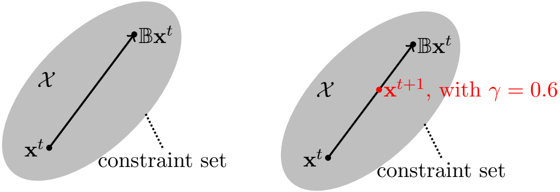
Iterative algorithms: variable update rule

Given $\mathbf{x}^t$ at iteration t:

- Step 1: We find a new point $\mathbb{B}\mathbf{x}^t \in \mathcal{X}$.
  - Notation: $\mathbb{B}\mathbf{x}^t$ is a function of $\mathbf{x}^t$ such that $\mathbb{B}\mathbf{x}^t - \mathbf{x}^t$ is a "good" direction.
- Step 2: We set $\mathbf{x}^{t+1}$ to be a point along that direction:

$$\mathbf{x}^{t+1} = \mathbf{x}^t + \underbrace{\gamma^t}_{\text{stepsize}} \cdot \underbrace{(\mathbb{B}\mathbf{x}^t - \mathbf{x}^t)}_{\text{update direction}}, \; 0 < \gamma \leq 1.$$

  - Since $\mathcal{X}$ is convex, $\mathbf{x}^{t+1} \in \mathcal{X}$ as long as $\mathbf{x}^t \in \mathcal{X}$ and $\mathbb{B}\mathbf{x}^t \in \mathcal{X}$.
- Step 3: $t \leftarrow t + 1$ and go back to Step 1.



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Iterative algorithms: variable update rule

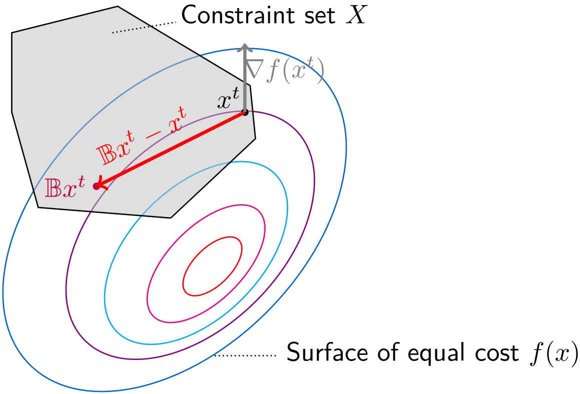
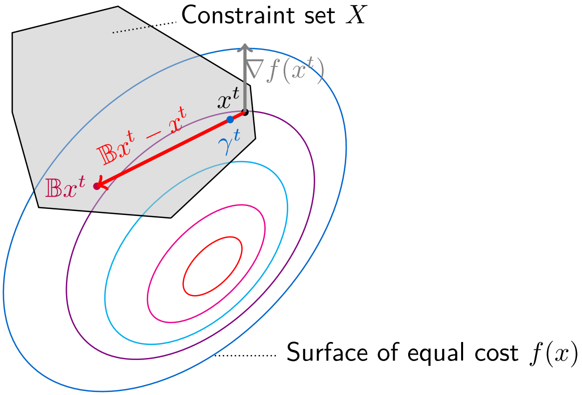


Figure: Descent direction step: $\mathbf{x}^{t+1} = \mathbf{x}^t + \gamma^t(\mathbb{B}\mathbf{x}^t - \mathbf{x}^t)$

Iterative algorithms: variable update rule



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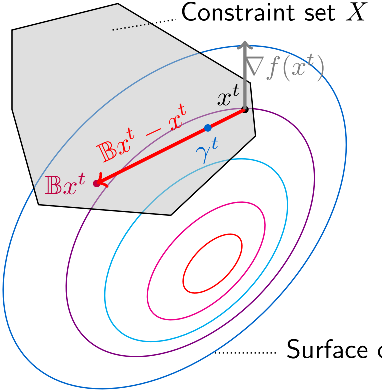

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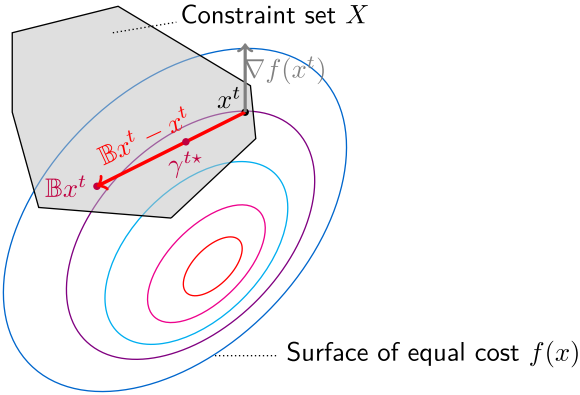

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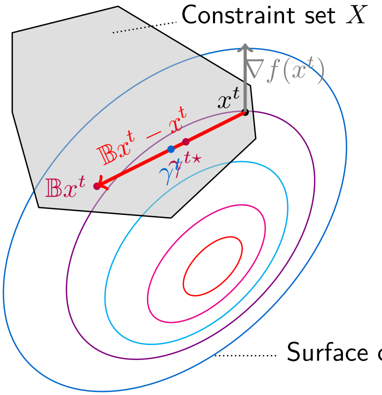

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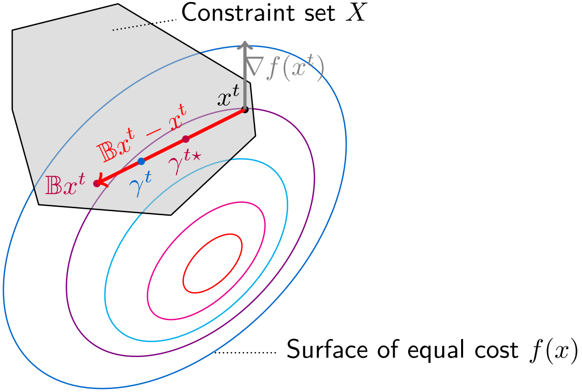

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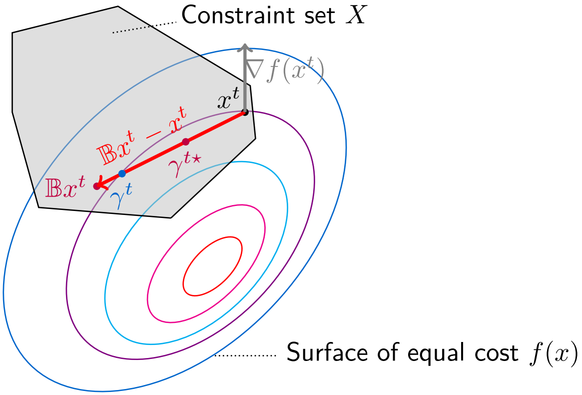

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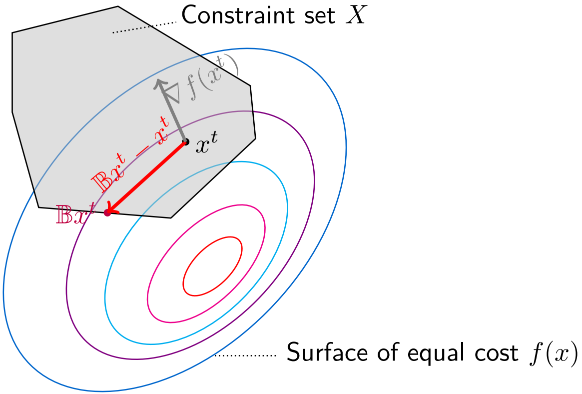

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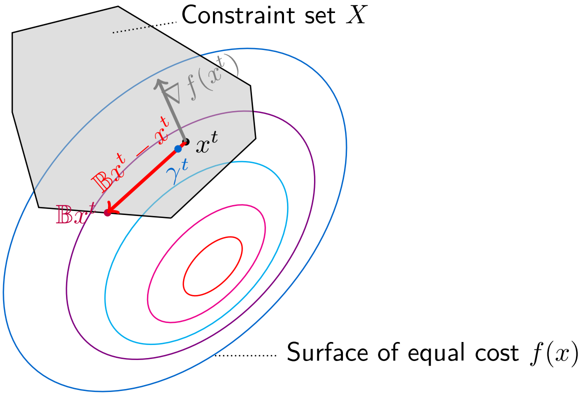

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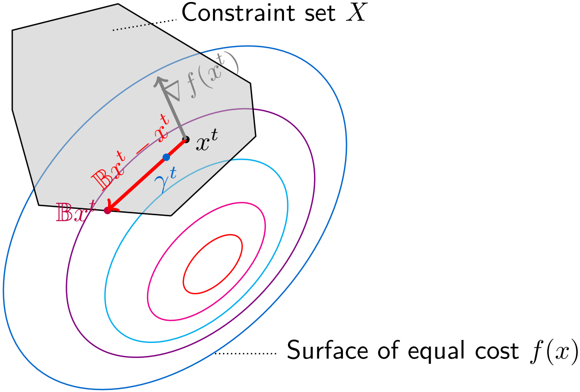

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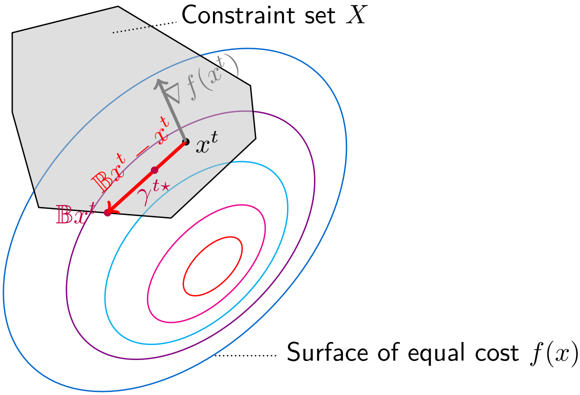

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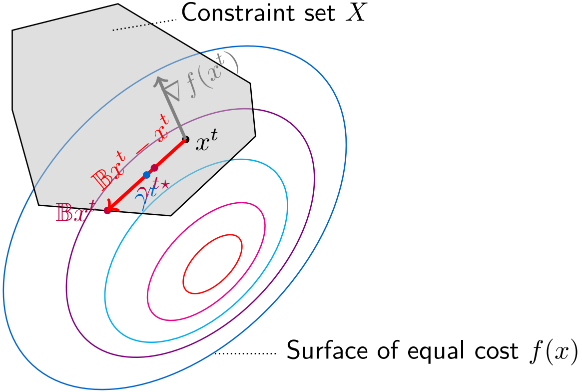

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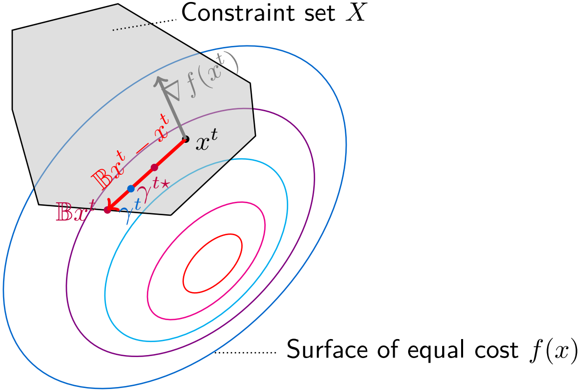

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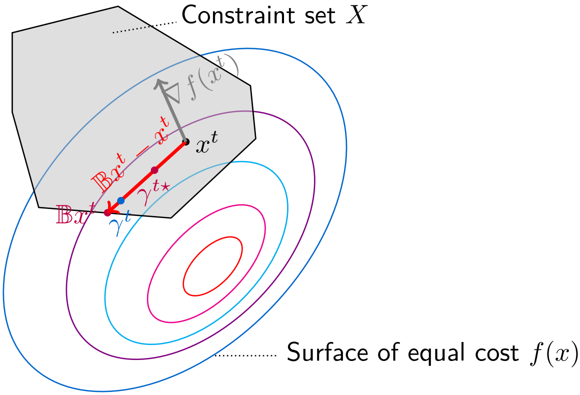

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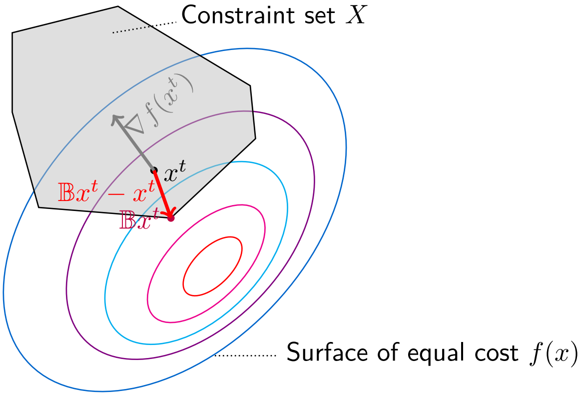

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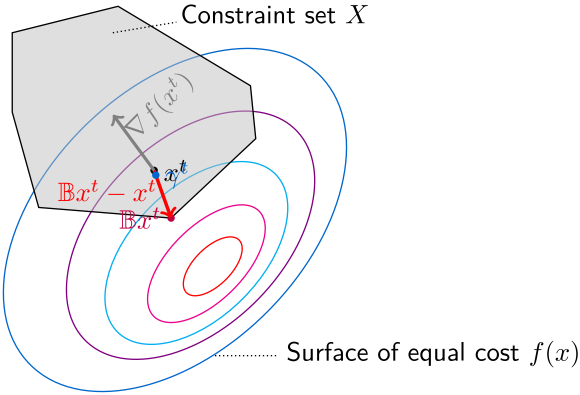

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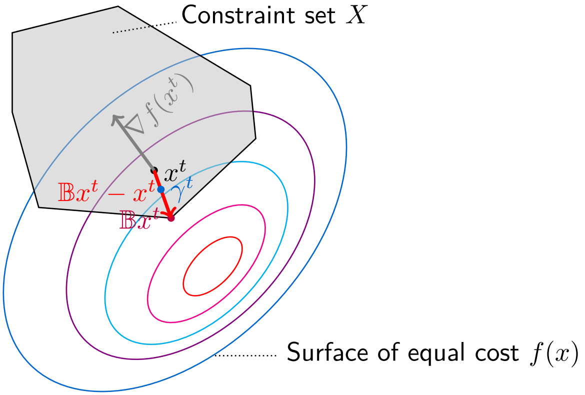

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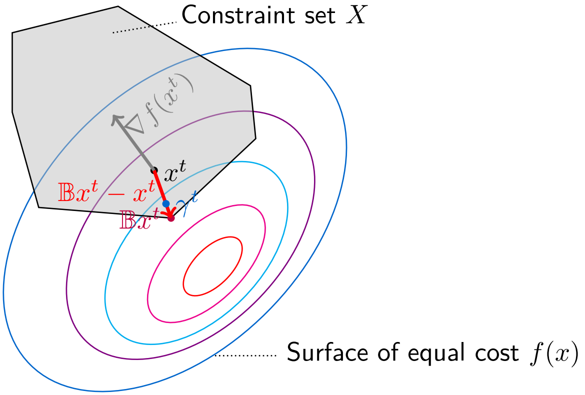

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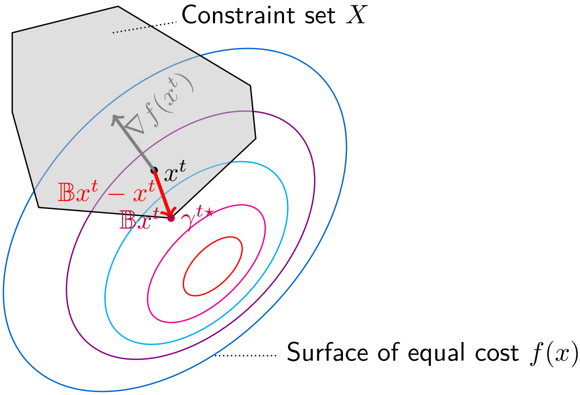

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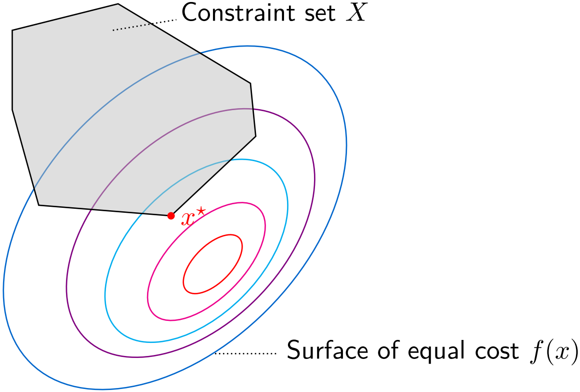

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The algorithms that we WILL and will NOT cover

<table>
<tr><td rowspan="7">We WILL cover</td><td>gradient (projection) method</td><td rowspan="7">for nonconvex problmes: $f(\mathbf{x}^{t+1}) < f(\mathbf{x}^t)$.</td></tr>
<tr><td>conditional gradient method</td></tr>
<tr><td>block coordinate descent (Gauss-Seidel) algorithm</td></tr>
<tr><td>Jacobi algorithm</td></tr>
<tr><td>proximal (gradient) method</td></tr>
<tr><td>successive convex approximation</td></tr>
<tr><td>block successive upper-bound minimization (BSUM) method</td></tr>
<tr><td rowspan="4">We will NOT cover</td><td>Newton's method</td><td rowspan="4">for convex problems: $\|\mathbf{x}^{t+1} - \mathbf{x}^\star\| < \|\mathbf{x}^t - \mathbf{x}^\star\|$.</td></tr>
<tr><td>primal-dual algorithm</td></tr>
<tr><td>interior point method</td></tr>
<tr><td>augmented Lagrangian and ADMM</td></tr>
</table>

Problem formulation: nondifferentiable problems

- Consider the following problem where $g(\mathbf{x})$ is nondifferentiable:

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & \underbrace{f(\mathbf{x})}_{\text{differentiable}} + \underbrace{g(\mathbf{x})}_{\text{nondifferentiable}} \\ \text{subject to} \quad & \mathbf{x} \in \mathcal{X}. \end{aligned}$$

- If $g(\mathbf{x})$ is convex, the above problem can be rewritten as:

$$\begin{aligned} \underset{\mathbf{x},y}{\text{minimize}} \quad & f(\mathbf{x}) + y \leftarrow \text{differentiable} \\ \text{subject to} \quad & \mathbf{x} \in \mathcal{X}, g(\mathbf{x}) \leq y \leftarrow \text{convex}. \end{aligned}$$

- Application: LASSO in sparse regularization

$$\underset{\mathbf{x}}{\text{minimize}} \quad \underbrace{\frac{1}{2}\left\|\mathbf{A}\mathbf{x} - \mathbf{b}\right\|_2^2}_{\triangleq f(\mathbf{x})} + \underbrace{\mu \left\|\mathbf{x}\right\|_1}_{\triangleq g(\mathbf{x})}.$$

Stationary point

- If $\mathbf{x}^\star$ is a local minimum of (P) (i.e., it is a local minimum of $f(\mathbf{x})$ over $\mathcal{X}$), then

$$\text{first-order optimality condition: } \nabla f(\mathbf{x}^\star)^T(\mathbf{x}-\mathbf{x}^\star) \geq 0, \ \forall \mathbf{x} \in \mathcal{X}.$$

  - For any arbitrary but fixed $\mathbf{x} \in \mathcal{X}$:

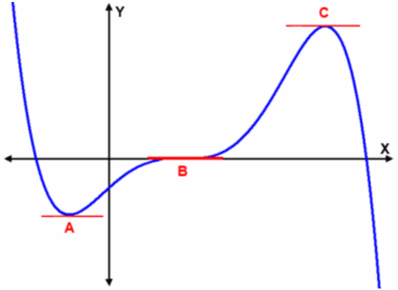
$$\begin{aligned} 0 &\leq \lim_{\gamma \downarrow 0} \frac{f(\mathbf{x}^\star + \gamma(\mathbf{x}-\mathbf{x}^\star)) - f(\mathbf{x}^\star)}{\gamma} \\ &= \nabla_\gamma f(\mathbf{x}^\star + \gamma(\mathbf{x}-\mathbf{x}^\star))\big|_{\gamma=0} \\ &= \nabla_{\mathbf{x}} f(\mathbf{x}^\star + \gamma(\mathbf{x}-\mathbf{x}^\star))\big|_{\gamma=0} \nabla_\gamma(\mathbf{x}^\star + \gamma(\mathbf{x}-\mathbf{x}^\star)) \\ &= \nabla_{\mathbf{x}} f(\mathbf{x}^\star)^T(\mathbf{x}-\mathbf{x}^\star) \end{aligned}$$

  - Note that $\mathbf{x}^\star + \gamma(\mathbf{x}-\mathbf{x}^\star)$ is in the vicinity of $\mathbf{x}^\star$ when $\gamma \downarrow 0$.
  - Necessary optimality condition. Sufficient if (P) is convex.

