Storage Constrained Linear Computation Coding

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Motivation

- Consider the matrix-vector multiplication

$$\mathbf{T}\mathbf{x},$$

where $\mathbf{T} \in \mathbb{R}^{N \times K}$, $\mathbf{x} \in \mathbb{R}^{K}$ and we assume $N = \log_2(K)$.

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- Ubiquitous usage in signal processing and machine learning, e.g. weights in connections between layers of neural networks.
- Efficient computation of matrix-vector multiplications of high relevance.

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- Entries of wiring matrices are integer powers of 2: multiplications are simple bit shifts.
- Sparse entries in wiring matrix limit number of required additions.

Ralf R. Müller, Bernhard Gäde, and Ali Bereyhi. Efficient Matrix Multiplication: The Sparse Power-of-2 Factorization. 2020

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- Formulation of an optimization problem constraining the cardinality $|\mathcal{I}_f^{(i)}|$.
- Proposal of sub-optimum search of $\mathcal{I}_f^{(i)}$ with reduced computational complexity.

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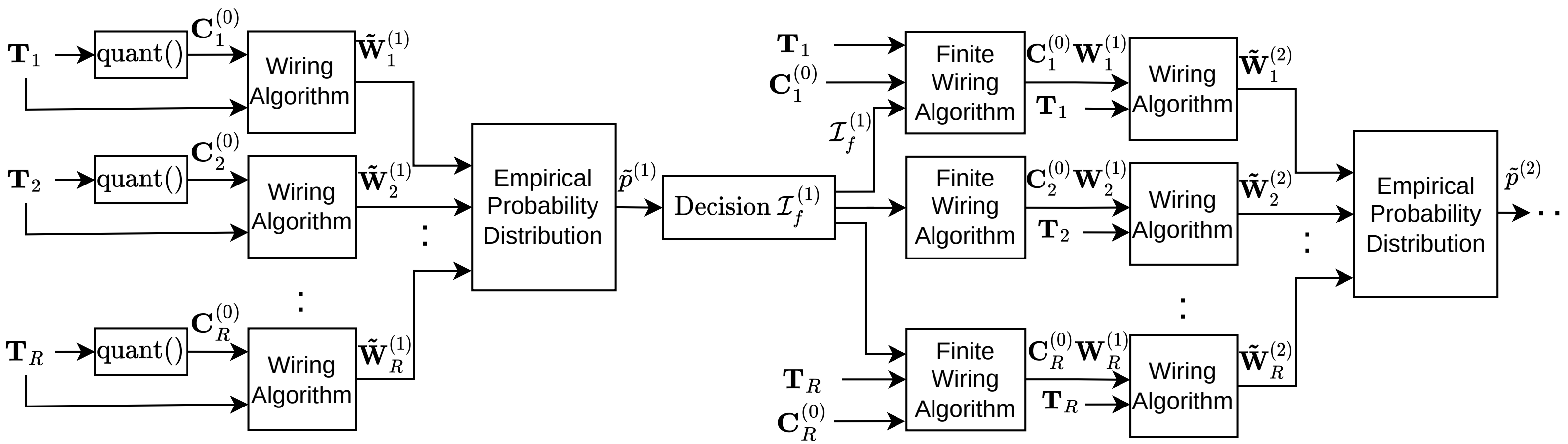
- Optimization, enforcing constraint on cardinality $|\mathcal{I}_f^{(i)}|$

$$\begin{aligned}
\underset{\mathcal{I}_f^{(1)},\dots,\mathcal{I}_f^{(L)}}{\text{minimize}} \quad & E(\mathcal{R}) = \frac{1}{KR}\sum_{r=1}^{R} \|\mathbf{T}_r - \hat{\mathbf{T}}_r\|_F^2 \\
\text{subject to} \quad & \hat{\mathbf{T}}_r = \mathbf{C}_r^{(0)} \prod_{i=1}^{L} \mathbf{W}_r^{(i)} \quad , \\
& \mathbf{W}_r^{(i)} \in (\mathcal{B}_f^{(i)})^{K\times K} = \{0, \pm 2^e | e \in \mathcal{I}_f^{(i)}\} \quad , \\
& |\mathcal{I}_f^{(i)}| = d \quad .
\end{aligned} \tag{1}$$

Data-Driven Approach



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- Cubic complexity in K and number of possible subsets $\binom{|\mathcal{I}_s|}{d}$ prohibitive.
- Surrogate Objective

$$\mathcal{I}_f^{(i)} = \underset{\tilde{\mathcal{I}}_f^{(i)} \subset \mathbb{Z}, |\tilde{\mathcal{I}}_f^{(i)}| = d}{\arg\max} \sum_{e \in \mathcal{I}_s^{(i)}} \mathrm{ReLU}\left(\max_{x \in \tilde{\mathcal{I}}_f^{(i)}} \hat{p}^{(i)}(e) \underbrace{\left(2^{2e} - (2^x - 2^e)^2\right)}_{\substack{\text{Approximate error reduction using} \\ \text{wiring exponent } x \text{ if wiring exponent } e \text{ occured}}} \right).$$

- Reduced Complexity $\mathcal{O}\left(\underbrace{LR|\mathcal{I}_s|dKs}_{\substack{\text{Greedy optimization of} \\ \text{surrogate objective}}} + \underbrace{RLK^3 s}_{\text{Unconstrained wiring}} \right).$