- If $\mathcal{X} = \mathbb{R}^n$, then the above optimality condition reduces to

$$\nabla f(\mathbf{x}^\star) = \mathbf{0}.$$

Stationary point



- A point $\mathbf{x}^\star$ satisfying the above optimality condition is called a stationary point of (P).
- It is the classic goal of algorithmic design in nonlinear programming.
- Stationary point and KKT point are generally not equivalent:
  - Even for convex problems, constraint qualifications must be satisfied to guarantee the existence of a KKT point.

Descent direction

- Variable update in iterative algorithms:

$$\mathbf{x}^{t+1} = \mathbf{x}^t + \gamma^t \cdot (\mathbb{B}\mathbf{x}^t - \mathbf{x}^t).$$

  - Notation: $\mathbb{B}\mathbf{x}^t$ is a function of $\mathbf{x}^t$ such that $\mathbb{B}\mathbf{x}^t - \mathbf{x}^t$ is a "good" direction.

- A vector $\mathbb{B}\mathbf{x}^t - \mathbf{x}^t$ is a descent direction of $f(\mathbf{x})$ at $\mathbf{x} = \mathbf{x}^t$ if

$$\nabla f(\mathbf{x}^t)^T (\mathbb{B}\mathbf{x}^t - \mathbf{x}^t) < 0.$$

- From the first-order Taylor expansion around $\mathbf{x}^t$:

$$f(\mathbf{x}^t + \gamma(\mathbb{B}\mathbf{x}^t - \mathbf{x}^t)) = f(\mathbf{x}^t) + \underbrace{\gamma \underbrace{\nabla f(\mathbf{x}^t)^T (\mathbb{B}\mathbf{x}^t - \mathbf{x}^t)}_{<0} + o(\gamma \|\mathbb{B}\mathbf{x}^t - \mathbf{x}^t\|)}_{<0 \text{ if } \gamma \text{ is sufficiently small}}$$

$$f(\mathbf{x}^{t+1}) < f(\mathbf{x}^t)$$

Descent direction

- Variable update in iterative algorithms:

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$$\nabla f(\mathbf{x}^t)^T (\mathbb{B}\mathbf{x}^t - \mathbf{x}^t) < 0.$$

- If $\mathbb{B}\mathbf{x}^t - \mathbf{x}^t$ is a descent direction, then there exists a $\bar{\gamma}^t > 0$ such that

$$f(\mathbf{x}^t + \gamma^t(\mathbb{B}\mathbf{x}^t - \mathbf{x}^t)) < f(\mathbf{x}^t), \forall \gamma^t \in (0, \bar{\gamma}^t].$$

  - See Ortega and Rheinboldt (1970, Ch. 8.2.1).

Descent direction

- In the (unconstrained) gradient method:

$$\mathbb{B}\mathbf{x}^t = \mathbf{x}^t - \nabla f(\mathbf{x}^t).$$

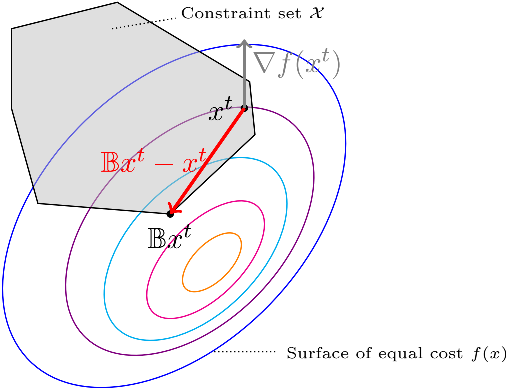
- $\mathbb{B}\mathbf{x}^t - \mathbf{x}^t$ is a descent direction:

$$\begin{aligned}\nabla f(\mathbf{x}^t)^T(\mathbb{B}\mathbf{x}^t - \mathbf{x}^t) &= \nabla f(\mathbf{x}^t)^T(\mathbf{x}^t - \nabla f(\mathbf{x}^t) - \mathbf{x}^t) \\ &= -\|\nabla f(\mathbf{x}^t)\|_2^2 < 0.\end{aligned}$$

Descent direction

- In the conditional gradient method:

$$\mathbb{B}\mathbf{x}^t = \arg\min_{\mathbf{x}\in\mathcal{X}} \nabla f(\mathbf{x}^t)^T(\mathbf{x}-\mathbf{x}^t).$$



Descent direction

- In the conditional gradient method:

$$\mathbb{B}\mathbf{x}^t = \arg\min_{\mathbf{x}\in\mathcal{X}} \nabla f(\mathbf{x}^t)^T(\mathbf{x}-\mathbf{x}^t).$$

- Assume $\mathbb{B}\mathbf{x}^t - \mathbf{x}^t$ and from the optimality of $\mathbb{B}\mathbf{x}^t$:

$$\nabla f(\mathbf{x}^t)^T(\mathbb{B}\mathbf{x}^t - \mathbf{x}^t) = \min_{\mathbf{x}\in\mathcal{X}} \nabla f(\mathbf{x}^t)^T(\mathbf{x}-\mathbf{x}^t) < \nabla f(\mathbf{x}^t)^T(\mathbf{x}^t-\mathbf{x}^t) = 0.$$

- Therefore $\mathbb{B}\mathbf{x}^t - \mathbf{x}^t$ is a descent direction:

$$\nabla f(\mathbf{x}^t)^T(\mathbb{B}\mathbf{x}^t - \mathbf{x}^t) < 0.$$

- At each iteration, an approximate problem, denoted as $\tilde{f}(\mathbf{x};\mathbf{x}^t)$, is solved:

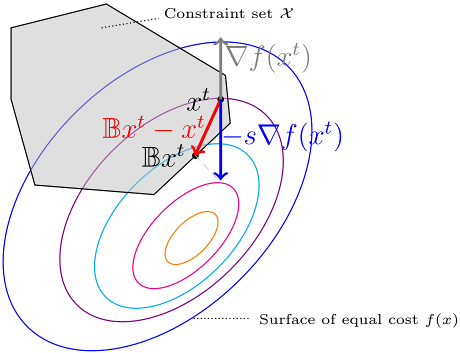
$$\tilde{f}(\mathbf{x};\mathbf{x}^t) \triangleq \nabla f(\mathbf{x}^t)^T(\mathbf{x}-\mathbf{x}^t).$$

Descent direction

- In the gradient projection method:

$$\mathbb{B}\mathbf{x}^t = [\mathbf{x}^t - s\nabla f(\mathbf{x}^t)]_{\mathcal{X}} = \arg\min_{\mathbf{x}\in\mathcal{X}} \underbrace{\left\|\mathbf{x} - (\mathbf{x}^t - s\nabla f(\mathbf{x}^t))\right\|_2^2}_{\triangleq \tilde{f}(\mathbf{x};\mathbf{x}^t)},$$

where $s > 0$ and $[\bullet]_{\mathcal{X}}$ is the projection operator.



Descent direction

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  where $s > 0$ and $[\bullet]_{\mathcal{X}}$ is the projection operator.
- From the first-order optimality condition (i.e., $\nabla f(\mathbf{x}^\star)^T(\mathbf{x} - \mathbf{x}^\star) \geq 0,\ \forall \mathbf{x} \in \mathcal{X}$):

$$(\mathbb{B}\mathbf{x}^t - (\mathbf{x}^t - s\nabla f(\mathbf{x}^t)))^T(\mathbf{x} - \mathbb{B}\mathbf{x}^t) \geq 0,\ \forall \mathbf{x} \in \mathcal{X}.$$

- Setting $\mathbf{x} = \mathbf{x}^t$ in the first-order optimality condition yields

$$(\mathbb{B}\mathbf{x}^t - (\mathbf{x}^t - s\nabla f(\mathbf{x}^t)))^T(\mathbf{x}^t - \mathbb{B}\mathbf{x}^t) \geq 0$$
$$\Downarrow$$
$$\nabla f(\mathbf{x}^t)^T(\mathbb{B}\mathbf{x}^t - \mathbf{x}^t) \leq -\frac{1}{s}\left\|\mathbb{B}\mathbf{x}^t - \mathbf{x}^t\right\|_2^2 < 0.$$

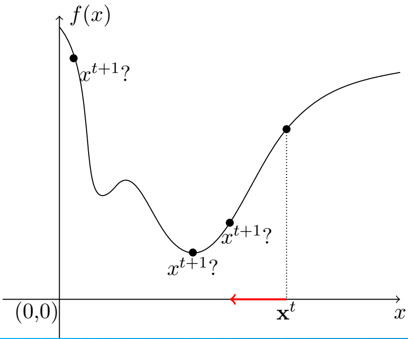
Stepsize

- Variable update in iterative algorithms:

$$\mathbf{x}^{t+1} = \mathbf{x}^t + \gamma^t \cdot (\mathbb{B}\mathbf{x}^t - \mathbf{x}^t).$$

- How far do we move along the descent direction?
  - If $\mathbb{B}\mathbf{x}^t - \mathbf{x}^t$ is a descent direction, then there exists a $\bar{\gamma}^t > 0$ such that

$$f(\mathbf{x}^t + \gamma^t(\mathbb{B}\mathbf{x}^t - \mathbf{x}^t)) < f(\mathbf{x}^t), \forall \gamma^t \in (0, \bar{\gamma}^t].$$

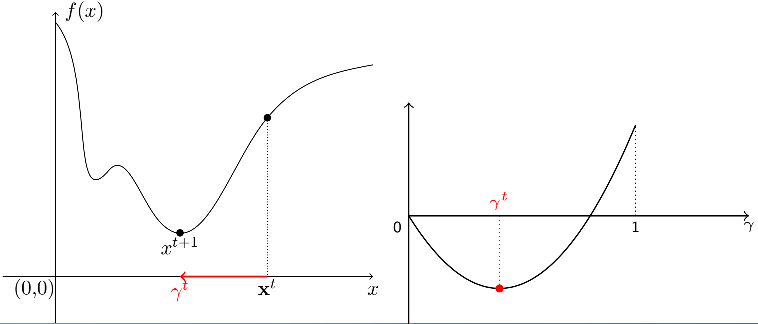


Stepsize

- Constant stepsize: $\gamma^t = \gamma$, where γ is sufficiently small.
- Decreasing stepsize: $\gamma^t \to 0$, $\sum_{t=0}^{\infty} \gamma^t = \infty$. For example, $\gamma^t = 1/t$.
- Exact line search:

$$\gamma^t \in \underset{0 \le \gamma \le 1}{\arg\min}\, f(\mathbf{x}^t + \gamma(\mathbb{B}\mathbf{x}^t - \mathbf{x}^t)).$$

  - Could be performed efficiently by bisection if $f(\mathbf{x})$ is convex.

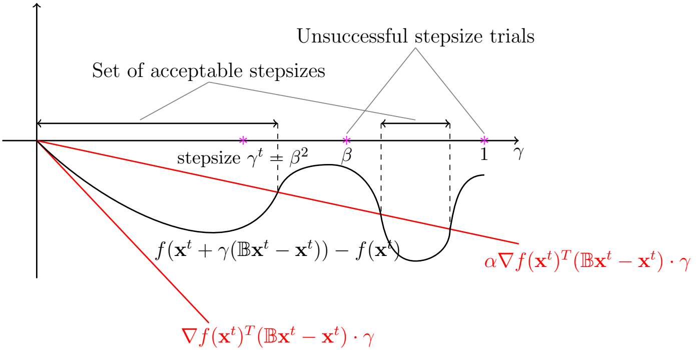


Stepsize

- Successive line search (Armijo rule): given two constants $0 < \alpha, \beta < 1$, we try $\gamma^t = 1, \beta^1, \beta^2, \ldots$ until the following is satisfied

$$f(\underbrace{\mathbf{x}^t + \beta^m(\mathbb{B}\mathbf{x}^t - \mathbf{x}^t)}_{\mathbf{x}^{t+1}}) - f(\mathbf{x}^t) \leq \alpha\beta^m \underbrace{\nabla f(\mathbf{x}^t)^T(\mathbb{B}\mathbf{x}^t - \mathbf{x}^t)}_{<0}.$$

Then $\gamma^t = \beta^{m_t}$ while m_t is the smallest integer of such a m.



Convex functions

- A function is (strictly) convex if

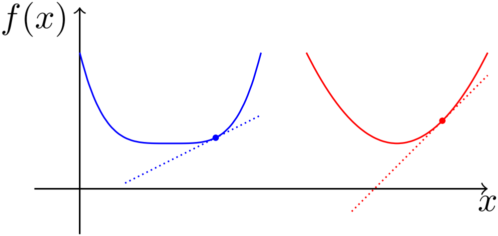
$$f(\mathbf{y})(>) \geq f(\mathbf{x}) + \nabla f(\mathbf{x})^T(\mathbf{y}-\mathbf{x}), \forall \mathbf{x}, \mathbf{y} \in \mathcal{X}.$$

  - A convex function is lower bounded by its first-order approximation.

- A function is strongly convex with constant $a > 0$ if

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \nabla f(\mathbf{x})^T(\mathbf{y}-\mathbf{x}) + \frac{a}{2}\|\mathbf{x}-\mathbf{y}\|_2^2, \forall \mathbf{x}, \mathbf{y} \in \mathcal{X}.$$

  - The difference must be big enough.



Convex functions

Characterizations of strongly convex functions

- A function $f(\mathbf{x})$ is strongly convex in $\mathcal{X}$ if and only if for some $a > 0$:

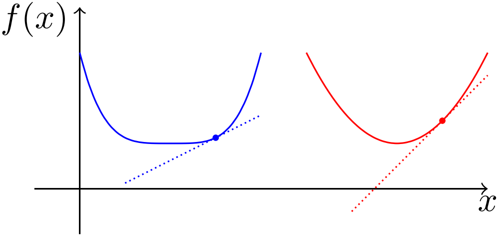
$$\nabla^2 f(\mathbf{x}) \succeq a\mathbf{I}, \forall \mathbf{x} \in \mathcal{X}.$$

  - Why the left function is not strongly convex?

- A function $f(\mathbf{x})$ is strongly convex in $\mathcal{X}$ if and only if for some $a > 0$:

$$(\nabla f(\mathbf{x}) - \nabla f(\mathbf{y}))^T (\mathbf{x} - \mathbf{y}) \geq a \|\mathbf{x} - \mathbf{y}\|_2^2, \forall \mathbf{x}, \mathbf{y} \in \mathcal{X}.$$

- See Boyd and Vandenberghe (2004, Ch. 9.1.2).

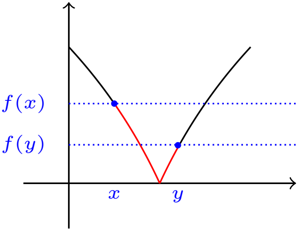


Quasiconvex functions

- A function $f(\mathbf{x})$ is (strictly) quasiconvex (or unimodal) if

$$f((1-\alpha)\mathbf{x}+\alpha\mathbf{y})(<) \leq \max(f(\mathbf{x}), f(\mathbf{y})), \forall\alpha \in (0,1).$$

  - The function value at any point between endpoints $\mathbf{x}$ and $\mathbf{y}$ is smaller than either of the two endpoints.
  - A function $f(\mathbf{x})$ is quasiconvex if and only if it is unimodal.



Pseudoconvex functions

- A function $f(\mathbf{x})$ is pseudoconvex if (Mangasarian, 1969):

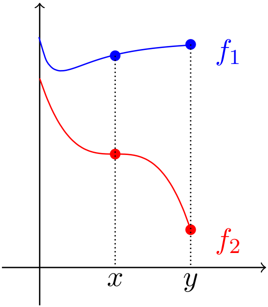
$$f(\mathbf{y}) < f(\mathbf{x}) \Longrightarrow \nabla f(\mathbf{x})^T(\mathbf{y}-\mathbf{x}) < 0.$$

- Function $f_1(x)$ is pseudoconvex, because

$$f_1(x) < f_1(y) \text{ and } \underbrace{\nabla f_1(y)}_{>0}\underbrace{(x-y)}_{<0} < 0.$$

- Function $f_2(x)$ is quasiconvex but not pseudoconvex, because

$$f_2(y) < f_2(x) \text{ but } \underbrace{\nabla f_2(x)}_{=0}\underbrace{(y-x)}_{>0} = 0.$$



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Pseudoconvex functions

- A function $f(\mathbf{x})$ is said to be pseudoconvex if

$$f(\mathbf{y}) < f(\mathbf{x}) \Longrightarrow \nabla f(\mathbf{x})^T(\mathbf{y} - \mathbf{x}) < 0.$$

- Similar to convex functions, every stationary point of a pseudoconvex function is global minimum. To see this:
  - Suppose $\mathbf{x}^\star$ is a stationary point but not a global minimum.
  - Suppose $\mathbf{y}^\star$ is a global minimum.
  - Since $\mathbf{x}^\star$ is a stationary point, then

$$\nabla f(\mathbf{x}^\star)^T(\mathbf{y} - \mathbf{x}^\star) \geq 0, \forall \mathbf{y} \in \mathcal{X}.$$

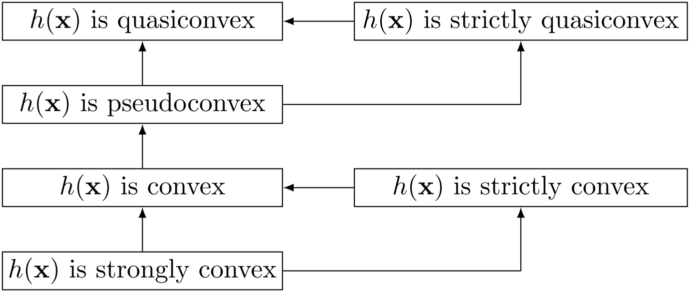
  - Since $\mathbf{y}^\star$ is a global minimum, then

$$f(\mathbf{y}^\star) < f(\mathbf{x}^\star) \Longrightarrow \nabla f(\mathbf{x}^\star)^T(\mathbf{y}^\star - \mathbf{x}^\star) < 0.$$

  - A contradiction is derived.
- Different from convex functions, the sum of pseudoconvex functions is not necessarily pseudoconvex (shown by examples later).

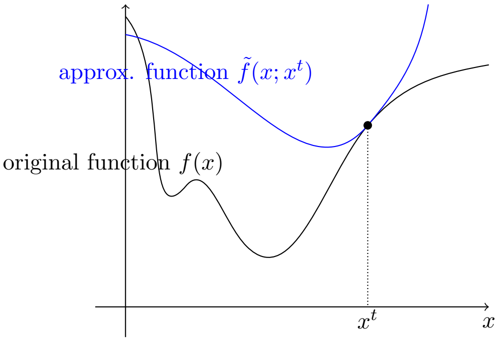
Convex functions

Relationship of functions with different degree of convexity:



A general framework

- Idea: The original problem (P) is solved by solving a sequence of successively refined approximate problems.
  - Each of the approximate problems is much easier to solve than (P), e.g., closed-form solution, parallel and distributed implementation...



A general framework: update direction

- Idea: The original problem (P) is solved by solving a sequence of successively refined approximate problems.
  - Each of the approximate problems is much easier to solve than (P), e.g., closed-form solution, parallel and distributed implementation...
- Suppose the approximate function around $\mathbf{x}^t$ is $\tilde{f}(\mathbf{x};\mathbf{x}^t)$:

$$\mathbb{B}\mathbf{x}^t \in \mathcal{S}(\mathbf{x}^t) \triangleq \left\{\mathbf{x}^\star \in \mathcal{X} : \ \tilde{f}(\mathbf{x}^\star;\mathbf{x}^t) = \min_{\mathbf{x}\in\mathcal{X}} \tilde{f}(\mathbf{x};\mathbf{x}^t)\right\}.$$

- Example: gradient projection method

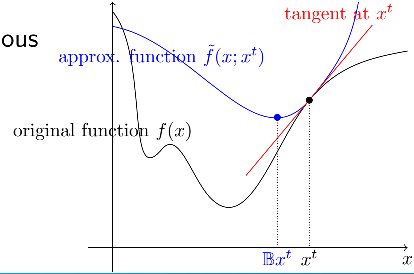
$$\begin{aligned}\mathbb{B}\mathbf{x}^t = \left[\mathbf{x}^t - s^t \nabla f(\mathbf{x}^t)\right]_{\mathcal{X}} &= \arg\min_{\mathbf{x}\in\mathcal{X}} \left\|\mathbf{x} - (\mathbf{x}^t - s^t \nabla f(\mathbf{x}^t))\right\|_2^2 \\ &= \arg\min_{\mathbf{x}\in\mathcal{X}} \Big\{\underbrace{\nabla f(\mathbf{x}^t)^T(\mathbf{x}-\mathbf{x}^t) + \frac{1}{2s^t}\left\|\mathbf{x}-\mathbf{x}^t\right\|_2^2}_{\triangleq \tilde{f}(\mathbf{x};\mathbf{x}^t)}\Big\}.\end{aligned}$$

A general framework: update direction

- Suppose the approximate function around $\mathbf{x}^t$ is $\tilde{f}(\mathbf{x};\mathbf{x}^t)$:

$$\mathbb{B}\mathbf{x}^t \in \mathcal{S}(\mathbf{x}^t) \triangleq \left\{\mathbf{x}^\star \in \mathcal{X} : \ \tilde{f}(\mathbf{x}^\star;\mathbf{x}^t) = \min_{\mathbf{x}\in\mathcal{X}} \tilde{f}(\mathbf{x};\mathbf{x}^t)\right\}.$$

- The approximate function $\tilde{f}(\mathbf{x};\mathbf{x}^t)$ satisfies some technical conditions (Yang and Pesavento, 2017):
  - (A1) Pseudoconvexity: $\tilde{f}(\mathbf{x};\mathbf{x}^t)$ is pseudoconvex in $\mathbf{x} \in \mathcal{X}$ for any given $\mathbf{x}^t \in \mathcal{X}$;
  - (A2) Gradient: $\tilde{f}(\mathbf{x};\mathbf{x}^t)$ is continuously differentiable in $\mathbf{x}$ for any given $\mathbf{x}^t$ and $\nabla\tilde{f}(\mathbf{x}^t;\mathbf{x}^t) = \nabla f(\mathbf{x}^t)$;
  - (A3) Continuity: $\tilde{f}(\mathbf{x};\mathbf{x}^t)$ is continuous in $\mathbf{x}^t$ for a fixed $\mathbf{x}$;



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A general framework: update direction

- The approximate function $\tilde{f}(\mathbf{x};\mathbf{x}^t)$ satisfies some technical conditions:
  - (A1) Pseudoconvexity: $\tilde{f}(\mathbf{x};\mathbf{x}^t)$ is pseudoconvex in $\mathbf{x} \in \mathcal{X}$ for any given $\mathbf{x}^t \in \mathcal{X}$;
  - (A2) Gradient: $\tilde{f}(\mathbf{x};\mathbf{x}^t)$ is continuously differentiable in $\mathbf{x}$ for a fixed $\mathbf{x}^t$ and $\nabla \tilde{f}(\mathbf{x}^t;\mathbf{x}^t) = \nabla f(\mathbf{x}^t)$;
  - (A3) Continuity: $\tilde{f}(\mathbf{x};\mathbf{x}^t)$ is continuous in $\mathbf{x}^t$ for a fixed $\mathbf{x}$;

Proposition (Stationary point and descent direction)

Provided that Assumptions (A1)-(A3) are satisfied:
(i) A point $\mathbf{x}^t$ is a stationary point of (P) if and only if $\mathbf{x}^t \in \mathcal{S}(\mathbf{x}^t)$;
(ii) If $\mathbf{x}^t$ is not a stationary point of (P), then $\mathbb{B}\mathbf{x}^t - \mathbf{x}^t$ is a descent direction of $f(\mathbf{x}^t)$: $\nabla f(\mathbf{x}^t)^T(\mathbb{B}\mathbf{x}^t - \mathbf{x}^t) < 0$.

A general framework: update direction

- The approximate function $\tilde{f}(\mathbf{x};\mathbf{x}^t)$ satisfies some technical conditions:
  - (A1) Pseudoconvexity: $\tilde{f}(\mathbf{x};\mathbf{x}^t)$ is pseudoconvex in $\mathbf{x} \in \mathcal{X}$ for any given $\mathbf{x}^t \in \mathcal{X}$;
  - (A2) Gradient: $\tilde{f}(\mathbf{x};\mathbf{x}^t)$ is continuously differentiable in $\mathbf{x}$ for a fixed $\mathbf{x}^t$ and $\nabla \tilde{f}(\mathbf{x}^t;\mathbf{x}^t) = \nabla f(\mathbf{x}^t)$;
  - (A3) Continuity: $\tilde{f}(\mathbf{x};\mathbf{x}^t)$ is continuous in $\mathbf{x}^t$ for a fixed $\mathbf{x}$;
- A point $\mathbf{x}^t$ is a stationary point of (P) if and only if $\mathbf{x}^t \in \mathcal{S}(\mathbf{x}^t)$:
  - $\mathbf{x}^t \in \mathcal{S}(\mathbf{x}^t) \Longrightarrow \mathbf{x}^t$ is a stationary point of (P):

$$\mathbf{x}^t = \arg\min_{\mathbf{x}\in\mathcal{X}} \tilde{f}(\mathbf{x};\mathbf{x}^t) \Longrightarrow \underbrace{0 \leq \nabla \tilde{f}(\mathbf{x}^t;\mathbf{x}^t)^T(\mathbf{x}-\mathbf{x}^t)}_{\text{first-order optimality condition}} \overset{(A2)}{=} \nabla f(\mathbf{x}^t)^T(\mathbf{x}-\mathbf{x}^t)$$

  - $\mathbf{x}^t$ is a stationary point of $(P) \Longrightarrow \mathbf{x}^t \in \mathcal{S}(\mathbf{x}^t)$:

$$0 \leq \nabla f(\mathbf{x}^t)^T(\mathbf{x}-\mathbf{x}^t) \overset{(A2)}{=} \nabla \tilde{f}(\mathbf{x}^t;\mathbf{x}^t)^T(\mathbf{x}-\mathbf{x}^t) \overset{(A1)}{\Longrightarrow} \mathbf{x}^t = \arg\min_{\mathbf{x}\in\mathcal{X}} \tilde{f}(\mathbf{x};\mathbf{x}^t)$$

A general framework: update direction

- The approximate function $\tilde{f}(\mathbf{x};\mathbf{x}^t)$ satisfies some technical conditions:
  - (A1) Pseudoconvexity: $\tilde{f}(\mathbf{x};\mathbf{x}^t)$ is pseudoconvex in $\mathbf{x} \in \mathcal{X}$ for any given $\mathbf{x}^t \in \mathcal{X}$;
  - (A2) Gradient: $\tilde{f}(\mathbf{x};\mathbf{x}^t)$ is continuously differentiable in $\mathbf{x}$ for a fixed $\mathbf{x}^t$ and $\nabla \tilde{f}(\mathbf{x}^t;\mathbf{x}^t) = \nabla f(\mathbf{x}^t)$;
  - (A3) Continuity: $\tilde{f}(\mathbf{x};\mathbf{x}^t)$ is continuous in $\mathbf{x}^t$ for a fixed $\mathbf{x}$;
- If $\mathbf{x}^t \notin \mathcal{S}(\mathbf{x}^t)$, $\mathbb{B}\mathbf{x}^t - \mathbf{x}^t$ is a descent direction of $f(\mathbf{x})$ at $\mathbf{x} = \mathbf{x}^t$:

$$\underbrace{\tilde{f}(\mathbb{B}\mathbf{x}^t;\mathbf{x}^t) = \min_{\mathbf{x}\in\mathcal{X}} \tilde{f}(\mathbf{x};\mathbf{x}^t) < \tilde{f}(\mathbf{x}^t;\mathbf{x}^t)}_{\text{optimality of } \mathbb{B}\mathbf{x}^t} \overset{(A1)}{\Longrightarrow}$$

$$0 > \nabla \tilde{f}(\mathbf{x}^t;\mathbf{x}^t)^T (\mathbb{B}\mathbf{x}^t - \mathbf{x}^t) \overset{(A2)}{=} \nabla f(\mathbf{x}^t)^T (\mathbb{B}\mathbf{x}^t - \mathbf{x}^t).$$

A general framework: stepsize

- The variable $\mathbf{x}^{t+1}$ is updated according to:

$$\mathbf{x}^{t+1} = \mathbf{x}^t + \gamma^t(\mathbb{B}\mathbf{x}^t - \mathbf{x}^t).$$

- Exact line search:

$$\gamma^t \in \underset{0\leq\gamma\leq1}{\arg\min}\ f(\mathbf{x}^t + \gamma(\mathbb{B}\mathbf{x}^t - \mathbf{x}^t)).$$

If $f(\mathbf{x})$ is furthermore convex, then γ^t can be computed by bisection.

- Successive line search: given two constants $0 < \alpha, \beta < 1$, $\gamma^t = \beta^{m_t}$ while m_t is the smallest integer of m such that

$$f(\mathbf{x}^t + \beta^m(\mathbb{B}\mathbf{x}^t - \mathbf{x}^t)) \leq f(\mathbf{x}^t) + \alpha\beta^m \nabla f(\mathbf{x}^t)^T(\mathbb{B}\mathbf{x}^t - \mathbf{x}^t).$$

A general framework: convergence

Algorithm: $\mathbf{x}^0 \in \mathcal{X}$; repeat the following steps until convergence:

- **S1:** Compute $\mathbb{B}\mathbf{x}^t \in \arg\min_{\mathbf{x}\in\mathcal{X}} \tilde{f}(\mathbf{x};\mathbf{x}^t)$.
- **S2:** Compute γ^t by exact or successive line search.
- **S3:** Update $\mathbf{x}$: $\mathbf{x}^{t+1} = \mathbf{x}^t + \gamma^t(\mathbb{B}\mathbf{x}^t - \mathbf{x}^t)$ and $t \leftarrow t+1$.

Theorem (Convergence to a stationary point)

Assume Assumptions (A1)-(A3) as well as the following assumptions are satisfied:

- *(A4): The set $\mathcal{S}(\mathbf{x}^t)$ is nonempty for $t = 1, 2, \ldots$;*
- *(A5): Given any convergent subsequence $\{\mathbf{x}^t\}_{t\in\mathcal{T}}$ where $\mathcal{T} \subseteq \{1, 2, \ldots\}$, the sequence $\{\mathbb{B}\mathbf{x}^t\}_{t\in\mathcal{T}}$ is bounded.*

Then $\{f(\mathbf{x}^t)\}$ is a decreasing sequence $(f(\mathbf{x}^{t+1}) < f(\mathbf{x}^t))$ and any limit point of $\{\mathbf{x}^t\}$ is a stationary point of (P) (but not a local maximum).

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A general framework: convergence

Proof of convergence to a stationary point.

To show $\mathbb{B}\mathbf{x}$ is a closed mapping: if $\mathbf{x}^t \to \mathbf{x}$ and $\mathbb{B}\mathbf{x}^t \to \mathbf{y}$, then $\mathbf{y} = \mathbb{B}\mathbf{x}$. $\square$

The assumption on pseudoconvexity cannot be further relaxed.

- Consider the following optimization problem:

$$\begin{aligned} \text{minimize} \quad & x^3 \\ \text{subject to} \quad & -1 \leq x \leq 1. \end{aligned}$$

- Consider the following approximate problem at $x^t = 0$:

$$\mathbb{B}x^t = \underset{-1\leq x\leq 1}{\arg\min}\, x^3 = -1,$$

$$0 = x^t \neq \underset{-1\leq x\leq 1}{\arg\min}\, x^3 = -1.$$

- However, $\mathbb{B}x^t - x^t$ is not a descent direction:

$$(\mathbb{B}x^t - x^t) \cdot \nabla f(x^t) = (-1 - 0) \cdot 0 = 0.$$

A general framework: convergence

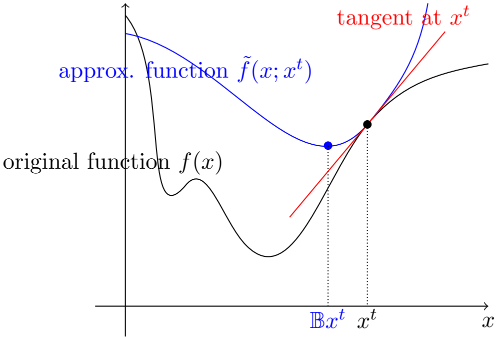
- In some existing iterative algorithms, the approximate function must be a global upper bound of the original function $f(\mathbf{x}^t)$, i.e., $\tilde{f}(\mathbf{x};\mathbf{x}^t) \geq f(\mathbf{x})$ and $\tilde{f}(\mathbf{x}^t;\mathbf{x}^t) = f(\mathbf{x}^t)$:
  - BSUM, see Razaviyayn, Hong, and Luo (2013);
  - majorization minimization, see Sun, Babu, and Palomar (2017);
  - sequential programming, see Zappone, Björnson, Sanguinetti, and Jorswieck (2017).
- Advantages of such an approximate function:
  - A constant unit stepsize can be chosen:

$$\mathbf{x}^{t+1} = \mathbf{x}^t + 1 \cdot (\mathbb{B}\mathbf{x}^t - \mathbf{x}^t) = \mathbb{B}\mathbf{x}^t = \arg\min_{\mathbf{x}\in\mathcal{X}} \tilde{f}(\mathbf{x};\mathbf{x}^t).$$

- Disadvantages of such an approximate function:
  - It does not always exists;
  - It may be difficult to be optimized (illustrated later by examples);
  - A constant unit stepsize may not be the optimal stepsize.

A general framework: convergence

- Note that the approximate function $\tilde{f}(\mathbf{x};\mathbf{x}^t)$ specified in (A1)-(A3) does not have to be a global upper bound of the original function $f(\mathbf{x})$, i.e., $\tilde{f}(\mathbf{x};\mathbf{x}^t) \ngeq f(\mathbf{x})$.
- Advantages: more flexibility to construct the approximate function.
- Disadvantages: more complexity to calculate the stepsize by the line search.



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A general framework: convergence

If, in addition, (A6) $\tilde{f}(\mathbf{x};\mathbf{x}^t) \geq f(\mathbf{x})$ and $\tilde{f}(\mathbf{x}^t;\mathbf{x}^t) = f(\mathbf{x}^t)$, the following constant stepsize can be adopted:

$$\gamma^t = 1,$$

because it yields sufficient decrease on $f(\mathbf{x})$ along $\mathbb{B}\mathbf{x}^t - \mathbf{x}^t$:

$$\begin{aligned}
f(\mathbf{x}^t + 1\cdot(\mathbb{B}\mathbf{x}^t - \mathbf{x}^t)) &= f(\mathbb{B}\mathbf{x}^t)\\
\text{(A6) upper bound property} \rightarrow &\leq \tilde{f}(\mathbb{B}\mathbf{x}^t;\mathbf{x}^t)\\
\text{optimality of } \mathbb{B}\mathbf{x}^t \rightarrow &= \min_{\mathbf{x}\in\mathcal{X}} \tilde{f}(\mathbf{x};\mathbf{x}^t)\\
\text{perform line search on } \tilde{f}(\mathbf{x};\mathbf{x}^t) \rightarrow &\leq \tilde{f}(\mathbf{x}^t + \beta^{m_t}(\mathbb{B}\mathbf{x}^t - \mathbf{x}^t);\mathbf{x}^t)\\
\text{definition of line search} \rightarrow &\leq \tilde{f}(\mathbf{x}^t;\mathbf{x}^t) + \alpha\beta^{m_t}\nabla\tilde{f}(\mathbf{x}^t;\mathbf{x}^t)^T(\mathbb{B}\mathbf{x}^t - \mathbf{x}^t)\\
\text{(A6) upper bound property} \rightarrow &= f(\mathbf{x}^t) + \alpha\beta^{m_t}\nabla f(\mathbf{x}^t)^T(\mathbb{B}\mathbf{x}^t - \mathbf{x}^t).
\end{aligned}$$

- However, $\gamma^t = 1$ may not be the optimal stepsize.

A general framework: special cases

- Conditional gradient method (Bertsekas, 2016):

$$\mathbb{B}\mathbf{x}^t = \arg\min_{\mathbf{x}\in\mathcal{X}} \Big\{ \underbrace{\nabla f(\mathbf{x}^t)^T(\mathbf{x}-\mathbf{x}^t)}_{=\tilde{f}(\mathbf{x};\mathbf{x}^t),\ \text{linear(convex)}} \Big\}.$$

- Gradient projection method (Bertsekas, 2016):

$$\begin{aligned}\mathbb{B}\mathbf{x}^t &= \Big[\mathbf{x}^t - s\nabla f(\mathbf{x}^t)\Big]_{\mathcal{X}} \\ &= \arg\min_{\mathbf{x}\in\mathcal{X}} \Big\|\mathbf{x} - (\mathbf{x}^t - s\nabla f(\mathbf{x}^t))\Big\|_2^2 \\ &= \arg\min_{\mathbf{x}\in\mathcal{X}} \Big\{ \underbrace{\nabla f(\mathbf{x}^t)^T(\mathbf{x}-\mathbf{x}^t) + \frac{1}{2s}\Big\|\mathbf{x}-\mathbf{x}^t\Big\|_2^2}_{=\tilde{f}(\mathbf{x};\mathbf{x}^t),\ \text{strongly convex}} \Big\}.\end{aligned}$$

  - Assumption verification:

$$\nabla_{\mathbf{x}}\big(\tilde{f}(\mathbf{x};\mathbf{x}^t)\big)\big|_{\mathbf{x}=\mathbf{x}^t} = \nabla f(\mathbf{x}^t) + \frac{1}{s}(\mathbf{x}-\mathbf{x}^t)\big|_{\mathbf{x}=\mathbf{x}^t} = \nabla f(\mathbf{x}^t)$$

A general framework: special cases

Proximal point algorithm (Parikh and Boyd, 2014):

- The approximate problem can be formulated as follows:

$$\underset{\mathbf{x}\in\mathcal{X}}{\text{minimize}} \quad \tilde{f}(\mathbf{x};\mathbf{x}^t) \triangleq f(\mathbf{x}) + \frac{c}{2}\left\|\mathbf{x}-\mathbf{x}^t\right\|_2^2.$$

- The approximate function is an upper bound function:

$$\tilde{f}(\mathbf{x};\mathbf{x}^t) = f(\mathbf{x}) + \frac{c}{2}\left\|\mathbf{x}-\mathbf{x}^t\right\|_2^2 \geq f(\mathbf{x}).$$

- $\tilde{f}(\mathbf{x};\mathbf{x}^t)$ is convex if $c + \lambda_{\min}(\nabla^2 f(\mathbf{x})) \geq 0$ for all $\mathbf{x} \in \mathcal{X}$.

Proximal gradient algorithm (Parikh and Boyd, 2014):

- The approximate problem is

$$\underset{\mathbf{x}\in\mathcal{X}}{\text{minimize}} \quad \tilde{f}(\mathbf{x};\mathbf{x}^t) \triangleq f(\mathbf{x}^t) + \nabla f(\mathbf{x}^t)^T(\mathbf{x}-\mathbf{x}^t) + \frac{c}{2}\left\|\mathbf{x}-\mathbf{x}^t\right\|_2^2.$$

- $\tilde{f}(\mathbf{x};\mathbf{x}^t)$ is convex and $\tilde{f}(\mathbf{x};\mathbf{x}^t) \geq f(\mathbf{x})$ if $\nabla f(\mathbf{x})$ is Lipschitz continuous.

A general framework: special cases

Jacobi algorithm:

- Consider the following optimization problem:

$$\begin{array}{ll} \underset{\mathbf{x}=(\mathbf{x}_k)_{k=1}^K}{\text{minimize}} & f(\mathbf{x}_1, \ldots, \mathbf{x}_K) \\ \text{subject to} & \mathbf{x} \in \mathcal{X}, \end{array} \quad \Leftarrow \text{nonconvex in } \mathbf{x}$$

  where $f(\mathbf{x})$ is individually convex in $\mathbf{x}_k$ for all k, but not jointly in $\mathbf{x}$.

- The approximate problem is (Yang and Pesavento, 2017)

$$\begin{array}{ll} \underset{\mathbf{x}}{\text{minimize}} & \tilde{f}(\mathbf{x}; \mathbf{x}^t) = \sum_{k=1}^K \underbrace{f(\mathbf{x}_k, \mathbf{x}_{-k}^t)}_{\text{convex in } \mathbf{x}_k} \quad \Leftarrow \text{convex in } \mathbf{x} \\ \text{subject to} & \mathbf{x} \in \mathcal{X}, \end{array}$$

  where $\mathbf{x}_{-k} \triangleq (\mathbf{x}_j)_{j \neq k}$.

A general framework: special cases

Jacobi algorithm:

- Consider the following optimization problem:

$$\begin{array}{ll} \underset{\mathbf{x}=(\mathbf{x}_k)_{k=1}^K}{\text{minimize}} & f(\mathbf{x}_1, \ldots, \mathbf{x}_K) \\ \text{subject to} & \mathbf{x} \in \mathcal{X}, \end{array} \quad \Leftarrow \text{nonconvex in } \mathbf{x}$$

  where $f(\mathbf{x})$ is individually convex in $\mathbf{x}_k$ for all k, but not jointly in $\mathbf{x}$.