Simulation Settings

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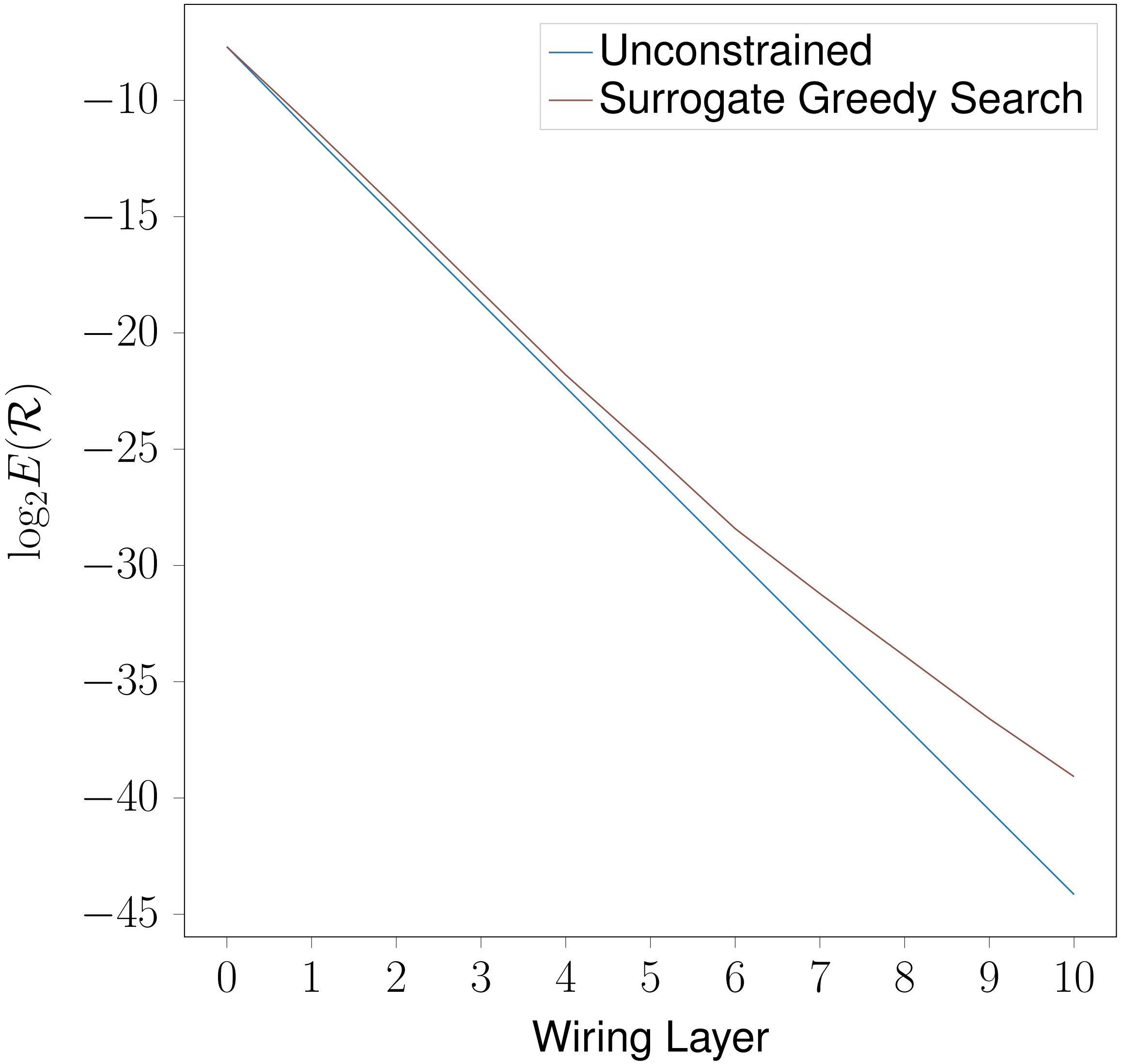
- i.i.d. zero mean Gaussian distributed target matrices $\mathbf{T} \in \mathbb{R}^{8\times 256}$ with variance $\frac{1}{8}$.
- Sample size $R = 1000$ matrices.
- Sparsity $s = 3$.
- Cardinality of finite wiring exponent sets $d = 4$.

Results


Unconstrained
Surrogate Greedy Search
-10
-15
-20
-25
-30
-35
-40
-45
$\log_2 E(\mathcal{R})$
0
1
2
3
4
5
6
7
8
9
10
Wiring Layer

Thank you for your attention!

1 Appendix

Codebook Matrix

- Codebook matrix in layer l

$$\mathbf{C}^{(l)} := \mathbf{C}^{(0)} \prod_{i=1}^{l} \mathbf{W}^{(i)} = \mathbf{C}^{(l-1)}\mathbf{W}^{(l)} \tag{2}$$

- Choice of initial codebook as $\mathbf{C}^{(0)} = q(\mathbf{T})$, with the quantization operator $q(\cdot)$ such that

$$q(T_{ij}) = \arg\min_{x \in \mathcal{B}_{\text{init}}} |T_{ij} - x|,$$

with $\mathcal{B}_{\text{init}} = \{0, \pm 2^e\}$, $e \in \mathcal{I}_{\text{init}} \subsetneq \mathbb{Z}$ and $\mathcal{I}_{\text{init}}$ finite.

- Defining L as the number of *wiring layers*,

$$\mathbf{T} \approx \hat{\mathbf{T}} = \mathbf{C}^{(L)}.$$

- According to (2), combination of column vectors of codebook matrix of preceding layer $\mathbf{C}^{(l-1)}$ by wiring matrix $\mathbf{W}^{(l)}$ for approximation of columns of $\mathbf{T}$.

General Optimization Problem

- Choice of wiring matrices $\{\mathbf{W}^{(i)}\}_{i\in\{1,\ldots,L\}}$ according to

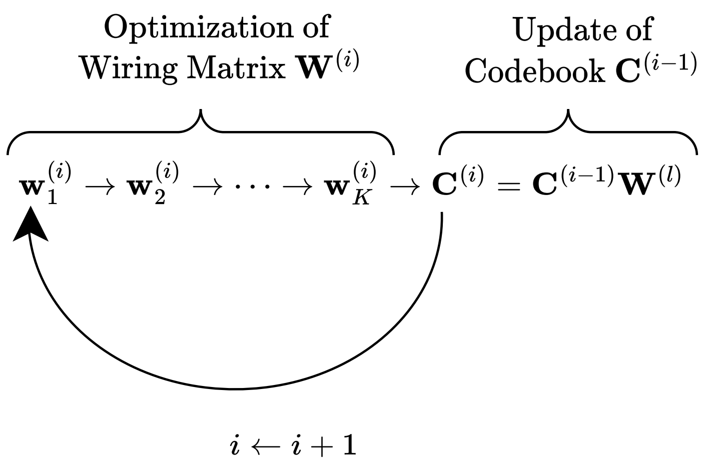
$$\{\mathbf{W}^{(i)}\}_{i\in\{1,\ldots,L\}} = \underset{\{\tilde{\mathbf{W}}^{(i)}\in\{0,\pm 2^e\}^{K\times K},\, e\in\mathcal{I}_f^{(i)},\, ||\mathbf{w}_k^{(i)}||_0 = s\,\forall k\}_{i\in\{1,\ldots,L\}}}{\arg\min} ||\mathbf{T} - \mathbf{C}^{(0)} \prod_{i=1}^{L} \tilde{\mathbf{W}}^{(i)}||_F. \quad (3)$$

Integer programming in LK^2 parameters and hence infeasible for large dimensions.

Split Optimization Problem

- Split optimization of the set of matrices $\{\mathbf{W}^{(i)}\}_{i\in\{1,\dots,L\}}$ into subsequent optimization of columns $\mathbf{w}_k^{(i)}$ of individual matrices $\mathbf{W}^{(i)}$