- Assumption verification:

$$\begin{aligned} & \nabla_{\mathbf{x}_k} \left(\textstyle\sum_{j=1}^K f(\mathbf{x}_j, \mathbf{x}_{-j}^t) \right) \Big|_{\mathbf{x}=\mathbf{x}^t} \\ = \; & \nabla_{\mathbf{x}_k} \left(f(\mathbf{x}_k, \mathbf{x}_{-k}^t) + \textstyle\sum_{j \neq k} f(\mathbf{x}_j, \mathbf{x}_{-j}^t) \right) \Big|_{\mathbf{x}=\mathbf{x}^t} \\ = \; & \nabla_{\mathbf{x}_k} f(\mathbf{x}_k, \mathbf{x}_{-k}^t) \Big|_{\mathbf{x}=\mathbf{x}^t} \\ = \; & \nabla_k f(\mathbf{x}^t). \end{aligned}$$

A general framework: special cases

Jacobi algorithm:

- The approximate problem is (Yang and Pesavento, 2017)

$$\begin{array}{ll} \underset{\mathbf{x}}{\text{minimize}} & \tilde{f}(\mathbf{x};\mathbf{x}^t) = \sum_{k=1}^{K} f(\mathbf{x}_k, \mathbf{x}_{-k}^t) \\ \text{subject to} & \mathbf{x} \in \mathcal{X}. \end{array}$$

- If $\mathcal{X} = \mathcal{X}_1 \times \ldots \times \mathcal{X}_K$, the approximate problem can be decomposed into K smaller optimization problems (Jacobi algorithms):

$$\underset{(\mathbf{x}_k \in \mathcal{X}_k)_{k=1}^K}{\text{minimize}} \quad \sum_{k=1}^{K} f(\mathbf{x}_k, \mathbf{x}_{-k}^t) \Rightarrow \begin{cases} \underset{\mathbf{x}_1 \in \mathcal{X}_1}{\text{minimize}} & f(\mathbf{x}_1, \mathbf{x}_{-1}^t) \\ & \vdots \\ \underset{\mathbf{x}_K \in \mathcal{X}_K}{\text{minimize}} & f(\mathbf{x}_K, \mathbf{x}_{-K}^t) \end{cases}$$

- The stepsize is determined by the line search.
- The Jacobi (best-response) algorithm exhibits fast convergence speed.
- Convergence conditions could be further relaxed (shown by examples)!

A general framework: special cases

Jacobi algorithm:

- If $\mathcal{X} = \{\mathbf{x} : \sum_{k=1}^{K} \mathbf{h}_k(\mathbf{x}_k) \leq \mathbf{0}\}$, the approximate problem is

$$\begin{array}{ll} \underset{\mathbf{x}}{\text{minimize}} & \tilde{f}(\mathbf{x};\mathbf{x}^t) = \sum_{k=1}^{K} f(\mathbf{x}_k, \mathbf{x}_{-k}^t) \\ \text{subject to} & \sum_{k=1}^{K} \mathbf{h}_k(\mathbf{x}_k) \leq \mathbf{0}. \end{array}$$

- It can be solved by the primal/dual decomposition techniques.
- For example, in dual decomposition, the Lagrangian is

$$\begin{aligned} L(\mathbf{x}, \boldsymbol{\mu}; \mathbf{x}^t) &= \sum_{k=1}^{K} f(\mathbf{x}_k, \mathbf{x}_{-k}^t) + \boldsymbol{\mu}^T \left(\sum_{k=1}^{K} \mathbf{h}_k(\mathbf{x}_k)\right) \\ &= \sum_{k=1}^{K} \left(f(\mathbf{x}_k, \mathbf{x}_{-k}^t) + \boldsymbol{\mu}^T \mathbf{h}_k(\mathbf{x}_k)\right). \end{aligned}$$

- The dual function $d(\boldsymbol{\mu})$ is

$$d(\boldsymbol{\mu}) = \min_{\mathbf{x}=(\mathbf{x}_k)_{k=1}^{K}} L(\mathbf{x}, \boldsymbol{\mu}; \mathbf{x}^t) \Rightarrow \min_{\mathbf{x}_k} \left\{ f(\mathbf{x}_k, \mathbf{x}_{-k}^t) + \boldsymbol{\mu}^T \mathbf{h}_k(\mathbf{x}_k) \right\}, \forall k.$$

- The dual variable $\boldsymbol{\mu}$ is updated by the subgradient method (bisection method if μ is a scalar); illustrated later by MIMO BC more details.

Dual variable update using subgradient

Dual variable update in inner iteration with iteration index m.

- Initialization: Choose initial dual variable $\boldsymbol{\mu}^{(0)}$.
- Step 1: Compute

$$\mathbb{B}\mathbf{x_k}^t(\boldsymbol{\mu}^{(m)}) \in \underset{\mathbf{x}=(\mathbf{x}_k)_{k=1}^K}{\arg\min} \left\{ f(\mathbf{x}_k, \mathbf{x}_{-k}^t) + \boldsymbol{\mu}^{(m)T} \mathbf{h}_k(\mathbf{x}_k) \right\}, \forall k.$$

- Note that $\sum_{k=1}^K \mathbf{h}_k(\mathbb{B}\mathbf{x}_k^t(\boldsymbol{\mu}^{(m)}))$ is subgradient of the dual function at $\boldsymbol{\mu}^{(m)}$ (Bertsekas, 2016, Sec. 7.1).
- Step 2: Update dual variable with subgradient:

$$\boldsymbol{\mu}^{(m+1)} = \boldsymbol{\mu}^{(m)} + \alpha^{(m)} \sum_{k=1}^K \mathbf{h}_k(\mathbb{B}\mathbf{x}_k^t(\boldsymbol{\mu}^{(m)})),$$

  where $\alpha^{(m)}$ is a properly chosen decreasing stepsize.
- Step 3: $m \leftarrow m + 1$ and go back to Step 1.

A general framework: special cases

Partial linearization method (Scutari, Facchinei, Song, Palomar, and Pang, 2014).

- In a multi-user communication network, the cost function for user k is $f_k(\mathbf{x}_k, \mathbf{x}_{-k})$ ($\mathbf{x}_{-k} = (\mathbf{x}_j)_{j \neq k}$) and it is convex in $\mathbf{x}_k$ only.
- Minimize the sum-cost function:

$$\begin{aligned} \underset{\mathbf{x}=(\mathbf{x}_k)_{k=1}^K}{\text{minimize}} \quad & \sum_{k=1}^K f_k(\mathbf{x}_k, \mathbf{x}_{-k}) \\ \text{subject to} \quad & \mathbf{x}_k \in \mathcal{X}_k, k = 1, \ldots, K. \end{aligned}$$

- Let us have a look at the objective function from $\mathbf{x}_k$'s perspective:

$$\sum_{j=1}^K f_j(\mathbf{x}) = \underbrace{f_k(\mathbf{x}_k, \mathbf{x}_{-k})}_{\text{convex in } \mathbf{x}_k} + \sum_{j \neq k} \underbrace{f_j(\mathbf{x}_j, \mathbf{x}_{-j})}_{\text{nonconvex in } \mathbf{x}_k} .$$

A general framework: special cases

Partial linearization method (Scutari, Facchinei, Song, Palomar, and Pang, 2014).

- The approximate problem can be formulated as follows:

$$\underset{(\mathbf{x}_k \in \mathcal{X}_k)_{k=1}^K}{\text{minimize}} \quad \tilde{f}(\mathbf{x};\mathbf{x}^t) \triangleq \sum_{k=1}^K \tilde{f}_k(\mathbf{x}_k;\mathbf{x}^t)$$

  where

$$\tilde{f}_k(\mathbf{x}_k;\mathbf{x}^t) = \underbrace{f_k(\mathbf{x}_k,\mathbf{x}_{-k}^t)}_{\text{convex}} + \underbrace{\left\langle \mathbf{x}_k - \mathbf{x}_k^t, \sum_{j\neq k} \nabla_{\mathbf{x}_k} f_j(\mathbf{x}^t) \right\rangle}_{\text{linear (convex)}}.$$

- Assumption verification:

$$\begin{aligned} \nabla_{\mathbf{x}_k} \sum_{j=1}^K \tilde{f}_j(\mathbf{x}_j;\mathbf{x}^t)\big|_{\mathbf{x}=\mathbf{x}^t} &= \nabla_{\mathbf{x}_k} \tilde{f}_k(\mathbf{x}_k^t;\mathbf{x}^t) + \sum_{j\neq k} \nabla_{\mathbf{x}_k} \tilde{f}_j(\mathbf{x}_j^t;\mathbf{x}^t) \\ &= \nabla_{\mathbf{x}_k} f_k(\mathbf{x}^t) + \sum_{j\neq k} \nabla_{\mathbf{x}_k} f_j(\mathbf{x}^t) \\ &= \nabla_{\mathbf{x}_k} \Big(\sum_{j=1}^K f_j(\mathbf{x})\Big)\Big|_{\mathbf{x}=\mathbf{x}^t}. \end{aligned}$$

A general framework: special cases

Partial linearization method (Scutari, Facchinei, Song, Palomar, and Pang, 2014).

- The approximate problem can be formulated as follows:

$$\underset{(\mathbf{x}_k \in \mathcal{X}_k)_{k=1}^K}{\text{minimize}} \quad \tilde{f}(\mathbf{x}; \mathbf{x}^t) \triangleq \sum_{k=1}^K \tilde{f}_k(\mathbf{x}_k; \mathbf{x}^t)$$

  where

$$\tilde{f}_k(\mathbf{x}_k; \mathbf{x}^t) = \underbrace{f_k(\mathbf{x}_k, \mathbf{x}_{-k}^t)}_{\text{convex}} + \underbrace{\left\langle \mathbf{x}_k - \mathbf{x}_k^t, \sum_{j \neq k} \nabla_{\mathbf{x}_k} f_j(\mathbf{x}^t) \right\rangle}_{\text{linear (convex)}}.$$

- The approximate problem can be solved in parallel among users:

$$\begin{cases} \underset{\mathbf{x}_1 \in \mathcal{X}_1}{\text{minimize}} & \tilde{f}_1(\mathbf{x}_1; \mathbf{x}^t) \\ & \vdots \\ \underset{\mathbf{x}_K \in \mathcal{X}_K}{\text{minimize}} & \tilde{f}_K(\mathbf{x}_K; \mathbf{x}^t) \end{cases}$$

A general framework: other cases

The block successive upper-bound minimization (BSUM) method (Razaviyayn et al., 2013) cannot be analyzed by the introduced framework.

- The approximate problem at iteration t is:

$$\underset{\mathbf{x}_k \in \mathcal{X}_k}{\text{minimize}} \quad \tilde{f}(\mathbf{x}_k; \mathbf{x}^t),$$

where $k = \text{mod}(t-1, K) + 1$,

$$\begin{aligned} \tilde{f}(\mathbf{x}_k; \mathbf{x}^t) &\geq f(\mathbf{x}_k, \mathbf{x}_{-k}^t), \\ \tilde{f}(\mathbf{x}_k^t; \mathbf{x}^t) &= f(\mathbf{x}_k^t, \mathbf{x}_{-k}^t), \\ \nabla_k \tilde{f}(\mathbf{x}_k; \mathbf{x}^t)\big|_{\mathbf{x}=\mathbf{x}^t} &= \nabla_k f(\mathbf{x}_k, \mathbf{x}_{-k}^t)\big|_{\mathbf{x}=\mathbf{x}^t}. \end{aligned}$$

- It does not satisfy the assumption on equal gradients (A2):

$$\begin{aligned} \nabla_k \tilde{f}(\mathbf{x}_k; \mathbf{x}^t)\big|_{\mathbf{x}=\mathbf{x}^t} &= \nabla_k f(\mathbf{x}_k, \mathbf{x}_{-k}^t)\big|_{\mathbf{x}=\mathbf{x}^t}, \\ \nabla_j \tilde{f}(\mathbf{x}_k; \mathbf{x}^t)\big|_{\mathbf{x}=\mathbf{x}^t} &= \mathbf{0} \neq \nabla_j f(\mathbf{x})\big|_{\mathbf{x}=\mathbf{x}^t}, \forall j \neq k. \end{aligned}$$

Outline

1 Theory
- Problem formulation and motivating examples
- Descent direction and stepsize
- Functions with different level of convexity
- Design of approximate functions

2 Applications
- MIMO MAC capacity maximization
- MIMO BC capacity maximization
- Global energy efficiency maximization in MIMO systems
- Sum energy efficiency maximization in MIMO systems
- Nondifferentiable problems and LASSO
- MD rank sparse regularization for MIMO channel estimation

MIMO MAC capacity maximization: sequential algorithm

- Suppose $\mathbf{H}_k$ is the channel from user k to the base station.
- The sum capacity of MIMO MAC is

$$\begin{aligned} \underset{\{\mathbf{Q}_k\}}{\text{maximize}} \quad & \log\left|\mathbf{I} + \textstyle\sum_{k=1}^{K} \mathbf{H}_k \mathbf{Q}_k \mathbf{H}_k^H\right| \\ \text{subject to} \quad & \mathbf{Q}_k \succeq \mathbf{0},\ \mathrm{tr}(\mathbf{Q}_k) \leq P_k,\ k = 1, \ldots, K. \end{aligned}$$

- The famous iterative (sequential) water-filling algorithm (Yu, Rhee, Boyd, and Cioffi, 2004) is essentially the block coordinate descent (BCD) algorithm (Bertsekas, 2016, Sec. 3.7):

$$\begin{aligned} \mathbf{Q}_1^{t+1} &= \underset{\mathrm{tr}(\mathbf{Q}_1) \leq P_1}{\arg\max} \ \log\left|\mathbf{I} + \mathbf{H}_1 \mathbf{Q}_1 \mathbf{H}_1^H + \textstyle\sum_{j=2}^{K} \mathbf{H}_j \mathbf{Q}_j^t \mathbf{H}_j^H\right| \\ \mathbf{Q}_2^{t+1} &= \underset{\mathrm{tr}(\mathbf{Q}_2) \leq P_2}{\arg\max} \ \log\left|\mathbf{I} + \mathbf{H}_1 \mathbf{Q}_1^{t+1} \mathbf{H}_1^H + \mathbf{H}_2 \mathbf{Q}_2 \mathbf{H}_2^H + \textstyle\sum_{j=3}^{K} \mathbf{H}_j \mathbf{Q}_j^t \mathbf{H}_j^H\right| \\ & \downarrow \\ \mathbf{Q}_K^{t+1} &= \underset{\mathrm{tr}(\mathbf{Q}_K) \leq P_K}{\arg\max} \ \log\left|\mathbf{I} + \textstyle\sum_{j=1}^{K-1} \mathbf{H}_j \mathbf{Q}_j^{t+1} \mathbf{H}_j^H + \mathbf{H}_K \mathbf{Q}_K \mathbf{H}_K^H\right| \end{aligned}$$

MIMO MAC capacity maximization: parallel algorithm

- Suppose $\mathbf{H}_k$ is the channel from user k to the base station.
  - The sum capacity of MIMO MAC is

$$\begin{array}{ll} \underset{\{\mathbf{Q}_k\}}{\text{maximize}} & \log\left|\mathbf{I} + \sum_{k=1}^K \mathbf{H}_k \mathbf{Q}_k \mathbf{H}_k^H\right| \\ \text{subject to} & \mathbf{Q}_k \succeq \mathbf{0},\ \text{tr}(\mathbf{Q}_k) \leq P_k,\ k = 1, \ldots, K. \end{array}$$

- Is there an iterative simultaneous water-filling algorithm? Yes.
- The approximate problem at the t-th iteration is

$$\begin{array}{ll} \underset{\{\mathbf{Q}_k\}_{k=1}^K}{\text{maximize}} & \sum_{k=1}^K \log\left|\mathbf{I} + \sum_{j\neq k} \mathbf{H}_j \mathbf{Q}_j^t \mathbf{H}_j^H + \mathbf{H}_k \mathbf{Q}_k \mathbf{H}_k^H\right| \\ \text{subject to} & \mathbf{Q}_k \succeq \mathbf{0},\ \text{tr}(\mathbf{Q}_k) \leq P_k,\ k = 1, \ldots, K. \end{array}$$

MIMO MAC capacity maximization: parallel algorithm

- Suppose $\mathbf{H}_k$ is the channel from user k to the base station.
  - The sum capacity of MIMO MAC is

$$\begin{array}{ll} \underset{\{\mathbf{Q}_k\}}{\text{maximize}} & \log\left|\mathbf{I} + \sum_{k=1}^K \mathbf{H}_k \mathbf{Q}_k \mathbf{H}_k^H\right| \\ \text{subject to} & \mathbf{Q}_k \succeq \mathbf{0},\ \mathrm{tr}(\mathbf{Q}_k) \leq P_k,\ k = 1, \ldots, K. \end{array}$$

- Is there an iterative simultaneous water-filling algorithm? Yes.

- The approximate problem at t-th iteration is

$$\Rightarrow \left\{ \begin{array}{ll} \underset{\mathbf{Q}_1 \succeq \mathbf{0}, \mathrm{tr}(\mathbf{Q}_1) \leq P_1}{\text{maximize}} & \log\left|\mathbf{I} + \sum_{j \neq 1} \mathbf{H}_j \mathbf{Q}_j^t \mathbf{H}_j^H + \mathbf{H}_1 \mathbf{Q}_1 \mathbf{H}_1^H\right| \\ & \vdots \\ \underset{\mathbf{Q}_K \succeq \mathbf{0}, \mathrm{tr}(\mathbf{Q}_K) \leq P_K}{\text{maximize}} & \log\left|\mathbf{I} + \sum_{j \neq K} \mathbf{H}_j \mathbf{Q}_j^t \mathbf{H}_j^H + \mathbf{H}_K \mathbf{Q}_K \mathbf{H}_K^H\right| \end{array} \right.$$

- It has a closed-form solution based on water-filling over noise and "interference" (pre-whitening): parallel implementation.

Stepsize computation based on exact line search

- Recall update procedure and exact line search:

$$\mathbf{x}^{t+1} = \mathbf{x}^t + \underbrace{\gamma^t}_{\text{stepsize}} \cdot \underbrace{(\mathbb{B}\mathbf{x}^t - \mathbf{x}^t)}_{\text{update direction}}$$

$$\gamma^t \in \underset{0 \leq \gamma \leq 1}{\arg\min}\ f(\mathbf{x}^t + \gamma(\mathbb{B}\mathbf{x}^t - \mathbf{x}^t)).$$

- With $\mathbf{Q}_k^{t+1} = \mathbf{Q}_k^t + \gamma^t(\mathbb{B}\mathbf{Q}_k^t - \mathbf{Q}_k^t)$ the exact line search consists in solving the convex optimization problem:

$$\gamma^t = \underset{0 \leq \gamma \leq 1}{\arg\max}\ \log\left|\mathbf{I} + \sum_{k=1}^K \mathbf{H}_k (\mathbf{Q}_k^t + \gamma(\mathbb{B}\mathbf{Q}_k^t - \mathbf{Q}_k^t))\mathbf{H}_k^H\right|$$

- This concave problem can be solved efficiently using bisection method (linear convergence).

Stepsize computation based on exact line search

- Alternative: Use efficient customized algorithm. (Bunch, Nielsen, and Sorensen (1978))

$$\gamma^t = \underset{0\le\gamma\le1}{\arg\max}\ \log\Big|\underbrace{\mathbf{I} + \textstyle\sum_{k=1}^K \mathbf{H}_k \mathbf{Q}_k^t \mathbf{H}_k^H}_{\mathbf{A}} + \gamma \underbrace{\textstyle\sum_{k=1}^K \mathbf{H}_k (\mathbb{B}\mathbf{Q}_k^t - \mathbf{Q}_k^t)\mathbf{H}_k^H}_{\mathbf{B}}\Big|$$

- Define $\lambda_1, \ldots, \lambda_r$ as the generalized eigenvalues of pair $(\mathbf{B}, \mathbf{A})$ partitioned in negative and positive sets $\{\lambda_1, \ldots, \lambda_{r'}\}$ and $\{\lambda_{r'+1}, \ldots, \lambda_r\}$, respectively.

$$\gamma^t = \underset{0\le\gamma\le1}{\arg\max} \sum_{i=1}^r \log(1+\gamma\lambda_i)$$

- Results in numerically rooting

$$\sum_{i=1}^r \frac{\lambda_i}{(1+\gamma\lambda_i)} = 0 \Rightarrow \sum_{i=1}^{r'} \frac{1}{(1/|\lambda_i| - \gamma)} = \sum_{i=r'+1}^{r} \frac{1}{(1/|\lambda_i| + \gamma)}$$

Stepsize computation based on exact line search

$$\psi(\gamma) \triangleq \sum_{i=1}^{r'} \frac{1}{(1/|\lambda_i| - \gamma)} = \sum_{i=r'+1}^{r} \frac{1}{(1/|\lambda_i| + \gamma)} \triangleq \phi(\gamma)$$

- Iteratively approximate each side by simple rational functions and root resulting quadratic function.

$$\tilde{\psi}_{p,q}(\gamma) \triangleq \frac{p}{q-\gamma} = u + \frac{v}{1/|\lambda_{r'+1}| - \gamma} \triangleq \tilde{\phi}_{u,v}(\gamma)$$

- At iteration $\tau + 1$ and point $\gamma^{(\tau)}$ choose p and q such that

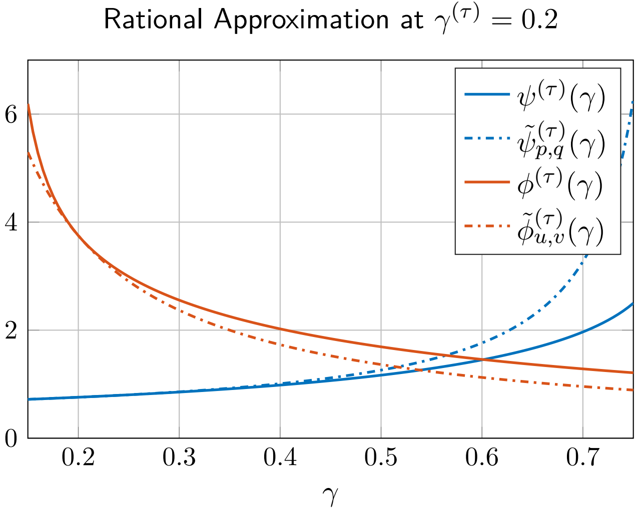
$$\tilde{\psi}_{p,q}(\gamma)\big|_{\gamma=\gamma^{(\tau)}} = \psi(\gamma)\big|_{\gamma=\gamma^{(\tau)}}, \qquad \nabla\tilde{\psi}_{p,q}(\gamma)\big|_{\gamma=\gamma^{(\tau)}} = \nabla\psi(\gamma)\big|_{\gamma=\gamma^{(\tau)}}.$$

- Similarly choose u and v such that

$$\tilde{\phi}_{u,v}(\gamma)\big|_{\gamma=\gamma^{(\tau)}} = \phi(\gamma)\big|_{\gamma=\gamma^{(\tau)}}, \qquad \nabla\tilde{\phi}_{u,v}(\gamma)\big|_{\gamma=\gamma^{(\tau)}} = \nabla\phi(\gamma)\big|_{\gamma=\gamma^{(\tau)}}.$$

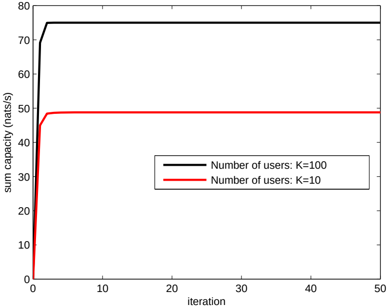
- Algorithm has quadratic convergence!

Stepsize computation based on exact line search



MIMO MAC capacity maximization

- Rayleigh channel, SNR=10dB.
- The number of Tx and Rx antennas: 5 and 10.
- The number of users: $K = 10$ and $K = 100$.



- The algorithm scales well with the number of users K.
- Complexity of the line search does not depend on the number of users K.

MIMO BC capacity maximization

- Suppose $\mathbf{H}_k$ is the channel from the base station to user k.
- The sum capacity of MIMO BC is (based on the downlink-uplink duality)

$$\begin{array}{ll} \underset{\{\mathbf{Q}_k\}}{\text{maximize}} & \log\left|\mathbf{I} + \sum_{k=1}^{K} \mathbf{H}_k^H \mathbf{Q}_k \mathbf{H}_k\right| \\ \text{subject to} & \mathbf{Q}_k \succeq \mathbf{0},\ k = 1, \ldots, K,\ \sum_{k=1}^{K} \text{tr}(\mathbf{Q}_k) \leq P. \end{array}$$

- The approximate problem at t-th iteration is

$$\begin{array}{ll} \underset{\{\mathbf{Q}_k\}}{\text{maximize}} & \sum_{k=1}^{K} \log\left|\mathbf{I} + \sum_{j \neq k} \mathbf{H}_j^H \mathbf{Q}_j^t \mathbf{H}_j + \mathbf{H}_k^H \mathbf{Q}_k \mathbf{H}_k\right| \\ \text{subject to} & \mathbf{Q}_k \succeq \mathbf{0},\ k = 1, \ldots, K, \quad \underbrace{\sum_{k=1}^{K} \text{tr}(\mathbf{Q}_k) \leq P}_{\text{coupling constraints, but seperable}} \quad . \end{array}$$

- A parallel implementation is possible based on dual decomposition ▸ more details.

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MIMO BC capacity maximization

- The Lagrangian is

$$L(\mathbf{Q},\mu) = \sum_{k=1}^{K} \log\left|\mathbf{I} + \sum_{j\neq k} \mathbf{H}_j^H \mathbf{Q}_j^t \mathbf{H}_j + \mathbf{H}_k^H \mathbf{Q}_k \mathbf{H}_k\right| + \mu\left(\sum_{k=1}^{K} \mathrm{tr}(\mathbf{Q}_k) - P\right).$$

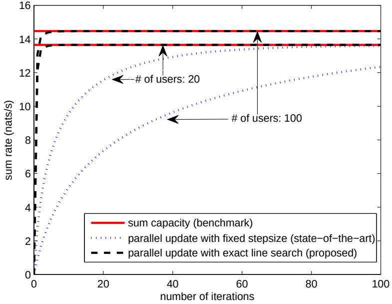
- The dual function is $d(\mu) = \max_{(\mathbf{Q}_k \succeq \mathbf{0})_{k=1}^K} \; L(\mathbf{Q},\mu)$:

$$\Rightarrow \begin{cases} \max\limits_{\mathbf{Q}_1 \succeq \mathbf{0}} & \log\left|\mathbf{I} + \sum_{j\neq 1} \mathbf{H}_j^H \mathbf{Q}_j^t \mathbf{H}_j + \mathbf{H}_1^H \mathbf{Q}_1 \mathbf{H}_1\right| + \mu \mathrm{tr}(\mathbf{Q}_1) \\ \max\limits_{\mathbf{Q}_2 \succeq \mathbf{0}} & \log\left|\mathbf{I} + \sum_{j\neq 2} \mathbf{H}_j^H \mathbf{Q}_j^t \mathbf{H}_j + \mathbf{H}_2^H \mathbf{Q}_2 \mathbf{H}_2\right| + \mu \mathrm{tr}(\mathbf{Q}_2) \\ & \vdots \\ \max\limits_{\mathbf{Q}_K \succeq \mathbf{0}} & \log\left|\mathbf{I} + \sum_{j\neq K} \mathbf{H}_j^H \mathbf{Q}_j^t \mathbf{H}_j + \mathbf{H}_K^H \mathbf{Q}_K \mathbf{H}_K\right| + \mu \mathrm{tr}(\mathbf{Q}_K) \end{cases}$$

- It has a closed-form solution based on water-filling.
- Optimal dual variable $\mu^\star$ can be found efficiently by the bisection method.

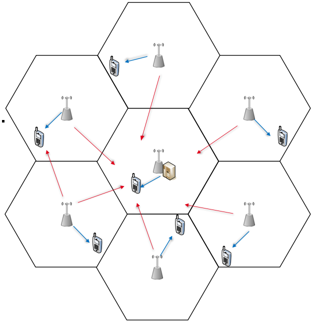
MIMO BC capacity maximization

- State-of-the-art: the iterative algorithm in Jindal, Rhee, Vishwanath, Jafar, and Goldsmith (2005):
  - Same approximate function, but with a constant stepsize: $\gamma^t = 1/K$.



- Complexity per iteration: 0.0023s water-filling+ 0.0018s exact line search

Global energy efficiency maximization in MIMO systems



- MIMO multi-cell network with K cells.
- Each cell serves a single user in the downlink.
- $\mathbf{Q}_k$: transmit covariance matrix of user k.
- σ_k^2: noise covariance at user k.
- $\{\mathbf{H}_{kj}\}_{k,j}$: channel from BS j to user k.
- Inter-cell interference is treated as noise.
- $\sum_{j\neq k} \mathbf{H}_{kj}\mathbf{Q}_j\mathbf{H}_{kj}^H$: interference covariance at user k.
- Achievable rate for link k (downlink):

$$r_k(\mathbf{Q}) \triangleq \log\left|\mathbf{I} + \left(\sigma_k^2\mathbf{I} + \sum_{j\neq k}\mathbf{H}_{kj}\mathbf{Q}_j\mathbf{H}_{kj}^H\right)^{-1}\mathbf{H}_{kk}\mathbf{Q}_k\mathbf{H}_{kk}^H\right|.$$

  - $r_k(\mathbf{Q})$ is concave in $\mathbf{Q}_k$, but not jointly concave in $\mathbf{Q} = (\mathbf{Q}_j)_{j=1}^K$.

Global energy efficiency maximization in MIMO systems

- Global Energy efficiency (GEE) of the network: Ratio of sum transmission rate divided by sum power consumption.

$$\begin{aligned} \underset{\mathbf{Q}}{\text{maximize}} \quad & \frac{\sum_{k=1}^{K} r_k(\mathbf{Q})}{\sum_{k=1}^{K}(P_{0,k} + \rho_k \text{tr}(\mathbf{Q}_k))} \triangleq f(\mathbf{Q}) \qquad \leftarrow \text{nonconcave} \\ \text{subject to} \quad & \mathbf{Q}_k \succeq \mathbf{0}, \text{tr}(\mathbf{Q}_k) \leq P_k, \forall k. \end{aligned}$$

- $P_{0,k}$: the power consumption at the zero RF output power.
- ρ_k: the slope of the load dependent power consumption.
- P_k: the maximum transmission power.
- The GEE maximization problem is nonconvex and NP-hard (even when maximizing the sum rate in the numerator, see Luo and Zhang (2008)).
- For the same problem, there exist different approximate functions:
  - The choice of approximate function depends on the problem structure.
  - Different choices of approximate functions lead to different algorithms.

Global energy efficiency maximization in MIMO systems

One choice of the approximate function is based on lower bound maximization, see Zappone, Björnson, Sanguinetti, and Jorswieck (2017).

- Achievable rate for link k (downlink):

$$\begin{aligned} r_k(\mathbf{Q}) &= \log\left|\mathbf{I} + \left(\sigma_k^2\mathbf{I} + \textstyle\sum_{j\neq k}\mathbf{H}_{kj}\mathbf{Q}_j\mathbf{H}_{kj}^H\right)^{-1}\mathbf{H}_{kk}\mathbf{Q}_k\mathbf{H}_{kk}^H\right| \\ &= \underbrace{\log\left|\sigma_k^2\mathbf{I} + \textstyle\sum_{j=1}^K\mathbf{H}_{kj}\mathbf{Q}_j\mathbf{H}_{kj}^H\right|}_{\triangleq r_k^+(\mathbf{Q}),\ \text{concave}} - \underbrace{\log\left|\sigma_k^2\mathbf{I} + \textstyle\sum_{j\neq k}\mathbf{H}_{kj}\mathbf{Q}_j\mathbf{H}_{kj}^H\right|}_{\triangleq r_k^-(\mathbf{Q}),\ \text{concave}} \end{aligned}$$

- Numerator function approximation $\sum_{k=1}^K \tilde{r}_k^{(\mathrm{a})}(\mathbf{Q};\mathbf{Q}^t)$ (choice (a)):

$$\tilde{r}_k^{(\mathrm{a})}(\mathbf{Q};\mathbf{Q}^t) \triangleq \underbrace{r_k^+(\mathbf{Q})}_{\text{concave in }\mathbf{Q}} - \underbrace{r_k^-(\mathbf{Q}^t)}_{\text{constant}} - \textstyle\sum_{j\neq k}\underbrace{(\mathbf{Q}_j - \mathbf{Q}_j^t)\bullet\nabla_j r_k^-(\mathbf{Q}^t)}_{\text{linear in }\mathbf{Q}_k},$$

  where $\mathbf{A}\bullet\mathbf{B} \triangleq \Re(\mathrm{Tr}(\mathbf{A}^H\mathbf{B}))$.

- $\sum_{k=1}^K\tilde{r}_k^{(\mathrm{a})}(\mathbf{Q};\mathbf{Q}^t)$ is a global lower bound of $\sum_{k=1}^K r_k(\mathbf{Q})$: $\tilde{r}_k^{(\mathrm{a})}(\mathbf{Q};\mathbf{Q}^t) \leq r_k(\mathbf{Q})$.

Global energy efficiency maximization in MIMO systems

In choice (a) the approximate function at $\mathbf{Q} = \mathbf{Q}^t$ is chosen as:

$$\tilde{f}^{(a)}(\mathbf{Q};\mathbf{Q}^t) = \frac{\tilde{r}^{(a)}(\mathbf{Q};\mathbf{Q}^t)}{P(\mathbf{Q})} \triangleq \frac{\sum_{k=1}^{K} \tilde{r}_k^{(a)}(\mathbf{Q};\mathbf{Q}^t)}{\sum_{k=1}^{K}(P_{0,k} + \rho_k \mathrm{tr}(\mathbf{Q}_k))}.$$

- Numerator function:

$$\tilde{r}_k^{(a)}(\mathbf{Q};\mathbf{Q}^t) \triangleq \underbrace{\underbrace{r_k^{+}(\mathbf{Q})}_{\text{concave in } \mathbf{Q}} - \underbrace{r_k^{-}(\mathbf{Q}^t)}_{\text{constant}} - \sum_{j\neq k} \underbrace{(\mathbf{Q}_j - \mathbf{Q}_j^t) \bullet \nabla_j r_k^{-}(\mathbf{Q}^t)}_{\text{linear in } \mathbf{Q}}}_{\text{concave in } \mathbf{Q}}.$$

- Denominator function:

$$P(\mathbf{Q}) \triangleq \underbrace{\sum_{j=1}^{K}(P_{0,k} + \rho_k \mathrm{tr}(\mathbf{Q}_k))}_{\text{convex in } \mathbf{Q}}.$$

- $\tilde{f}^{(a)}(\mathbf{Q};\mathbf{Q}^t)$ is a global lower bound of $f(\mathbf{Q})$ with equality at $\mathbf{Q} = \mathbf{Q}^t$:

$$\tilde{f}^{(a)}(\mathbf{Q};\mathbf{Q}^t) \leq f(\mathbf{Q}) \text{ and } \tilde{f}^{(a)}(\mathbf{Q}^t;\mathbf{Q}^t) = f(\mathbf{Q}^t).$$

Global energy efficiency maximization in MIMO systems

For approximate function (a) and defining $\tilde{r}^{(\mathrm{a})}(\mathbf{Q};\mathbf{Q}^t) \triangleq \sum_{k=1}^{K} \tilde{r}_k^{(\mathrm{a})}(\mathbf{Q};\mathbf{Q}^t)$ the following properties hold:

equal function value at $\mathbf{Q}^t$: $\quad \tilde{r}^{(\mathrm{a})}(\mathbf{Q};\mathbf{Q}^t)\big|_{\mathbf{Q}=\mathbf{Q}^t} = r(\mathbf{Q})\big|_{\mathbf{Q}=\mathbf{Q}^t}$

equal gradient at $\mathbf{Q}^t$: $\quad \nabla\tilde{r}^{(\mathrm{a})}(\mathbf{Q};\mathbf{Q}^t)\big|_{\mathbf{Q}=\mathbf{Q}^t} = \nabla r(\mathbf{Q})\big|_{\mathbf{Q}=\mathbf{Q}^t}$

such that

equal function value at $\mathbf{Q}^t$: $\tilde{f}^{(\mathrm{a})}(\mathbf{Q};\mathbf{Q}^t)\big|_{\mathbf{Q}=\mathbf{Q}^t} = \frac{\tilde{r}^{(\mathrm{a})}(\mathbf{Q};\mathbf{Q}^t)}{P(\mathbf{Q})}\big|_{\mathbf{Q}=\mathbf{Q}^t} = f(\mathbf{Q}^t)$,

quotient rule: $\nabla\tilde{f}^{(\mathrm{a})}(\mathbf{Q};\mathbf{Q}^t) = \frac{\nabla\tilde{r}^{(\mathrm{a})}(\mathbf{Q};\mathbf{Q}^t)P(\mathbf{Q}) - \tilde{r}^{(\mathrm{a})}(\mathbf{Q};\mathbf{Q}^t)\nabla P(\mathbf{Q})}{P^2(\mathbf{Q})}$,

equal gradient at $\mathbf{Q}^t$: $\nabla\tilde{f}^{(\mathrm{a})}(\mathbf{Q};\mathbf{Q}^t)\big|_{\mathbf{Q}=\mathbf{Q}^t} = \nabla f(\mathbf{Q}^t)$,

Global energy efficiency maximization in MIMO systems

We define the approximate function at $\mathbf{Q} = \mathbf{Q}^t$ as:

$$\tilde{f}^{(\mathrm{a})}(\mathbf{Q};\mathbf{Q}^t) = \underbrace{\frac{\sum_{k=1}^{K} \tilde{r}_k^{(\mathrm{a})}(\mathbf{Q};\mathbf{Q}^t) \leftarrow \text{concave in } \mathbf{Q}}{\sum_{k=1}^{K}(P_{0,k} + \rho_k \mathrm{tr}(\mathbf{Q}_k)) \leftarrow \text{convex in } \mathbf{Q}}}_{\text{pseudoconcave in } \mathbf{Q}}.$$

- The ratio of a concave function and a convex function is generally not concave.
- However, $\tilde{f}^{(\mathrm{a})}(\mathbf{Q};\mathbf{Q}^t)$, the ratio of a nonnegative concave function and a positive convex function, is pseudoconcave (A1).
- The lower bound assumption (A6) is satisfied and a constant unit stepsize can be used:

$$\gamma^t = 1 \Longrightarrow \mathbf{Q}^{t+1} = \underset{(\mathbf{Q}_k \succeq \mathbf{0}, \mathrm{tr}(\mathbf{Q}_k) \leq P_k)_{k=1}^K}{\arg\max} \tilde{f}^{(\mathrm{a})}(\mathbf{Q};\mathbf{Q}^t).$$

- A general-purpose optimization solver should be used to solve the approximate problem, which may be computationally expensive.

Global energy efficiency maximization in MIMO systems

Another choice of the approximate function, denoted as choice (b), is based on the best-response (Yang and Pesavento, 2017).

- Given the original function $f(\mathbf{Q})$ as:

$$f(\mathbf{Q}) = \frac{\sum_{k=1}^{K} r_k(\mathbf{Q})}{P(\mathbf{Q})}.$$

- We first define the approximate function at $\mathbf{Q} = \mathbf{Q}^t$ as:

$$\bar{f}^{(\mathrm{b})}(\mathbf{Q}; \mathbf{Q}^t) = \frac{\sum_{k=1}^{K} \bar{r}_k^{(\mathrm{b})}(\mathbf{Q}_k; \mathbf{Q}_{-k}^t)}{P(\mathbf{Q})}.$$

- Design approximate numerator functions $\bar{r}_k^{(\mathrm{b})}(\mathbf{Q}_k; \mathbf{Q}_{-k}^t) = r_k(\mathbf{Q}_k, \mathbf{Q}_{-k}^t) + \sum_{j \neq k}^{K} r_j(\mathbf{Q}_k, \mathbf{Q}_{-k}^t)$ to approximate $\sum_{k=1}^{K} r_k(\mathbf{Q})$ for variable $\mathbf{Q}_k$ and for $k = 1, \ldots, K$ using Jacobi-type approach.