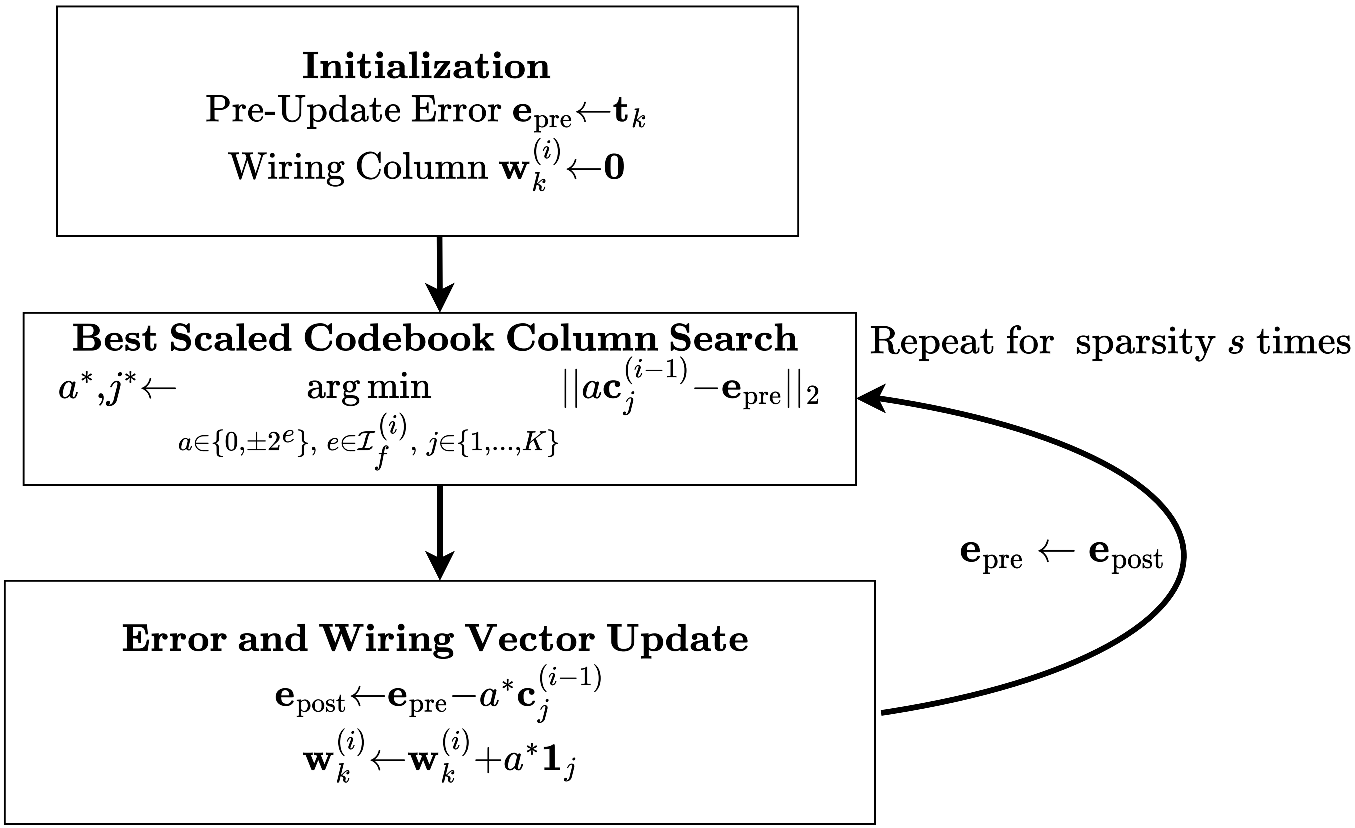
$$\mathbf{w}_k^{(i)} = \underset{\mathbf{w}\in\{0,\pm 2^e\}^K,\, e\in\mathcal{I}_f^{(i)},\, ||\mathbf{w}||_0 = s}{\arg\min} ||\mathbf{t}_k - \mathbf{C}^{(i-1)}\mathbf{w}||_2.$$



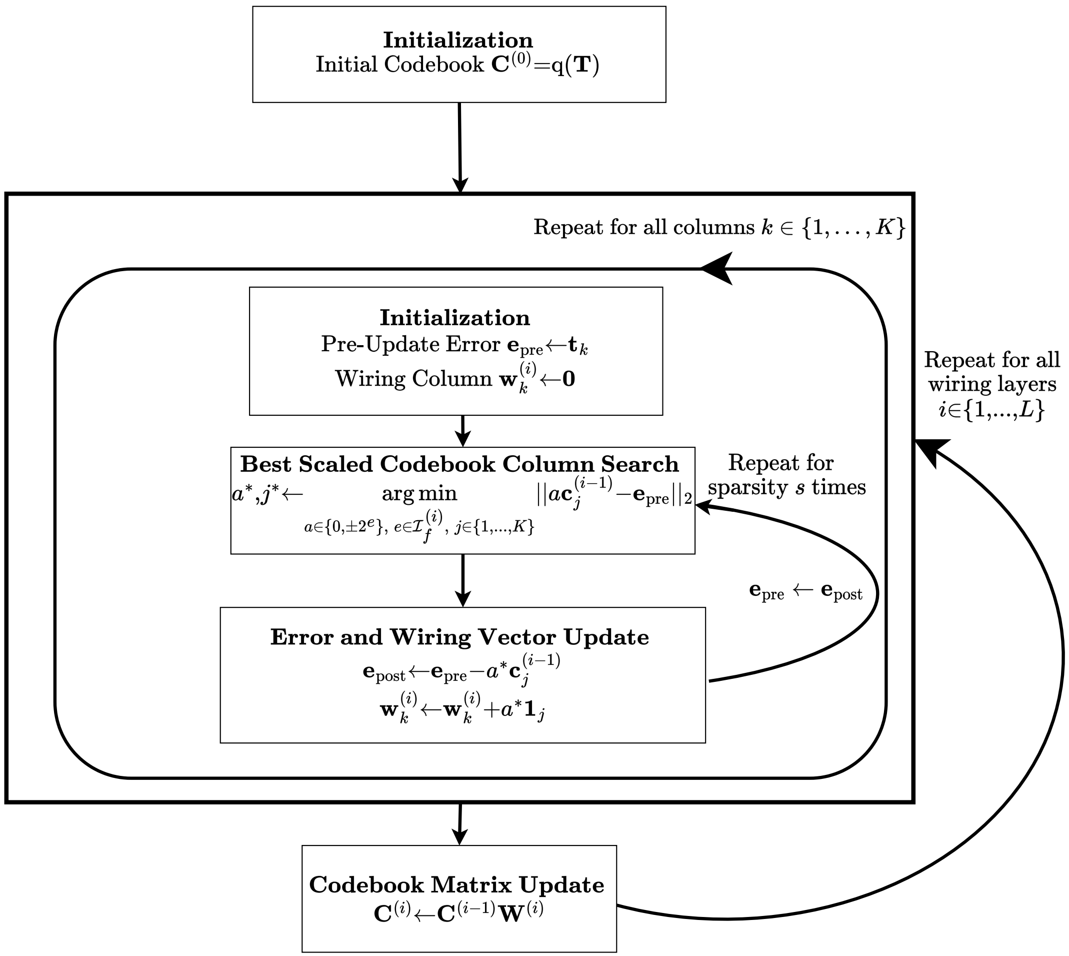
- Integer programming in K unknowns, limited feasibility.

Greedy Search

- Improvement of feasibility by greedy search of the components of $\mathbf{w}_k^{(i)}$



Example of Greedy Search



Numerical Example

$$\mathbf{T} = [\mathbf{t}_1 \quad \mathbf{t}_2 \quad \mathbf{t}_3] = \begin{bmatrix} 0.760 & 0.259 & -0.500 \\ 0.650 & 0.966 & 0.866 \end{bmatrix}, ||\mathbf{t}_i||_2 = 1$$

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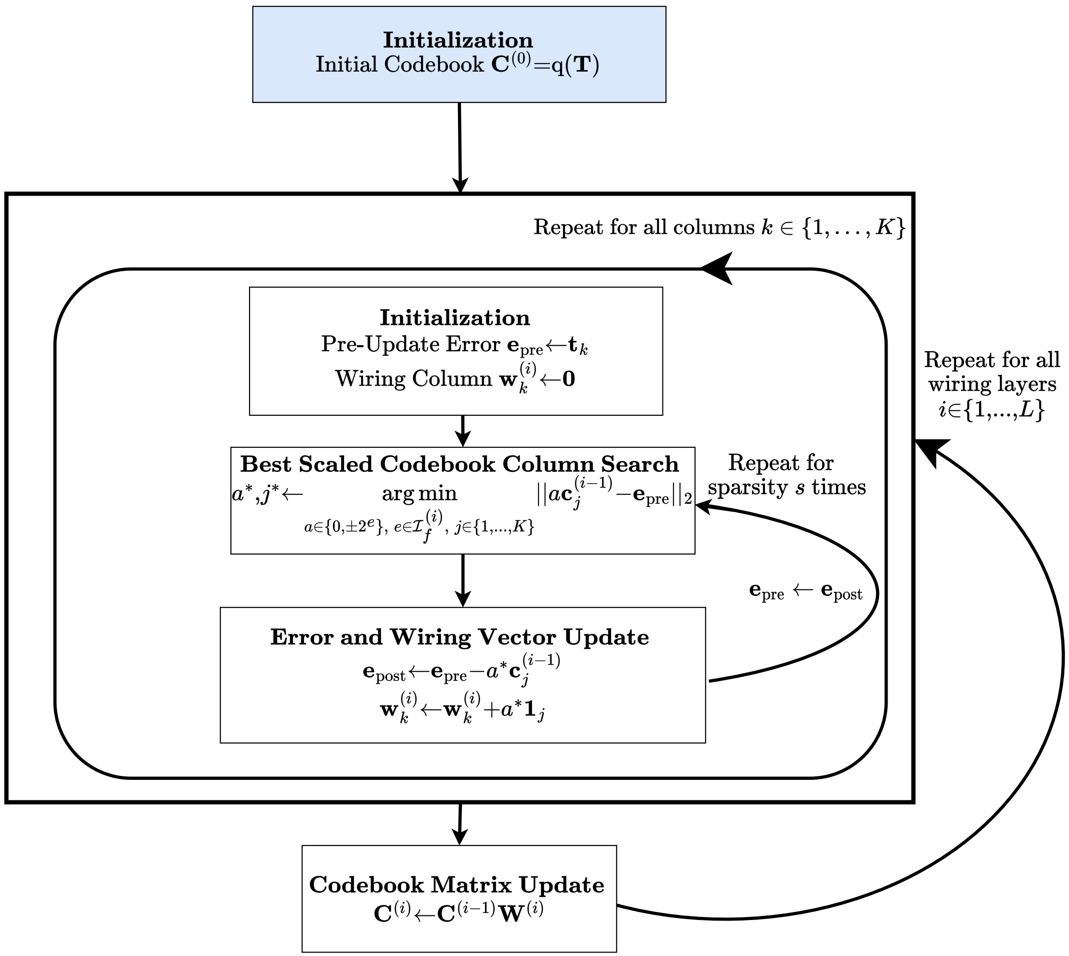

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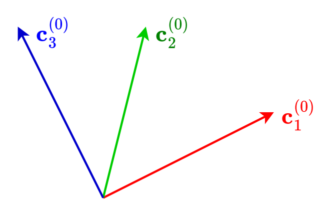

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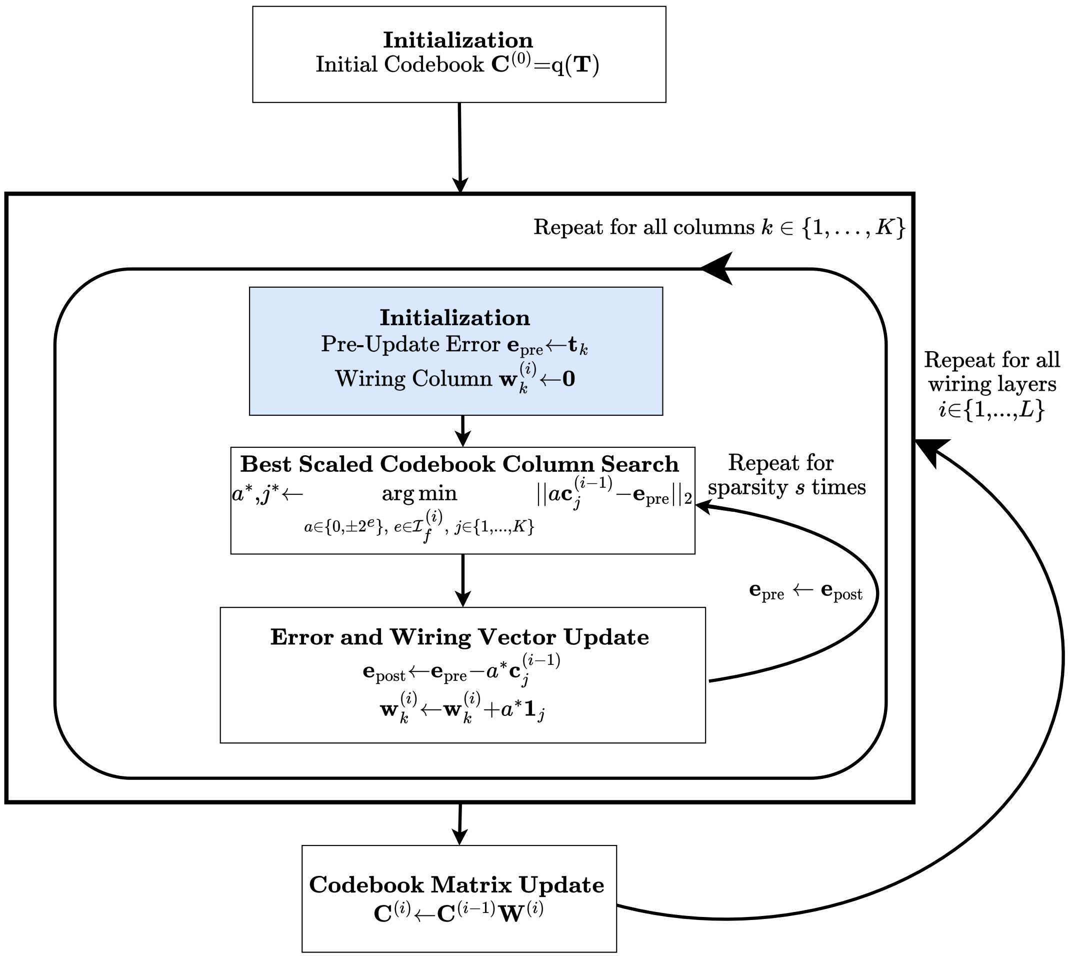