Global energy efficiency maximization in MIMO systems

- Linearize non-concave part of approx. numerator funct. $\bar{r}_k^{(\mathrm{b})}(\mathbf{Q}_k;\mathbf{Q}^t)$ such that equal grad. cond. (A2): $\nabla\tilde{f}^{(\mathrm{b})}(\mathbf{Q}^t;\mathbf{Q}^t) = \nabla f^{(\mathrm{b})}(\mathbf{Q}^t)$ holds:

$$\tilde{r}_k^{(\mathrm{b})}(\mathbf{Q}_k;\mathbf{Q}^t) \triangleq \underbrace{\underbrace{r_k(\mathbf{Q}_k,\mathbf{Q}_{-k}^t)}_{\text{concave in } \mathbf{Q}_k} + \underbrace{(\mathbf{Q}_k - \mathbf{Q}_k^t) \bullet \sum_{j\neq k} \nabla_k r_j(\mathbf{Q}_k^t,\mathbf{Q}_{-k}^t)}_{\text{linear in } \mathbf{Q}_k}}_{\text{concave in } \mathbf{Q}_k},$$

- From quotient rule:

$$\nabla\tilde{f}^{(\mathrm{b})}(\mathbf{Q};\mathbf{Q}^t) = \frac{\nabla\tilde{r}^{(\mathrm{b})}(\mathbf{Q};\mathbf{Q}^t)P(\mathbf{Q}) - \tilde{r}^{(\mathrm{b})}(\mathbf{Q};\mathbf{Q}^t)\nabla P(\mathbf{Q})}{P^2(\mathbf{Q})}.$$

- The following zero- and first-order conditions must hold:

$$\tilde{r}_k^{(\mathrm{b})}(\mathbf{Q}_k;\mathbf{Q}_{-k}^t)\big|_{\mathbf{Q}=\mathbf{Q}^t} = r_k(\mathbf{Q})\big|_{\mathbf{Q}=\mathbf{Q}^t},$$

$$\nabla_k\tilde{r}_k^{(\mathrm{b})}(\mathbf{Q}_k;\mathbf{Q}_{-k}^t)\big|_{\mathbf{Q}=\mathbf{Q}^t} = \nabla_k\big(\textstyle\sum_{j=1}^K r_j(\mathbf{Q})\big)\big|_{\mathbf{Q}=\mathbf{Q}^t}.$$

Global energy efficiency maximization in MIMO systems

Choice (b) defines the approximate function at $\mathbf{Q} = \mathbf{Q}^t$ as:

$$\tilde{f}^{(b)}(\mathbf{Q}; \mathbf{Q}^t) = \underbrace{\frac{\sum_{k=1}^{K} \tilde{r}_k^{(b)}(\mathbf{Q}_k; \mathbf{Q}^t) \leftarrow \text{concave in } \mathbf{Q}}{\sum_{k=1}^{K}(P_{0,k} + \rho_k \text{tr}(\mathbf{Q}_k)) \leftarrow \text{convex in } \mathbf{Q}}}_{\text{pseudoconcave in } \mathbf{Q}} \neq \tilde{f}^{(a)}(\mathbf{Q}; \mathbf{Q}^t).$$

- The ratio of a concave function and a convex function is generally not concave.
- However, $\tilde{f}(\mathbf{Q}; \mathbf{Q}^t)$, the ratio of a nonnegative concave function and a positive convex function, is pseudoconcave.
- Stepsize: successive line search (GEE function is nonconcave).
- Fast convergence: problem structure is preserved as much as possible, while making problem separable.

Global energy efficiency maximization in MIMO systems

We define the approximate function at $\mathbf{Q} = \mathbf{Q}^t$ as:

$$\mathbb{B}\mathbf{Q}^t = \underset{(\mathbf{Q}_k \succeq \mathbf{0}, \mathrm{tr}(\mathbf{Q}_k) \leq P_k)_{k=1}^K}{\arg\max} \frac{\sum_{k=1}^K \tilde{r}_k^{(\mathrm{b})}(\mathbf{Q}_k; \mathbf{Q}^t)}{\sum_{k=1}^K (P_{0,k} + \rho_k \mathrm{tr}(\mathbf{Q}_k))}.$$

- It represents a fractional programming problem and can be solved by the **Dinkelbach's algorithm**:

$$\underset{(\mathbf{Q}_k \succeq \mathbf{0}, \mathrm{tr}(\mathbf{Q}_k) \leq P_k)_{k=1}^K}{\arg\max} \sum_{k=1}^K \tilde{r}_k^{(\mathrm{b})}(\mathbf{Q}_k; \mathbf{Q}^t) - \lambda \sum_{k=1}^K (P_{0,k} + \rho_k \mathrm{tr}(\mathbf{Q}_k)).$$

- This problem is separable among different variables $(\mathbf{Q}_k)_{k=1}^K$.

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Global energy efficiency maximization in MIMO systems

We define the approximate function at $\mathbf{Q} = \mathbf{Q}^t$ as:

$$\mathbb{B}\mathbf{Q}^t = \underset{(\mathbf{Q}_k \succeq \mathbf{0}, \mathrm{tr}(\mathbf{Q}_k) \le P_k)_{k=1}^K}{\arg\max} \frac{\sum_{k=1}^K \tilde{r}_k^{(\mathrm{b})}(\mathbf{Q}_k; \mathbf{Q}^t)}{\sum_{k=1}^K (P_{0,k} + \rho_k \mathrm{tr}(\mathbf{Q}_k))}.$$

- It represents a fractional programming problem and can be solved by the **Dinkelbach's algorithm**:

$$\begin{cases} \underset{\mathbf{Q}_1 \succeq \mathbf{0}, \mathrm{tr}(\mathbf{Q}_1) \le P_1}{\arg\max} & \tilde{r}_1^{(\mathrm{b})}(\mathbf{Q}_1; \mathbf{Q}^t) - \lambda(P_{0,1} + \rho_1 \mathrm{tr}(\mathbf{Q}_1)) \\ & \vdots \\ \underset{\mathbf{Q}_K \succeq \mathbf{0}, \mathrm{tr}(\mathbf{Q}_K) \le P_K}{\arg\max} & \tilde{r}_K^{(\mathrm{b})}(\mathbf{Q}_K; \mathbf{Q}^t) - \lambda(P_{0,K} + \rho_K \mathrm{tr}(\mathbf{Q}_K)) \end{cases}$$

  This solution can be computed in closed-form.
- Easy implementation: closed-form expression + parallel decomposition.

Global energy efficiency maximization in MIMO systems

at $\tau = 0$: $\lambda^{t,0} = 0$ (or other value).
Step 1: Given $\lambda^{t,\tau}$ at iteration $\tau + 1$ solve the following concave problem:

$$\mathbf{Q}_k(\lambda^{t,\tau}) \triangleq \underset{\mathbf{Q}_k \succeq \mathbf{0}, \mathrm{tr}(\mathbf{Q}_k) \leq P_k}{\arg\max} \tilde{r}_k^{(\mathrm{b})}(\mathbf{Q}_k; \mathbf{Q}^t) - \lambda^{t,\tau}(P_{0,k} + \rho_k \mathrm{tr}(\mathbf{Q}_k)).$$

This solution can be computed in closed-form.
Step 2: The variable $\lambda^{t,\tau}$ is then updated in iteration $\tau + 1$ as

$$\lambda^{t,\tau+1} = \frac{\sum_{k=1}^K \tilde{r}_k^{(\mathrm{b})}(\mathbf{Q}_k(\lambda^{t,\tau}); \mathbf{Q}^t)}{\sum_{k=1}^K P_{0,k} + \rho_k \mathrm{tr}(\mathbf{Q}_k(\lambda_k^{t,\tau}))}.$$

- The Dinkelbach's algorithm converges $\lim_{\tau \to \infty} \mathbf{Q}(\lambda^{t,\tau}) = \mathbb{B}\mathbf{Q}^t$ at superlinear rate.

Global energy efficiency maximization in MIMO systems

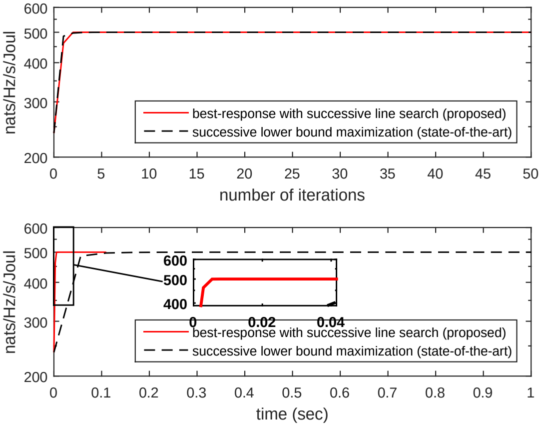

Figure: Energy Efficiency Maximization: achieved energy efficiency versus the number of iterations, Number of Tx antennas: $M = 50$, Number of cells: $K = 10$

State-of-the-art: A. Zappone, L. Sanguinetti, G. Bacci, E. Jorswieck, and M. Debbah, "Energy-efficient power control: A look at 5G wireless technologies," *IEEE Transactions on Signal Processing*, vol. 64, no. 7, pp. 1668-1683, April 2016.

Global energy efficiency maximization in MIMO systems

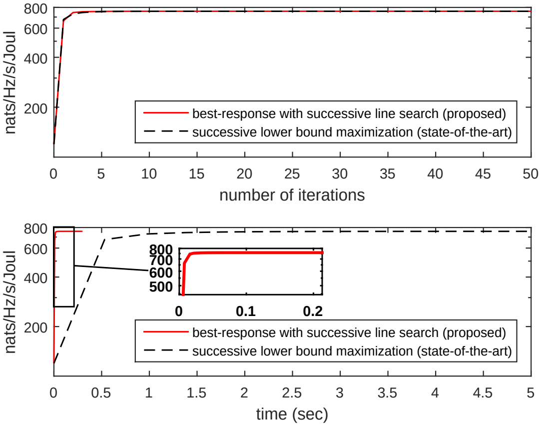

Figure: Energy Efficiency Maximization: achieved energy efficiency versus the number of iterations, Number of Tx antennas: $M = 50$, Number of cells: $K = 50$

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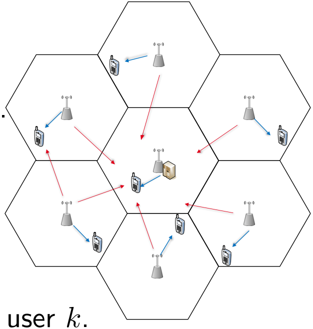
Global energy efficiency maximization in MIMO systems

- Global Energy efficiency (GEE): Ratio of sum transmission rate divided by sum power consumption

$$\begin{aligned}\underset{\mathbf{Q}}{\text{maximize}} \quad & f(\mathbf{Q}) \triangleq \frac{\sum_{k=1}^{K} r_k(\mathbf{Q})}{\sum_{k=1}^{K}(P_{0,k} + \rho_k \text{tr}(\mathbf{Q}_k))} \\ \text{subject to} \quad & \mathbf{Q}_k \succeq \mathbf{0}, \text{tr}(\mathbf{Q}_k) \leq P_k, \forall k, \\ & r_k(\mathbf{Q}) \geq R_k, \forall k. \leftarrow \text{QoS constraints}\end{aligned}$$

- $P_{0,k}$: the power consumption at the zero RF output power.
- ρ_k: the slope of the load dependent power consumption.
- P_k: the maximum transmission power.
- R_k: minimum guaranteed transmission rate for user k.
- Iterative algorithm based on the approximation of
  - the nonconcave objective function: pseudoconvex approximation;
  - the nonconvex constraint set: inner approximation.

Sum energy efficiency maximization in MIMO systems



- MIMO multi-cell network with K cells.
- Each cell serves a single user in the downlink.
- $\mathbf{Q}_k$: transmit covariance matrix of user k.
- σ_k^2: noise covariance at user k.
- $\{\mathbf{H}_{kj}\}_{k,j}$: channel from BS j to user k.
- Inter-cell interference is treated as noise.
- $\sum_{j\neq k}\mathbf{H}_{kj}\mathbf{Q}_j\mathbf{H}_{kj}^H$: interference covariance at user k.
- Achievable rate for link k (downlink):

$$\begin{aligned} r_k(\mathbf{Q}) &\triangleq \log\left|\mathbf{I} + \left(\sigma_k^2\mathbf{I} + \textstyle\sum_{j\neq k}\mathbf{H}_{kj}\mathbf{Q}_j\mathbf{H}_{kj}^H\right)^{-1}\mathbf{H}_{kk}\mathbf{Q}_k\mathbf{H}_{kk}^H\right| \\ &= \log\left|\sigma_k^2\mathbf{I} + \textstyle\sum_{j=1}^K\mathbf{H}_{kj}\mathbf{Q}_k\mathbf{H}_{kj}^H\right| - \log\left|\sigma_k^2\mathbf{I} + \textstyle\sum_{j\neq k}\mathbf{H}_{kj}\mathbf{Q}_j\mathbf{H}_{kj}^H\right|. \end{aligned}$$

  - $r_k(\mathbf{Q})$ is concave in $\mathbf{Q}_k$, but not jointly concave in $\mathbf{Q} = (\mathbf{Q}_j)_{j=1}^K$.

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Sum energy efficiency maximization in MIMO systems

- Sum Energy efficiency (SEE): Sum of individual energy efficiency.

$$\begin{aligned} \underset{\mathbf{Q}}{\text{maximize}} \quad & f(\mathbf{Q}) \triangleq \sum_{k=1}^{K} \frac{r_k(\mathbf{Q})}{P_{0,k} + \rho_k \text{tr}(\mathbf{Q}_k)} \leftarrow \text{nonconcave} \\ \text{subject to} \quad & \mathbf{Q}_k \succeq \mathbf{0}, \text{tr}(\mathbf{Q}_k) \leq P_k, \forall k. \end{aligned}$$

- $P_{0,k}$: the power consumption at the zero RF output power.
- ρ_k: the slope of the load dependent power consumption.
- P_k: the maximum transmission power.
- The SEE maximization problem is nonconvex and generally difficult to solve (NP-hard).

Sum energy efficiency maximization in MIMO systems

We derive approximate function at $\mathbf{Q} = \mathbf{Q}^t$ using Jacobi method as:

$$\bar{f}^{(\mathrm{c})}(\mathbf{Q};\mathbf{Q}^t) = \sum_{j=1}^{K}\Bigg(\underbrace{\underbrace{\frac{r_j(\mathbf{Q}_j,\mathbf{Q}_{-j}^t)}{P_j(\mathbf{Q}_j)}}_{\text{pseudoconcave}} + \underbrace{\sum_{k\neq j}^{K}\frac{r_j(\mathbf{Q}_k,\mathbf{Q}_{-k}^t)}{P_j(\mathbf{Q}_j^t)}}_{\text{non (pseudo)concave}}}_{\text{not pseudoconcave}}\Bigg).$$

Problem: Approx. function $\bar{f}^{(\mathrm{c})}(\mathbf{Q};\mathbf{Q}^t)$ is generally not pseudoconvex.
Straightforward idea: Linearize numerator of non-pseudoconcave term.

$$\tilde{f}^{(\mathrm{c}')}(\mathbf{Q};\mathbf{Q}^t)=\sum_{k=1}^{K}\Bigg(\underbrace{\frac{r_k(\mathbf{Q}_k,\mathbf{Q}_{-k}^t)}{P_k(\mathbf{Q}_k)}}_{\text{pseudoconvex}}+\underbrace{\sum_{j\neq k}^{K}\frac{r_j(\mathbf{Q}^t)+(\mathbf{Q}_k-\mathbf{Q}_k^t)\bullet\nabla_k r_j(\mathbf{Q}_k^t,\mathbf{Q}_{-k}^t)}{P_j(\mathbf{Q}_j^t)}}_{\text{linear}}\Bigg).$$

Problem: Sum of pseudoconcave and linear function is generally not pseudoconcave.

Sum energy efficiency maximization in MIMO systems

Solution: Introduce common denominator before linearizing numerator.

$$\bar{f}^{(c)}(\mathbf{Q},\mathbf{Q}^t)=\sum_{k=1}^{K}\Bigg(\underbrace{\frac{r_k(\mathbf{Q}_k,\mathbf{Q}_{-k}^t)}{P_k(\mathbf{Q}_k)}}_{\text{pseudoconcave}}+\sum_{j\neq k}^{K}\underbrace{\frac{r_j(\mathbf{Q}_k,\mathbf{Q}_{-k}^t)P_k(\mathbf{Q}_k)}{P_j(\mathbf{Q}_j^t)P_k(\mathbf{Q}_k)}}_{\text{linearize numerator}}\Bigg).$$

Linearizing the numerator the approximate function becomes pseudoconvex:

$$\tilde{f}^{(c)}(\mathbf{Q};\mathbf{Q}^t)=\sum_{k=1}^{K}\tilde{f}_k^{(c)}(\mathbf{Q}_k;\mathbf{Q}^t),$$

where $\tilde{f}_k^{(c)}(\mathbf{Q}_k;\mathbf{Q}^t)$ is a nonnegative concave over positive convex (thus pseudoconcave!) function:

$$\tilde{f}_k^{(c)}(\mathbf{Q}_k;\mathbf{Q}^t)=\frac{r_k(\mathbf{Q}_k,\mathbf{Q}_{-k}^t)}{P_k(\mathbf{Q}_k)}+\sum_{j\neq k}^{K}\frac{r_j(\mathbf{Q}^t)P_k(\mathbf{Q}_k^t)+(\mathbf{Q}_k-\mathbf{Q}_k^t)\bullet\nabla_k\Big(r_j(\mathbf{Q}_k,\mathbf{Q}^t)P_k(\mathbf{Q}_k)\Big)\Big|_{\mathbf{Q}_k=\mathbf{Q}_k^t}}{P_j(\mathbf{Q}_j^t)P_k(\mathbf{Q}_k)}$$

Sum energy efficiency maximization in MIMO systems

- Equal gradient condition at $\mathbf{Q} = \mathbf{Q}^t$:

$$\nabla_k \tilde{f}(\mathbf{Q}; \mathbf{Q}^t)\big|_{\mathbf{Q}=\mathbf{Q}^t} = \nabla_k f(\mathbf{Q})\big|_{\mathbf{Q}=\mathbf{Q}^t}.$$

- Follows directly from Jacobi approach and partial linearization.
- The approximate problem is:

$$\underset{(\mathbf{Q}_k)_{k=1}^K}{\text{maximize}} \quad \underbrace{\sum_{k=1}^{K} \underbrace{\tilde{f}_k^{(\mathrm{c})}(\mathbf{Q}_k; \mathbf{Q}^t)}_{\text{pseudoconcave in } \mathbf{Q}_k}}_{\text{not necessarily pseudoconcave in } \mathbf{Q}=(\mathbf{Q}_\mathbf{k})_{k=1}^K}$$

$$\text{subject to} \quad \mathbf{Q}_k \succeq \mathbf{0}, \mathrm{tr}(\mathbf{Q}_k) \leq P_k,\ k = 1, \ldots, K.$$

- The sum of pseudoconcave functions is not necessarily pseudoconcave.

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Sum energy efficiency maximization in MIMO systems

- The approximate problem can be decomposed into K subproblems:

$$\Longrightarrow \begin{cases} \underset{\mathbf{Q}_1 \succeq \mathbf{0}, \mathrm{tr}(\mathbf{Q}_1) \leq P_1}{\text{maximize}} & \tilde{f}_1^{(\mathrm{c})}(\mathbf{Q}_1; \mathbf{Q}^t) \\ & \vdots \\ \underset{\mathbf{Q}_K \succeq \mathbf{0}, \mathrm{tr}(\mathbf{Q}_K) \leq P_K}{\text{maximize}} & \tilde{f}_K^{(\mathrm{c})}(\mathbf{Q}_K; \mathbf{Q}^t) \end{cases}$$

- Each subproblem is pseudoconcave!

$$\mathbb{B}_k \mathbf{Q}^t = \underset{\mathbf{Q}_k \succeq \mathbf{0}, \mathrm{tr}(\mathbf{Q}_k) \leq P_k}{\arg\max} \quad \tilde{f}_k^{(\mathrm{c})}(\mathbf{Q}_k; \mathbf{Q}^t), k = 1, \ldots, K.$$

$$\Longrightarrow \tilde{f}_k^{(\mathrm{c})}(\mathbb{B}_k \mathbf{Q}_k^t; \mathbf{Q}^t) > \tilde{f}_k^{(\mathrm{c})}(\mathbf{Q}_k^t; \mathbf{Q}^t).$$

Sum energy efficiency maximization in MIMO systems

- A descent direction for the original problem is obtained as:

$$0 < (\mathbb{B}_k \mathbf{Q}^t - \mathbf{Q}_k^t) \bullet \nabla_k \left(\tilde{f}_k^{(\mathrm{c})}(\mathbf{Q}_k; \mathbf{Q}^t) \right) \big|_{\mathbf{Q}_k = \mathbf{Q}_k^t} = (\mathbb{B}_k \mathbf{Q}^t - \mathbf{Q}_k^t) \bullet \nabla_k f(\mathbf{Q}^t).$$

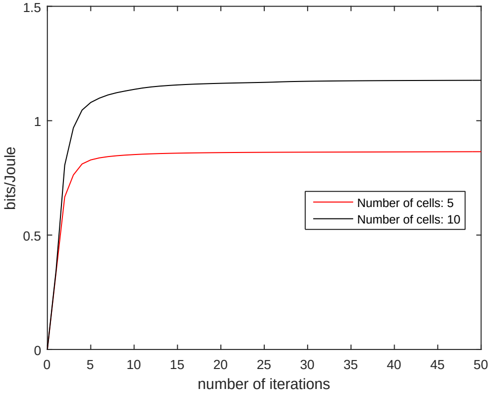
- Adding up over $k = 1, \ldots, K$, we obtain $(\mathbb{B}\mathbf{Q}^t - \mathbf{Q}^t) \bullet \nabla f(\mathbf{Q}^t) > 0$, where $\mathbb{B}\mathbf{Q}^t = (\mathbb{B}_k \mathbf{Q}^t)_{k=1}^K$.
- Stepsize: successive line search.