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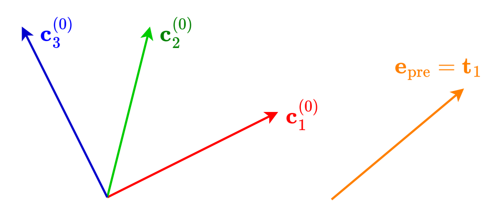

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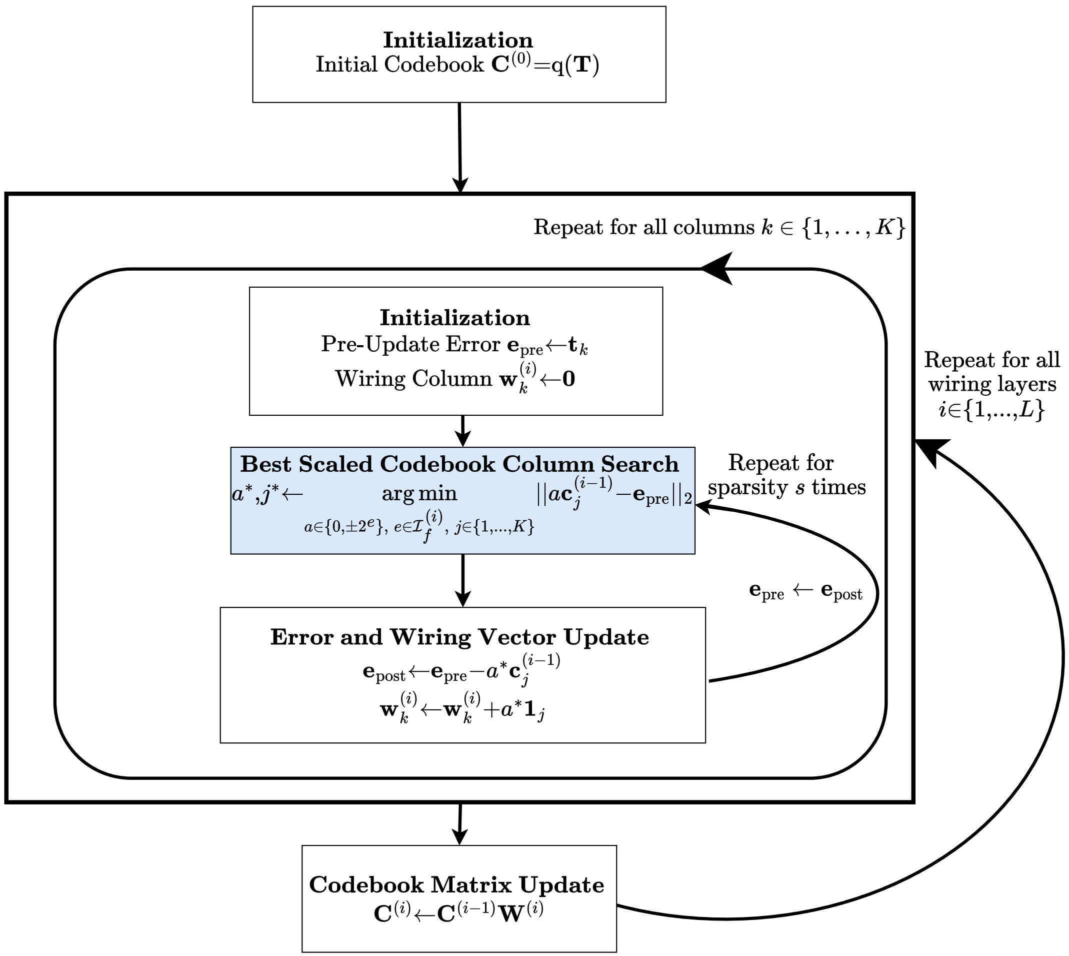

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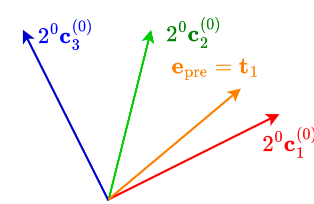

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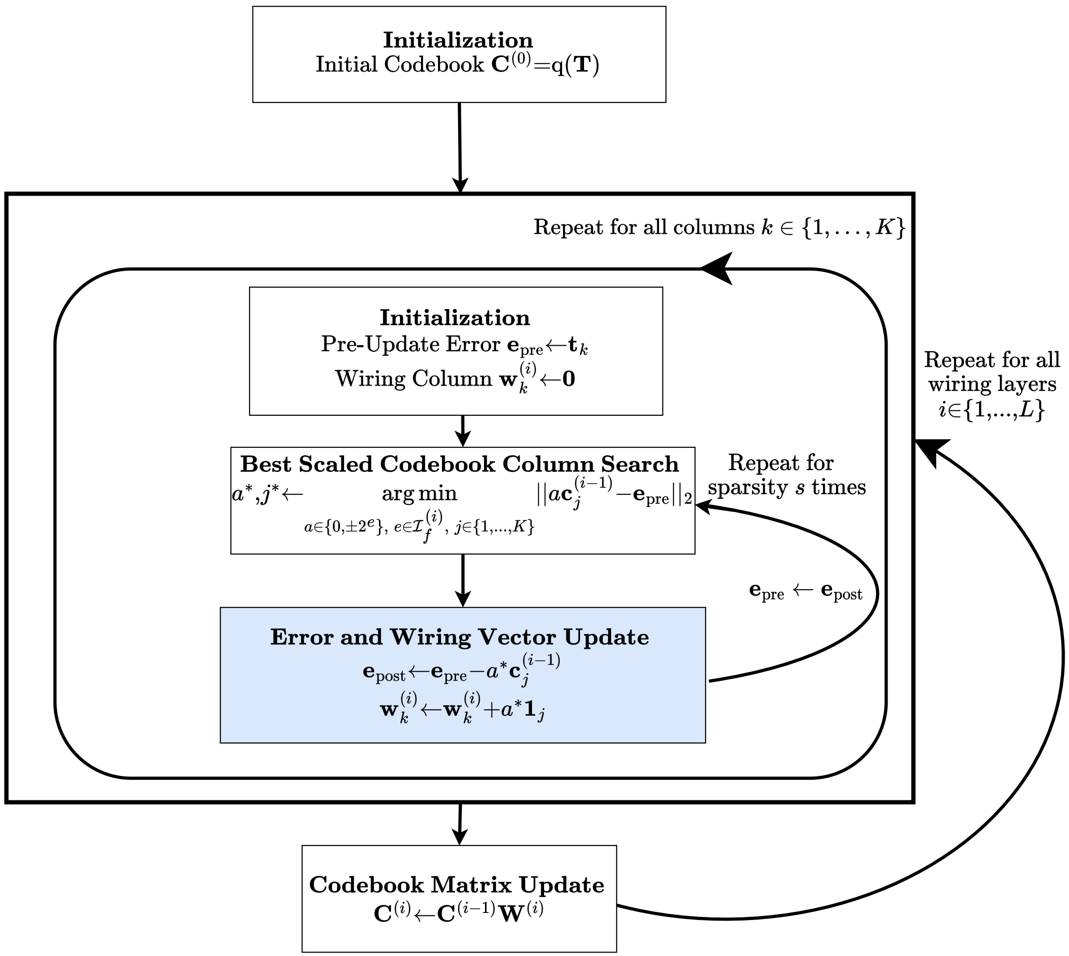

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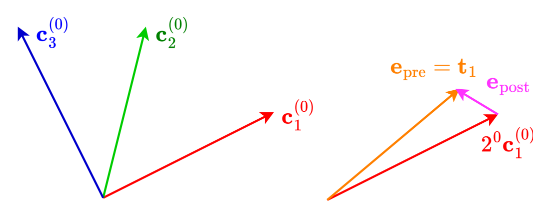

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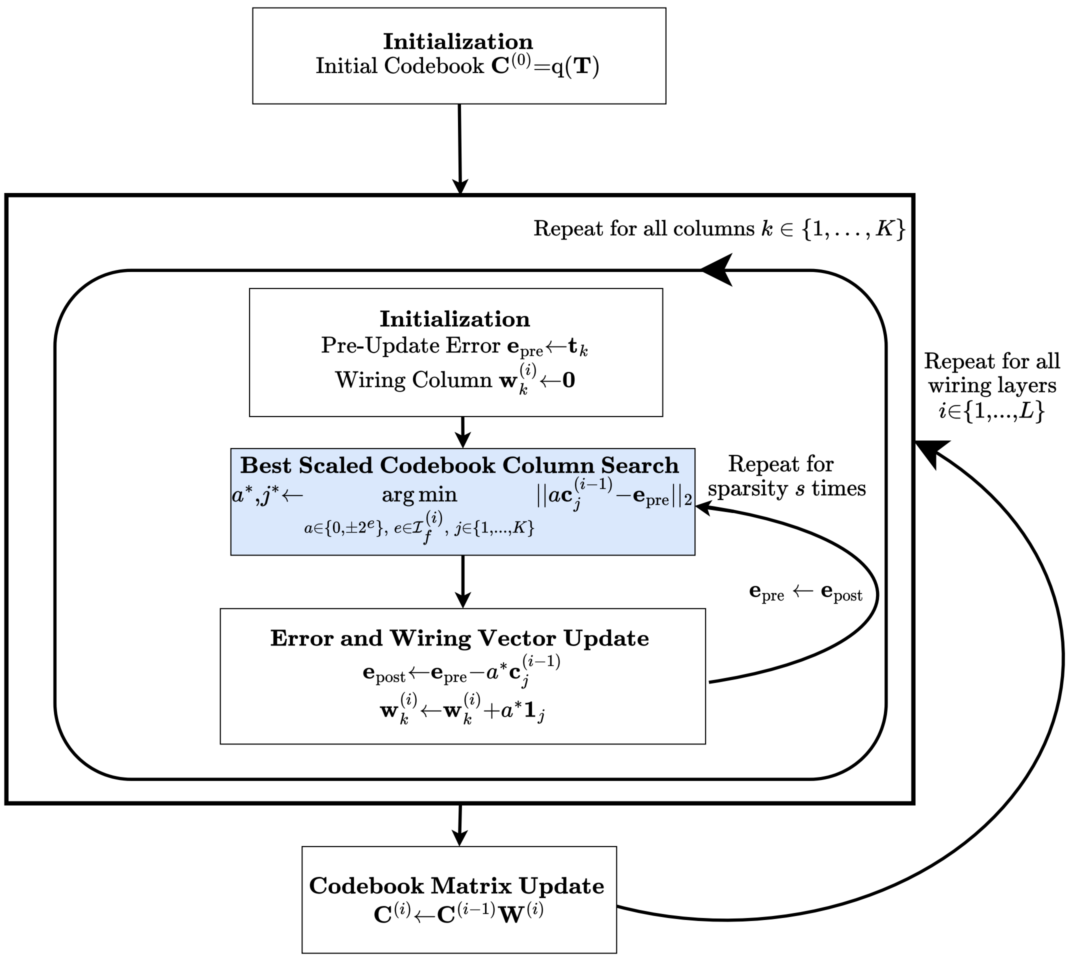

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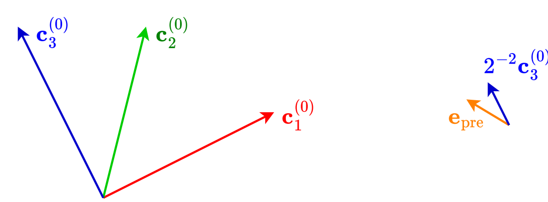

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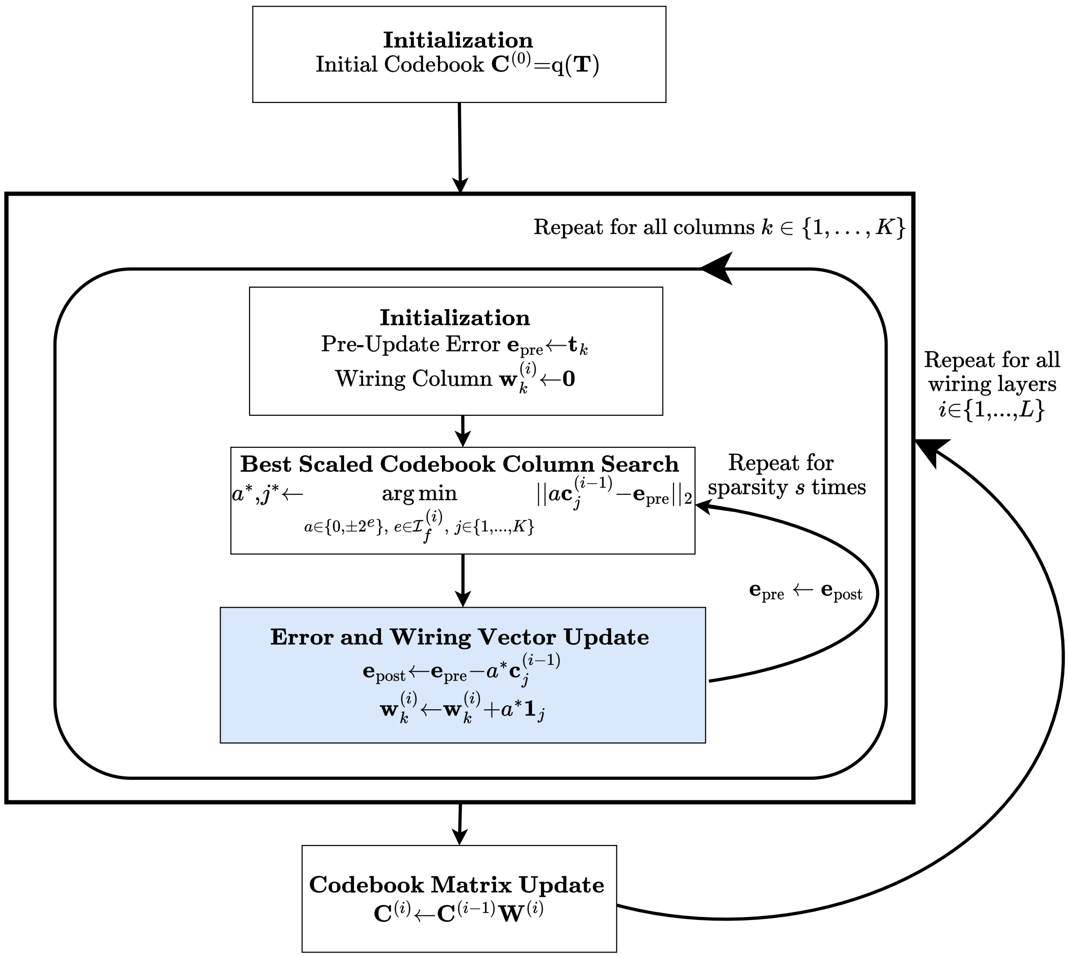