The proposed algorithm has the following advantages:

- Easy implementation: closed-form solution + parallel decomposition.
- Fast convergence: problem structure is preserved as much as possible, while decoupling into subproblems.
- Guaranteed convergence: despite the fact that approximate problem is not even pseudoconvex.

Sum energy efficiency maximization in MIMO systems

- The number of Tx and Rx antennas: 4 and 4.
- The number of cells: $K = 5$ and 10.
- Noise covariance $\sigma^2 = 1$, and power budget $P_k = 36$dBm.



A general framework: nonsmooth problems

- Consider the following nonsmooth optimization problem:

$$\begin{array}{ll} \underset{\mathbf{x}}{\text{minimize}} & f(\mathbf{x}) + g(\mathbf{x}) \\ \text{subject to} & \mathbf{x} \in \mathcal{X}. \end{array}$$

A general framework: nonsmooth problems

- Consider the following nonsmooth optimization problem:

$$\begin{aligned} \underset{\mathbf{x},y}{\text{minimize}} \quad & f(\mathbf{x}) + y \\ \text{subject to} \quad & \mathbf{x} \in \mathcal{X}, g(\mathbf{x}) \leq y. \end{aligned}$$

- The approximate problem is

$$\begin{aligned} \underset{\mathbf{x},y}{\text{minimize}} \quad & \tilde{f}(\mathbf{x}; \mathbf{x}^t) + y \\ \text{subject to} \quad & \mathbf{x} \in \mathcal{X}, g(\mathbf{x}) \leq y, \end{aligned}$$

  with optimal solution denoted as $(\mathbb{B}\mathbf{x}^t, y^\star(\mathbf{x}^t))$ and $y^\star(\mathbf{x}^t) = g(\mathbb{B}\mathbf{x}^t)$.

- Assumption verification:

$$\nabla_{\mathbf{x}}(\tilde{f}(\mathbf{x}; \mathbf{x}^t) + y)\big|_{\mathbf{x}=\mathbf{x}^t, y=y^t} = \nabla_{\mathbf{x}} f(\mathbf{x}^t) = \nabla_{\mathbf{x}}(f(\mathbf{x}) + y)\big|_{\mathbf{x}=\mathbf{x}^t},$$
$$\nabla_y(\tilde{f}(\mathbf{x}; \mathbf{x}^t) + y) = 1 = \nabla_y(f(\mathbf{x}) + y).$$

A general framework: nonsmooth problems

- Consider the following nonsmooth optimization problem:

$$\begin{array}{ll} \underset{\mathbf{x},y}{\text{minimize}} & f(\mathbf{x}) + y \\ \text{subject to} & \mathbf{x} \in \mathcal{X}, g(\mathbf{x}) \leq y. \end{array}$$

- The approximate problem is

$$\begin{array}{ll} \underset{\mathbf{x},y}{\text{minimize}} & \tilde{f}(\mathbf{x}; \mathbf{x}^t) + y \\ \text{subject to} & \mathbf{x} \in \mathcal{X}, g(\mathbf{x}) \leq y, \end{array}$$

with optimal solution denoted as $(\mathbb{B}\mathbf{x}^t, y^\star(\mathbf{x}^t))$ and $y^\star(\mathbf{x}^t) = g(\mathbb{B}\mathbf{x}^t)$.

- Exact line search over $f(\mathbf{x}) + y$:

$$\gamma^t \in \underset{0 \leq \gamma \leq 1}{\arg\min} \{ f(\mathbf{x}^t + \gamma(\mathbb{B}\mathbf{x}^t - \mathbf{x}^t)) + y^t + \gamma(y^\star(\mathbf{x}^t) - y^t)) \},$$

where $y^t \geq g(\mathbf{x}^t)$ and $y^\star(\mathbf{x}^t) = g(\mathbb{B}\mathbf{x}^t)$.

A general framework: nonsmooth problems

- Consider the following nonsmooth optimization problem:

$$\begin{array}{ll} \underset{\mathbf{x},y}{\text{minimize}} & f(\mathbf{x}) + y \\ \text{subject to} & \mathbf{x} \in \mathcal{X}, g(\mathbf{x}) \leq y. \end{array}$$

- The approximate problem is

$$\begin{array}{ll} \underset{\mathbf{x},y}{\text{minimize}} & \tilde{f}(\mathbf{x}; \mathbf{x}^t) + y \\ \text{subject to} & \mathbf{x} \in \mathcal{X}, g(\mathbf{x}) \leq y, \end{array}$$

with optimal solution denoted as $(\mathbb{B}\mathbf{x}^t, y^\star(\mathbf{x}^t))$ and $y^\star(\mathbf{x}^t) = g(\mathbb{B}\mathbf{x}^t)$.

- Exact line search:

$$\gamma^t \in \underset{0 \leq \gamma \leq 1}{\arg\min} \{ f(\mathbf{x}^t + \gamma(\mathbb{B}\mathbf{x}^t - \mathbf{x}^t)) + \cancel{g(\mathbf{x}^t)} + \gamma(g(\mathbb{B}\mathbf{x}^t) - g(\mathbf{x}^t))) \}.$$

A general framework: nonsmooth problems

- Consider the following nonsmooth optimization problem:

$$\begin{aligned} \underset{\mathbf{x},y}{\text{minimize}} \quad & f(\mathbf{x}) + y \\ \text{subject to} \quad & \mathbf{x} \in \mathcal{X}, g(\mathbf{x}) \leq y. \end{aligned}$$

- The approximate problem is

$$\begin{aligned} \underset{\mathbf{x},y}{\text{minimize}} \quad & \tilde{f}(\mathbf{x}; \mathbf{x}^t) + y \\ \text{subject to} \quad & \mathbf{x} \in \mathcal{X}, g(\mathbf{x}) \leq y, \end{aligned}$$

with optimal solution denoted as $(\mathbb{B}\mathbf{x}^t, y^\star(\mathbf{x}^t))$ and $y^\star(\mathbf{x}^t) = g(\mathbb{B}\mathbf{x}^t)$.

- Exact line search:

$$\gamma^t \in \underset{0 \leq \gamma \leq 1}{\arg\min} \{ f(\mathbf{x}^t + \gamma(\mathbb{B}\mathbf{x}^t - \mathbf{x}^t)) + \gamma(g(\mathbb{B}\mathbf{x}^t) - g(\mathbf{x}^t)) \}.$$

A general framework: nonsmooth problems

- Consider the following nonsmooth optimization problem:

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & f(\mathbf{x}) + g(\mathbf{x}) \\ \text{subject to} \quad & \mathbf{x} \in \mathcal{X}. \end{aligned}$$

- The approximate problem is

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & \tilde{f}(\mathbf{x}; \mathbf{x}^t) + g(\mathbf{x}) \\ \text{subject to} \quad & \mathbf{x} \in \mathcal{X}, \end{aligned}$$

- Exact line search is over a differentiable function:

$$\gamma^t \in \underset{0 \le \gamma \le 1}{\arg\min} \left\{ f(\mathbf{x}^t + \gamma(\mathbb{B}\mathbf{x}^t - \mathbf{x}^t)) + \gamma(g(\mathbb{B}\mathbf{x}^t) - g(\mathbf{x}^t)) \right\}.$$

- Traditional exact line search suffers from a high complexity:

$$\gamma^t \in \underset{0 \le \gamma \le 1}{\arg\min} \left\{ f(\mathbf{x}^t + \gamma(\mathbb{B}\mathbf{x}^t - \mathbf{x}^t)) + g(\mathbf{x}^t + {\color{red}\gamma}(\mathbb{B}\mathbf{x}^t - \mathbf{x}^t)) \right\}.$$

LASSO

- An important and widely studied problem in sparse regularization:

$$\underset{\mathbf{x}}{\text{minimize}} \quad \underbrace{\tfrac{1}{2}\left\|\mathbf{A}\mathbf{x}-\mathbf{b}\right\|_2^2}_{\triangleq f(\mathbf{x})} + \underbrace{\mu\left\|\mathbf{x}\right\|_1}_{\triangleq g(\mathbf{x})}.$$

- The approximate problem with a scalar decomposition $\mathbf{x} = (x_k)_{k=1}^K$ is:

$$\begin{aligned}\mathbb{B}\mathbf{x}^t &= \arg\min_{\mathbf{x}} \underbrace{\textstyle\sum_{k=1}^K f(x_k, \mathbf{x}_{-k}^t)}_{=\tilde{f}(\mathbf{x};\mathbf{x}^t)} + g(\mathbf{x}) \\ &= \mathbf{D}^{-1}\mathcal{S}_{\mu\mathbf{1}}(\mathbf{D}\mathbf{x}^t - \mathbf{A}^T(\mathbf{A}\mathbf{x}^t - \mathbf{b}))\end{aligned}$$

  where $\mathbf{D} \triangleq \text{diag}(\mathbf{A}^T\mathbf{A})$, $\mathcal{S}_{\mathbf{a}}(\mathbf{b}) = [\mathbf{b}-\mathbf{a}]^+ - [-\mathbf{b}-\mathbf{a}]^+$.
- Closed-form expression known as soft-thresholding operator.
- This choice of approximate function is well known, see Elad (2006); Facchinei, Scutari, and Sagratella (2015).

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LASSO

- An important and widely studied problem in sparse signal recovery:

$$\underset{\mathbf{x}}{\text{minimize}} \quad \underbrace{\frac{1}{2}\left\|\mathbf{A}\mathbf{x}-\mathbf{b}\right\|_2^2}_{\triangleq f(\mathbf{x})} + \underbrace{\mu\left\|\mathbf{x}\right\|_1}_{\triangleq g(\mathbf{x})}.$$

- The exact line search has a closed-form solution:

$$\begin{aligned}
\gamma^t &= \underset{0\leq\gamma\leq 1}{\arg\min}\ \left\{f(\mathbf{x}^t+\gamma(\mathbb{B}\mathbf{x}^t-\mathbf{x}^t))+\gamma\big(g(\mathbb{B}\mathbf{x}^t)-g(\mathbf{x}^t)\big)\right\}\\
&= \underset{0\leq\gamma\leq 1}{\arg\min}\ \left\{\begin{array}{c}\frac{1}{2}\left\|\mathbf{A}(\mathbf{x}^t+\gamma(\mathbb{B}\mathbf{x}^t-\mathbf{x}^t))-\mathbf{b}\right\|_2^2\\ +\gamma\mu(\left\|\mathbb{B}\mathbf{x}^t\right\|_1-\left\|\mathbf{x}^t\right\|_1)\end{array}\right\}\\
&= \left[-\frac{\langle\mathbf{A}\mathbf{x}^t-\mathbf{b},\mathbf{A}(\mathbb{B}\mathbf{x}^t-\mathbf{x}^t)\rangle+\mu(\left\|\mathbb{B}\mathbf{x}^t\right\|_1-\left\|\mathbf{x}^t\right\|_1)}{\langle\mathbf{A}(\mathbb{B}\mathbf{x}^t-\mathbf{x}^t),\mathbf{A}(\mathbb{B}\mathbf{x}^t-\mathbf{x}^t)\rangle}\right]_0^1.
\end{aligned}$$

- Soft-Thresholding with Exact Line search Algorithm (STELA, Yang and Pesavento (2017)).

LASSO

- An important and widely studied problem in sparse signal recovery:

$$\underset{\mathbf{x}}{\text{minimize}} \quad \underbrace{\tfrac{1}{2}\|\mathbf{A}\mathbf{x}-\mathbf{b}\|_2^2}_{\triangleq f(\mathbf{x})} + \underbrace{\mu\|\mathbf{x}\|_1}_{\triangleq g(\mathbf{x})}.$$

- The exact line search has a closed-form solution:

$$\gamma^t = \left[-\frac{\langle \mathbf{A}\mathbf{x}^t - \mathbf{b}, \mathbf{A}(\mathbb{B}\mathbf{x}^t - \mathbf{x}^t)\rangle + \mu(\|\mathbb{B}\mathbf{x}^t\|_1 - \|\mathbf{x}^t\|_1)}{\langle \mathbf{A}(\mathbb{B}\mathbf{x}^t - \mathbf{x}^t), \mathbf{A}(\mathbb{B}\mathbf{x}^t - \mathbf{x}^t)\rangle}\right]_0^1.$$

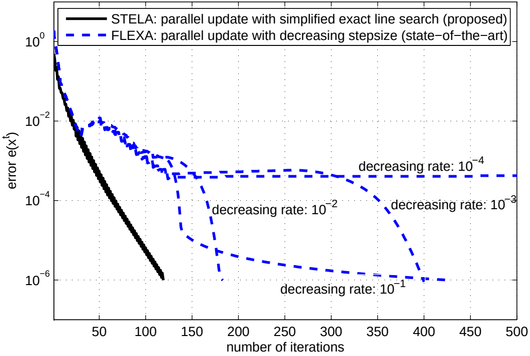
- By comparison, the traditional exact line search is:

$$\begin{aligned}\gamma^t &= \underset{0\leq\gamma\leq 1}{\arg\min} \left\{f(\mathbf{x}^t + \gamma(\mathbb{B}\mathbf{x}^t - \mathbf{x}^t)) + g(\mathbf{x}^t + \gamma(\mathbb{B}\mathbf{x}^t - \mathbf{x}^t)\right\} \\ &= \underset{0\leq\gamma\leq 1}{\arg\min} \left\{\begin{array}{c}\frac{1}{2}\|\mathbf{A}(\mathbf{x}^t + \gamma(\mathbb{B}\mathbf{x}^t - \mathbf{x}^t)) - \mathbf{b}\|_2^2 \\ +\mu\sum_{k=1}^K |x_k^t + \gamma(\mathbb{B}_k\mathbf{x}^t - x_k^t)|\end{array}\right\}.\end{aligned}$$

LASSO

State-of-the-art algorithms: FLEXA (Facchinei, Scutari, and Sagratella, 2015), where

- same approximate function as STELA;
- the decreasing stepsizes are used: $\gamma^{t+1} = \gamma^t(1 - \alpha\gamma^t)$, where α controls the decreasing rate.

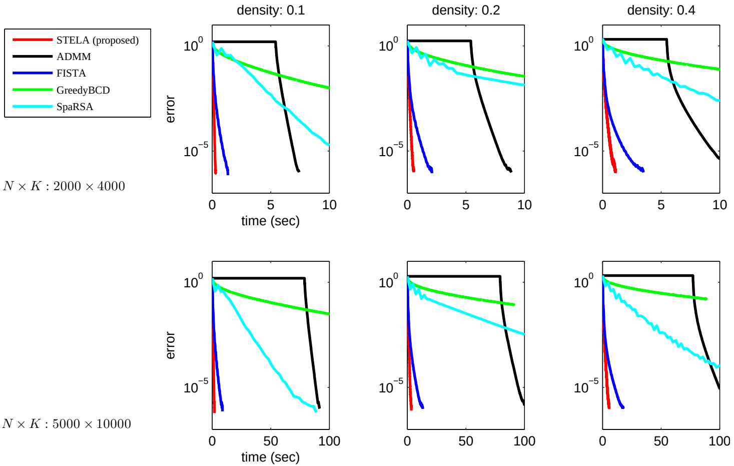


LASSO

State-of-the-art algorithms:

- FISTA (Beck and Teboulle, 2009)
- ADMM (Boyd, Parikh, Chu, Peleato, and Eckstein, 2010)
- GreedyBCD (Peng, Yan, and Yin, 2013)
- SpaRSA (Wright, Nowak, and Figueiredo, 2009)

LASSO



$N \times K : 2000 \times 4000$

$N \times K : 5000 \times 10000$

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STELA extension to basis pursuit (BP)

- The basis pursuit (BP) problem is:

$$\begin{aligned} \text{minimize} \quad & \|\mathbf{x}\|_1 \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b}. \end{aligned}$$

- It can be solved by the augmented Lagrangian approach.
- The augmented Lagrangian is

$$L_c(\mathbf{x}, \boldsymbol{\mu}) = \|\mathbf{x}\|_1 + \boldsymbol{\mu}^T(\mathbf{A}\mathbf{x} - \mathbf{b}) + \frac{c}{2}\|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2.$$

- Using STELA to minimize the augmented Lagrangian:

$$\mathbf{x}^\star(\boldsymbol{\mu}^t) \triangleq \arg\min_{\mathbf{x}} L_c(\mathbf{x}, \boldsymbol{\mu}^t).$$

- The dual variable $\boldsymbol{\mu}$ is updated as follows:

$$\boldsymbol{\mu}^{t+1} = \boldsymbol{\mu}^t + c(\mathbf{A}\mathbf{x}^\star(\boldsymbol{\mu}^t) - \mathbf{b}).$$

STELA extension to basis pursuit (BP)

- STELA for BP is tested on the benchmarking platform developed by the Optimization Group in Technische Universität Darmstadt.
- A total number of 100 instances of $\mathbf{A}$ and $\mathbf{b}$ are designed, reflecting different levels of solution difficulty.
- STELA is compared with the following solvers:
  - CPLEX (http://www.cplex.com)
  - ℓ_1-Homotopy (http://users.ece.gatech.edu/sasif/homotopy/)
  - SPGL1 (https://www.math.ucdavis.edu/~mpf/spgl1/)
  - ℓ_1-Magic (http://statweb.stanford.edu/~candes/l1magic/)
  - YALL1 (http://yall1.blogs.rice.edu/)
  - ISAL1 (http://wwwopt.mathematik.tu-darmstadt.de/spear/)

STELA extension to basis pursuit (BP)

Table 1: Let x be the solution of an instance returned by some solver, and let x^* be the exact solution. We regard an instance as solved if $\|x-x^*\|_2 < 10^{-6}$, acceptable if $10^{-6} \leq \|x-x^*\|_2 < 10^{-1}$ and unsolved if $10^{-1} \leq \|x-x^*\|_2$, respectively.

Solver	HOC	% solved HDR	% solved LDR	% solved all	% acceptable HDR	% acceptable LDR	% acceptable all	% unacceptable HDR	% unacceptable LDR	% unacceptable all
CPLEX	-	100.00	100.00	100.00	0.00	0.00	0.00	0.00	0.00	0.00
ℓ_1-Hom.	woHOC	100.00	100.00	100.00	0.00	0.00	0.00	0.00	0.00	0.00
	wHOC	100.00	100.00	100.00	0.00	0.00	0.00	0.00	0.00	0.00
STELA	woHOC	100.00	100.00	100.00	0.00	0.00	0.00	0.00	0.00	0.00
	wHOC	100.00	100.00	100.00	0.00	0.00	0.00	0.00	0.00	0.00
SPGL1	woHOC	0.00	0.00	0.00	83.94	90.51	87.23	16.06	9.49	12.77
	wHOC	70.07	71.17	70.62	14.96	19.34	17.15	14.96	9.49	12.23
ℓ_1-Magic	woHOC	0.00	15.33	7.66	54.01	82.48	68.25	45.99	2.19	24.09
	wHOC	62.04	75.91	68.98	5.11	21.90	13.50	32.85	2.19	17.52
YALL1	woHOC	0.00	1.09	0.55	7.66	79.56	43.61	92.34	19.34	55.84
	wHOC	60.95	63.87	62.41	0.00	17.52	8.76	39.05	18.61	28.83
SolveBP	woHOC	0.00	0.00	0.00	85.04	99.27	92.15	14.96	0.73	7.85
	wHOC	70.44	0.00	35.22	14.60	99.27	56.93	14.96	0.73	7.85
ISAL1	woHOC	44.89	94.89	69.89	28.47	0.36	14.42	26.64	4.74	15.69
	wHOC	82.12	97.81	89.96	5.84	0.73	3.28	12.04	1.46	6.75

- STELA is one of the three solvers that solve all problem instances.
- Note that HOC stands for "Heuristic Optimality Check" proposed in Lorenz, Pfetsch, and Tillmann (2015).

STELA extension to basis pursuit (BP)

Table 3: Runtime comparison for the three solvers that solved every single instance of our test set.

			without HOC		
n	Instances		ℓ_1-Homotopy	STELA	CPLEX
1024	72	avg. t	0.16	0.67	1.56
		max t	0.70	8.78	9.30
1536	72	avg. t	0.19	1.26	4.79
		max t	0.52	10.28	50.54
2048	144	avg. t	0.57	4.43	13.34
		max t	3.26	182.43	119.22
3072	72	avg. t	0.96	8.78	33.74
		max t	2.89	232.79	203.81
4096	94	avg. t	1.34	18.83	31.34
		max t	8.07	397.10	176.32
6144	16	avg. t	3.26	0.66	0.08
		max t	22.35	3.24	0.14
8192	22	avg. t	3.52	46.43	11.36
		max t	15.73	658.77	129.12
12288	4	avg. t	2.79	2.12	0.15
		max t	5.90	4.41	0.22
16384	16	avg. t	29.79	1.96	0.68
		max t	125.44	7.82	2.29
24576	16	avg. t	132.03	10.09	0.63
		max t	726.47	50.03	1.40
32768	16	avg. t	189.56	15.71	1.51
		max t	850.40	109.10	4.22
49152	4	avg. t	143.50	24.93	3.34
		max t	342.68	72.66	4.38

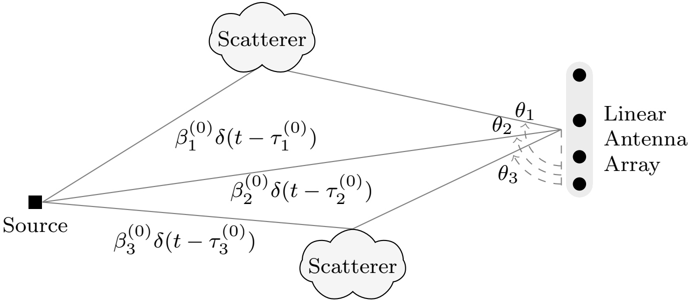
			with HOC		
n	Instances		ℓ_1-Homotopy	STELA	CPLEX
1024	72	avg. t	0.22	0.24	1.56
		max t	1.15	4.14	9.30
1536	72	avg. t	0.26	0.59	4.79
		max t	0.72	9.15	50.54
2048	144	avg. t	0.66	2.68	13.34
		max t	3.41	123.22	119.22
3072	72	avg. t	1.07	6.31	33.74
		max t	4.05	113.83	203.81
4096	94	avg. t	1.15	16.00	31.34
		max t	5.99	323.51	176.32
6144	16	avg. t	2.66	0.45	0.08
		max t	13.69	2.53	0.14
8192	22	avg. t	2.83	35.12	11.36
		max t	13.17	495.24	129.12
12288	4	avg. t	2.60	0.99	0.15
		max t	4.80	2.31	0.22
16384	16	avg. t	37.46	0.95	0.68
		max t	171.78	4.66	2.29
24576	16	avg. t	139.95	3.97	0.63
		max t	735.77	19.16	1.40
32768	16	avg. t	186.09	6.82	1.51
		max t	1236.41	47.89	4.22
49152	4	avg. t	109.40	10.37	3.34
		max t	206.87	27.35	4.38

- Although STELA must be called multiple times (one for each augmented Lagrangian), it is the second fastest solver for BP.
- More simulation results are reported in Kuske and Tillmann (2016).

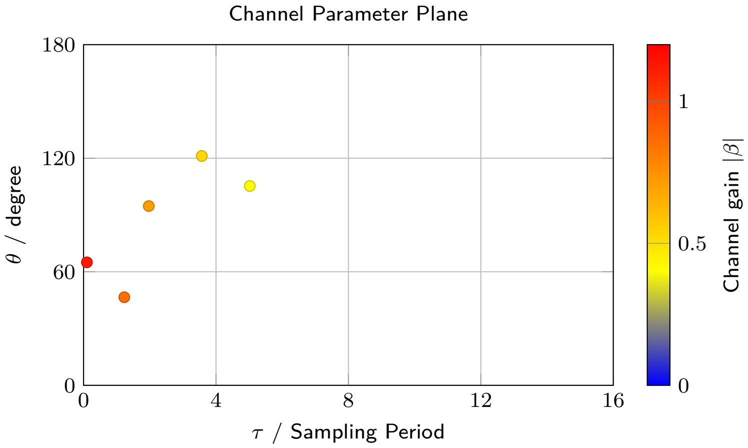
STELA extension to MD rank-sparse regularization

Multipath parameter estimation (Steffens, Yang, and Pesavento (2016))

- P propagation paths with ...
  - complex channel gain coefficient $\beta_p^{(0)}$,
  - propagation delay $\tau_p^{(0)}$ and
  - angle-of-arrival (AoA) $\theta_p^{(0)}$



Multipath parameter estimation



Multipath propagation model

- Signal received by the antenna array in time instant ℓ (OFDM symbol)

$$\boldsymbol{Y}(\ell) = \boldsymbol{A}^{(0)}\boldsymbol{H}^{(0)}(\ell)\boldsymbol{B}^{(0)\mathsf{T}}\boldsymbol{X}(\ell) + \boldsymbol{W}(\ell) \quad \in \mathbb{C}^{M\times N}$$

  with $[\boldsymbol{Y}(\ell)]_{m,n}$ denoting the signal at antenna m on subcarrier n
- **Array steering matrix**:
  $\boldsymbol{A}^{(0)} = \left[\boldsymbol{a}(\theta_1^{(0)}), \ldots, \boldsymbol{a}(\theta_P^{(0)})\right] \in \mathbb{C}^{M\times P}, \qquad a_m(\theta_p^{(0)}) = e^{-j\ r_m \frac{2\pi}{\lambda}\cos\theta_p^{(0)}}$
- **Frequency response matrix**:
  $\boldsymbol{B}^{(0)} = \left[\boldsymbol{b}(\tau_1^{(0)}), \ldots, \boldsymbol{b}(\tau_P^{(0)})\right] \in \mathbb{C}^{N\times P}, \qquad b_n(\tau_p^{(0)}) = e^{-j\ \frac{2\pi}{NT_\mathrm{s}}(n-1)\tau_p^{(0)}}$
- **Channel gain matrix**: $\boldsymbol{H}^{(0)}(\ell) = \mathrm{diag}\left(\beta_1^{(0)}(\ell), \cdots, \beta_P^{(0)}(\ell)\right) \in \mathbb{C}^{P\times P}$
- **Reference signal matrix**: $\boldsymbol{X}(\ell) = \mathrm{diag}\left(x_1(\ell), \ldots, x_N(\ell)\right) \in \mathbb{C}^{N\times N}$
- **AWGN matrix**: $\boldsymbol{W}(\ell)$

Multipath propagation model

- Collect L snapshots and define

$$\begin{aligned}\boldsymbol{Z} &= \left[\text{vec}\left(\boldsymbol{Y}(1)\boldsymbol{X}^{-1}(1)\right), \ldots, \text{vec}\left(\boldsymbol{Y}(L)\boldsymbol{X}^{-1}(L)\right)\right] \\ \bar{\boldsymbol{G}}^{(0)} &= \left[\text{vecd}\left(\boldsymbol{H}^{(0)}(1)\right), \ldots, \text{vecd}\left(\boldsymbol{H}^{(0)}(L)\right)\right] \\ \boldsymbol{W} &= \left[\text{vec}\left(\boldsymbol{W}(1)\boldsymbol{X}^{-1}(1)\right), \ldots, \text{vec}\left(\boldsymbol{W}(L)\boldsymbol{X}^{-1}(L)\right)\right]\end{aligned}$$

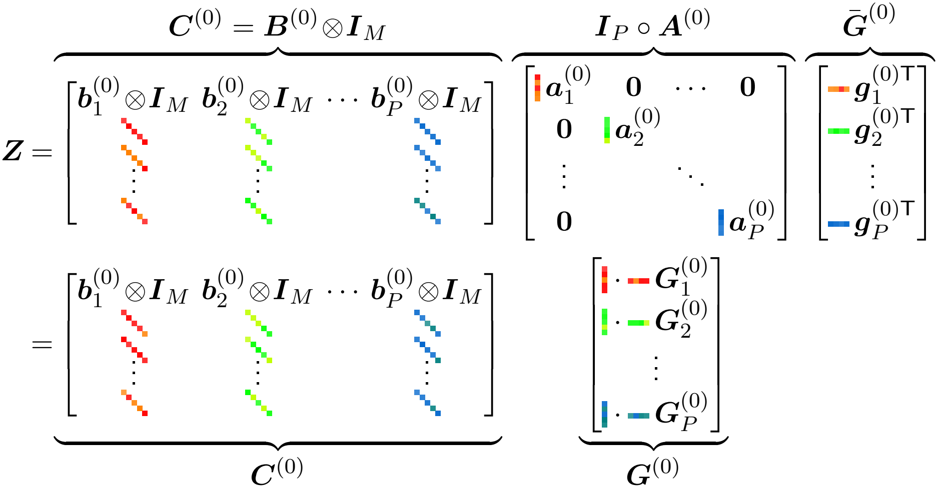
- Multiple snapshot signal model:

$$\boldsymbol{Z} = (\boldsymbol{B}^{(0)} \circ \boldsymbol{A}^{(0)})\, \bar{\boldsymbol{G}}^{(0)} + \boldsymbol{W} \quad \in \mathbb{C}^{MN \times L}$$

with Khatri-Rao product $\circ$, i.e. columnwise Kronecker product $\otimes$

Multipath propagation model: structure

- Structure in reformulated signal model (neglecting noise)



Rank-sparse regularization

- The rank-sparse (nuclear norm) regularization problem is given as:

$$\hat{\boldsymbol{G}} = \arg\min_{\boldsymbol{G}} \ \frac{1}{2}\|\boldsymbol{Z} - \boldsymbol{C}\boldsymbol{G}\|_{\mathsf{F}}^2 + \mu \sum_{q=1}^{Q} \|\boldsymbol{G}_q\|_*$$

- Decompose problem into Q approximate subproblems (Steffens, Yang, and Pesavento (2016)):

$$\mathbb{B}\boldsymbol{G}_q^{(t)} = \arg\min_{\boldsymbol{G}_q} \frac{1}{2}\left\|\boldsymbol{Z} - \boldsymbol{C}_{-q}\boldsymbol{G}_{-q}^{(t)} - \boldsymbol{C}_q\boldsymbol{G}_q\right\|_{\mathsf{F}}^2 + \mu\left(\left\|\boldsymbol{G}_{-q}^{(t)}\right\|_* + \left\|\boldsymbol{G}_q\right\|_*\right)$$

with $\boldsymbol{C}_{-q} = [\boldsymbol{C}_1, \ldots, \boldsymbol{C}_{q-1}, \boldsymbol{C}_{q+1}, \ldots, \boldsymbol{C}_Q]$
and $\boldsymbol{G}_{-q}^{(t)} = [\boldsymbol{G}_1^{(t)\mathsf{T}}, \ldots, \boldsymbol{G}_{q-1}^{(t)\mathsf{T}}, \boldsymbol{G}_{q+1}^{(t)\mathsf{T}}, \ldots, \boldsymbol{G}_Q^{(t)\mathsf{T}}]^{\mathsf{T}}$

The approximate function

- Closed-form solution for approximate problems:

$$\mathbb{B}\boldsymbol{G}_q^{(t)} = \mathcal{S}_\mu(\boldsymbol{\Gamma}_q^{(t)})$$

  where

$$\boldsymbol{\Gamma}_q^{(t)} = (\boldsymbol{C}_q^{\mathsf{H}}\boldsymbol{C}_q)^{-1}\boldsymbol{C}_q^{\mathsf{H}}(\boldsymbol{Z} - \boldsymbol{C}_{-q}\boldsymbol{G}_{-q}^{(t)})$$

  is the Least-Squares estimate of $\boldsymbol{G}_q$, with compact singular value decomposition $\boldsymbol{\Gamma}_q^{(t)} = \boldsymbol{U}_q^{(t)}\boldsymbol{\Omega}_q^{(t)}\boldsymbol{V}_q^{(t)\mathsf{H}}$, and

$$\mathcal{S}_\mu(\boldsymbol{\Gamma}_q^{(t)}) = \boldsymbol{U}_q^{(t)}\left(\boldsymbol{\Omega}_q^{(t)} - \mu\boldsymbol{I}\right)_+\boldsymbol{V}_q^{(t)\mathsf{H}}$$

  is the singular value thresholding operator, with $[(\boldsymbol{X})_+]_{ij} = \max([\boldsymbol{X}]_{ij}, 0)$

Update step

- Denote solutions to approximate problems as best-response matrix

$$\mathbb{B}\boldsymbol{G}^{(t)} = [\mathbb{B}\boldsymbol{G}_1^{(t)\mathsf{T}}, \ldots, \mathbb{B}\boldsymbol{G}_Q^{(t)\mathsf{T}}]^{\mathsf{T}}$$

- Update of approximate solution $\boldsymbol{G}^{(t)}$ in iteration t by

$$\boldsymbol{G}^{(t+1)} = \boldsymbol{G}^{(t)} + \gamma^{(t)}(\mathbb{B}\boldsymbol{G}^{(t)} - \boldsymbol{G}^{(t)})$$

- Closed-form solution for update step size

$$\begin{aligned}\gamma^{(t)} &= \underset{0 \leq \gamma \leq 1}{\arg\min} \frac{1}{2}\left\|\boldsymbol{Z} - \boldsymbol{C}(\boldsymbol{G}^{(t)} + \gamma(\mathbb{B}\boldsymbol{G}^{(t)} - \boldsymbol{G}^{(t)}))\right\|_{\mathsf{F}}^2 + \gamma\mu\sum_{q=1}^{Q}(\|\mathbb{B}\boldsymbol{G}_q^{(t)}\|_* - \|\boldsymbol{G}_q^{(t)}\|_*) \\ &= \left[\frac{\mathrm{Re}\left\{\mathrm{Tr}\left((\boldsymbol{C}\boldsymbol{G}^{(t)} - \boldsymbol{Z})^{\mathsf{H}}\boldsymbol{C}(\mathbb{B}\boldsymbol{G}^{(t)} - \boldsymbol{G}^{(t)})\right)\right\} + \mu\sum_{q=1}^{Q}(\|\mathbb{B}\boldsymbol{G}_q^{(t)}\|_* - \|\boldsymbol{G}_q^{(t)}\|_*)}{\left\|\boldsymbol{C}(\mathbb{B}\boldsymbol{G}^{(t)} - \boldsymbol{G}^{(t)})\right\|_{\mathsf{F}}^2}\right]_0^1\end{aligned}$$

Questions? Thank you!

J. M. Ortega and W. C. Rheinboldt, *Iterative solution of nonlinear equations in several variables*. Academic, New York, 1970.

S. Boyd and L. Vandenberghe, *Convex optimization*. Cambridge Univ Pr, 2004.

O. L. Mangasarian, *Nonlinear programming*. McGraw-Hill, 1969.

Y. Yang and M. Pesavento, "A Unified Successive Pseudoconvex Approximation Framework," *IEEE Transactions on Signal Processing*, vol. 65, no. 13, pp. 3313–3328, 2017.

M. Razaviyayn, M. Hong, and Z.-q. Luo, "A Unified Convergence Analysis of Block Successive Minimization Methods for Nonsmooth Optimization," *SIAM Journal on Optimization*, vol. 23, no. 2, pp. 1126–1153, jan 2013. [Online]. Available: http://epubs.siam.org/doi/10.1137/120891009

Y. Sun, P. Babu, and D. P. Palomar, "Majorization-Minimization Algorithms in Signal Processing, Communications, and Machine Learning," *IEEE Transactions on Signal Processing*, vol. 65, no. 3, pp. 794–816, 2017.

A. Zappone, E. Björnson, L. Sanguinetti, and E. Jorswieck, "Globally Optimal Energy-Efficient Power Control and Receiver Design in Wireless

Networks," *Ieee Transactions on Signal Processing*, vol. 65, no. 11, pp. 2844–2859, 2017.

D. P. Bertsekas, *Nonlinear programming*, 3rd ed. Athena Scientific, 2016.

N. Parikh and S. Boyd, "Proximal Algorithms," *Foundations and Trends in Optimization*, vol. 1, no. 3, pp. 127–239, 2014. [Online]. Available: http://www.nowpublishers.com/articles/foundations-and-trends-in-optimization/OPT-003

G. Scutari, F. Facchinei, P. Song, D. P. Palomar, and J.-S. Pang, "Decomposition by Partial Linearization: Parallel Optimization of Multi-Agent Systems," *IEEE Transactions on Signal Processing*, vol. 62, no. 3, pp. 641–656, feb 2014. [Online]. Available: http://ieeexplore.ieee.org/lpdocs/epic03/wrapper.htm?arnumber=6675875

W. Yu, W. Rhee, S. Boyd, and J. Cioffi, "Iterative Water-Filling for Gaussian Vector Multiple-Access Channels," *IEEE Transactions on Information Theory*, vol. 50, no. 1, pp. 145–152, jan 2004. [Online]. Available: http://ieeexplore.ieee.org/lpdocs/epic03/wrapper.htm?arnumber=1262622

J. Bunch, C. Nielsen, and D. Sorensen, "Rank-one modification of the symmetric eigenproblem," *Numerische Mathematik*, vol. 31, pp. 31–48, 1978.

N. Jindal, W. Rhee, S. Vishwanath, S. Jafar, and A. Goldsmith, "Sum Power Iterative Water-Filling for Multi-Antenna Gaussian Broadcast Channels," *IEEE Transactions on Information Theory*, vol. 51, no. 4, pp. 1570–1580, apr 2005. [Online]. Available: http://ieeexplore.ieee.org/lpdocs/epic03/wrapper.htm?arnumber=1412050

Z.-Q. Luo and S. Zhang, "Dynamic Spectrum Management: Complexity and Duality," *IEEE Journal of Selected Topics in Signal Processing*, vol. 2, no. 1, pp. 57–73, feb 2008. [Online]. Available: http://ieeexplore.ieee.org/lpdocs/epic03/wrapper.htm?arnumber=4453890

M. Elad, "Why simple shrinkage is still relevant for redundant representations?" *IEEE Transactions on Information Theory*, vol. 52, no. 12, pp. 5559–5569, 2006.

F. Facchinei, G. Scutari, and S. Sagratella, "Parallel Selective Algorithms for Nonconvex Big Data Optimization," *IEEE Transactions on Signal Processing*, vol. 63, no. 7, pp. 1874–1889, nov 2015. [Online]. Available: http://arxiv.org/abs/1402.5521

A. Beck and M. Teboulle, “A Fast Iterative Shrinkage-Thresholding Algorithm for Linear Inverse Problems,” *SIAM Journal on Imaging Sciences*, vol. 2, no. 1, pp. 183–202, jan 2009. [Online]. Available: http://epubs.siam.org/doi/abs/10.1137/080716542

S. Boyd, N. Parikh, E. Chu, B. Peleato, and J. Eckstein, “Distributed Optimization and Statistical Learning via the Alternating Direction Method of Multipliers,” *Foundations and Trends in Machine Learning*, vol. 3, no. 1, 2010. [Online]. Available: http://www.nowpublishers.com/product.aspx?product=MAL{&}doi=2200000016

Z. Peng, M. Yan, and W. Yin, “Parallel and distributed sparse optimization,” *2013 Asilomar Conference on Signals, Systems and Computers*, pp. 659–646, nov 2013. [Online]. Available: http://ieeexplore.ieee.org/lpdocs/epic03/wrapper.htm?arnumber=6810364

S. Wright, R. Nowak, and M. Figueiredo, “Sparse Reconstruction by Separable Approximation,” *IEEE Transactions on Signal Processing*, vol. 57, no. 7, pp. 2479–2493, jul 2009. [Online]. Available: http://ieeexplore.ieee.org/lpdocs/epic03/wrapper.htm?arnumber=4799134

D. Lorenz, M. Pfetsch, and A. Tillmann, "Solving Basis Pursuit: Heuristic Optimality Check and Solver Comparison," *ACM Transactions on Mathematical Software*, vol. 41, no. 2, pp. 1–29, 2015.

J. Kuske and A. Tillmann, "Solving basis pursuit: Update," Technische Universität Darmstadt, Tech. Rep., April 2016. [Online]. Available: http://www.mathematik.tu-darmstadt.de/~tillmann/docs/SolvingBPupdateApr2016.pdf

C. Steffens, Y. Yang, and M. Pesavento, "Multidimensional sparse recovery for mimo channel parameter estimation," in *2016 24th European Signal Processing Conference (EUSIPCO)*, Aug 2016, pp. 66–70.