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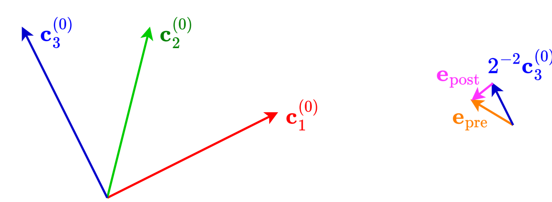
$$\mathbf{e}_{\text{post}} = \begin{bmatrix} -0.115 \\ -0.100 \end{bmatrix}$$



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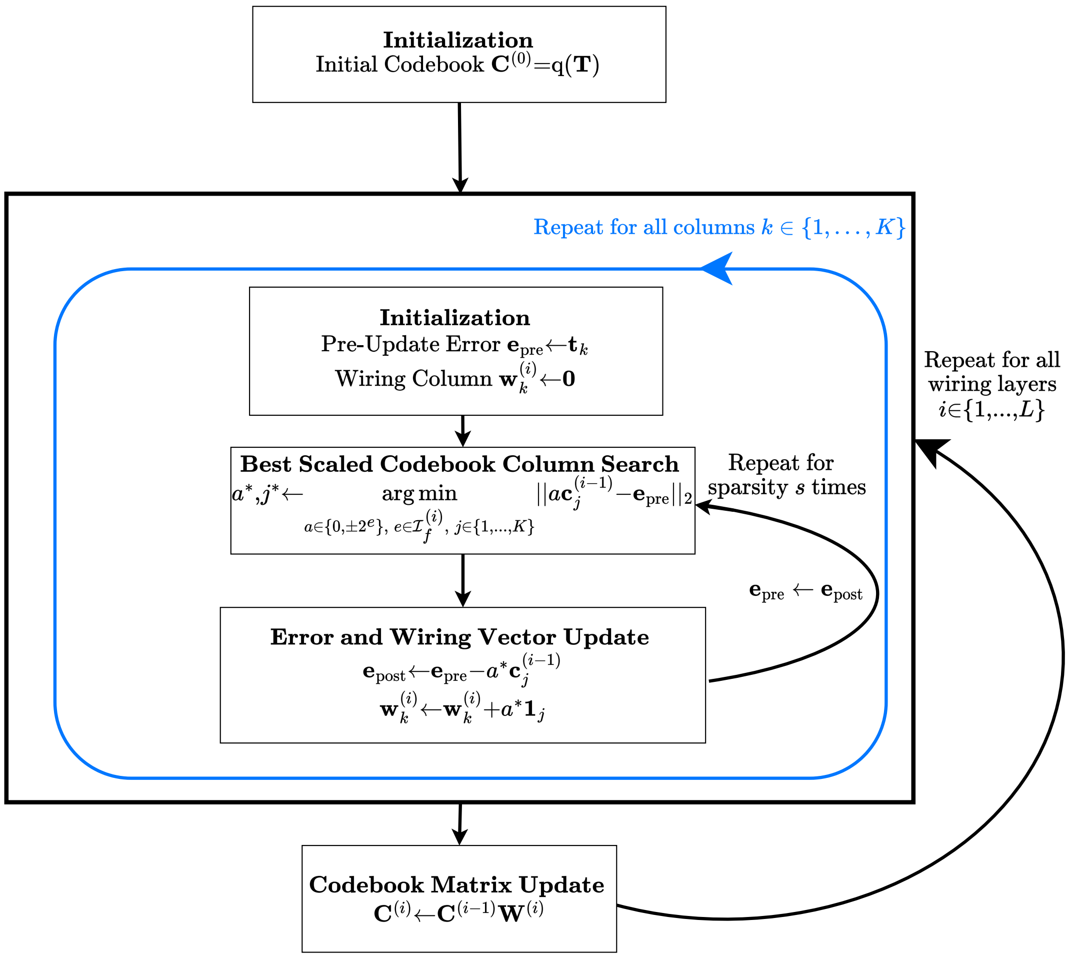

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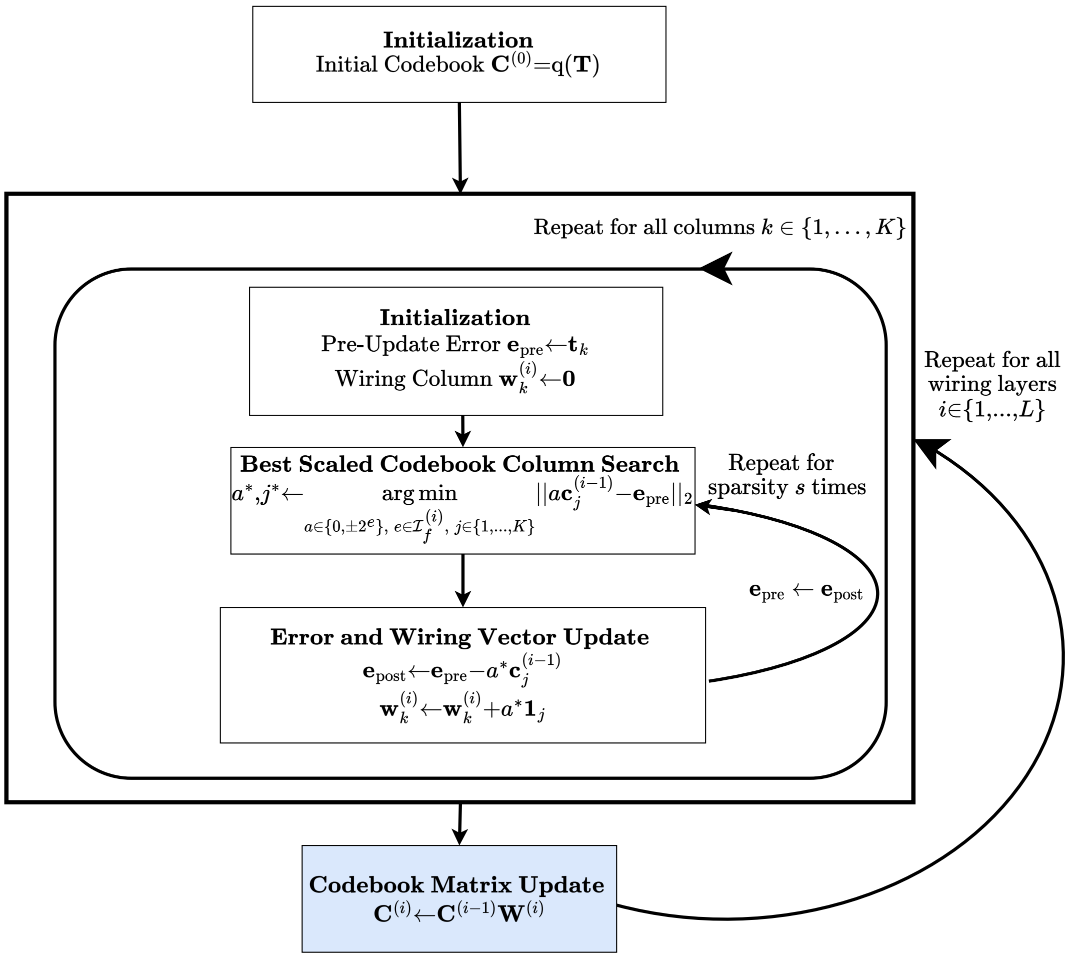

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$$||\mathbf{C}^{(0)} - \mathbf{T}||_F = 0.315$$

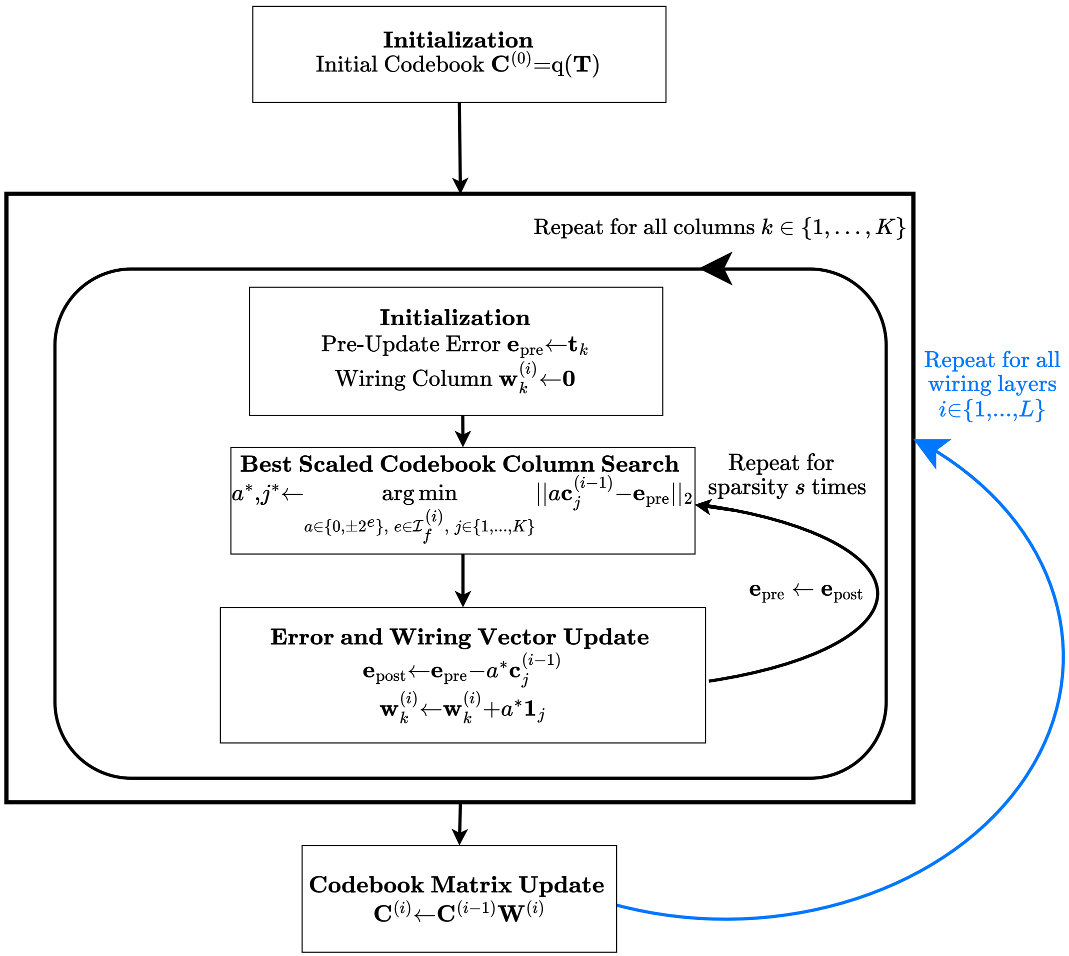
$$\mathbf{W}^{(1)} = \begin{bmatrix} 2^0 & 0 & 0 \\ 0 & 2^0 & -2^{-3} \\ 2^{-2} & -2^{-5} & 2^0 \end{bmatrix}$$

Finite set of wiring exponents $\mathcal{I}_f^{(i)} = \{0, -1, \ldots, -10\}$

$$\mathbf{C}^{(1)} = \begin{bmatrix} 0.875 & 0.266 & -0.531 \\ 0.750 & 0.969 & 0.875 \end{bmatrix}$$

$$||\mathbf{C}^{(1)} - \mathbf{T}||_F = 0.156$$

Example of Greedy Search



Numerical Example

$$\mathbf{T} = [\mathbf{t}_1 \quad \mathbf{t}_2 \quad \mathbf{t}_3] = \begin{bmatrix} 0.760 & 0.259 & -0.500 \\ 0.650 & 0.966 & 0.866 \end{bmatrix}, ||\mathbf{t}_i||_2 = 1$$

$$\mathbf{C}^{(0)} = \text{q}(\mathbf{T}) = \left[\mathbf{c}_1^{(0)} \quad \mathbf{c}_2^{(0)} \quad \mathbf{c}_3^{(0)}\right] = \begin{bmatrix} 2^0 & 2^{-2} & -2^{-1} \\ 2^{-1} & 2^0 & 2^0 \end{bmatrix}$$

$$||\mathbf{C}^{(0)} - \mathbf{T}||_F = 0.315$$

$$\mathbf{W}^{(1)} = \begin{bmatrix} 2^0 & 0 & 0 \\ 0 & 2^0 & -2^{-3} \\ 2^{-2} & -2^{-5} & 2^0 \end{bmatrix}$$

Finite set of wiring exponents $\mathcal{I}_f^{(i)} = \{0, -1, \ldots, -10\}$

$$\mathbf{C}^{(1)} = \begin{bmatrix} 0.875 & 0.266 & -0.531 \\ 0.750 & 0.969 & 0.875 \end{bmatrix}$$

$$||\mathbf{C}^{(1)} - \mathbf{T}||_F = 0.156$$

$$\mathbf{W}^{(2)} = \begin{bmatrix} 2^0 & -2^{-7} & 2^{-6} \\ -2^{-3} & 2^0 & 0 \\ 0 & 0 & 2^0 \end{bmatrix}$$

$$\mathbf{C}^{(2)} = \begin{bmatrix} 0.842 & 0.259 & -0.518 \\ 0.629 & 0.963 & 0.887 \end{bmatrix}$$

$$||\mathbf{C}^{(2)} - \mathbf{T}||_F = 0.089$$