New Results on the Weibull Distribution and Weibull Sums, with Application to Radar Sea Clutter

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Background

Sums of independent Weibull random variables (RVs) arise in many contexts, including

- in wireless communications;
- for spectral estimation in time-series analysis;
- as models for sums of waiting times;
- for modelling value-at-risk-efficient portfolios;
- for modelling rough surfaces;
- as models for radar sea clutter, including clutter spikes, and for synthetic aperture radar images.

But Nadarajah's 2008 review of results on sums of random variables states that no results are known for the Weibull case. A small number of *ad hoc* approximate results have emerged subsequently.

Present motivation: Modelling of high-resolution sea clutter as a Weibull distribution, the main application being threshold estimation for airborne maritime surveillance radar systems that employ scan-to-scan feedback integration, which entails the tail behaviour of discounted sums of Weibull + noise RVs.

Outline

The following strategy will be pursued:

- Characterize the (heavy-tailed) Weibull distribution as a compound clutter model
- Derive a representation of its texture distribution:
  - → Facilitates inclusion of thermal noise and speckle correlation
- Extract texture quadrature nodes and weights:
  - → Clutter distribution well approximated by an exponential mixture
  - → Rational moment generating function (MGF)
  - → Positively weighted sum of clutter RVs also has rational MGF
  - → Distribution recovered via generic saddle-point methods

Outcome: The compound representation of the Weibull distribution renders tractable the computation of Weibull sums with high accuracy extending far into the tail region.

Weibull Texture

Weibull survival function (SF) as a compound clutter model:

$$\bar{F}_X(x) = e^{-(x/\lambda)^\alpha} = \int_0^\infty du\, f_\alpha(u) e^{-x/u} \simeq \sum_{\ell=1}^{L} w_\ell e^{-x/u_\ell} \,.$$

Shape parameter: $\alpha > 0$. Heavy tailed for $0 < \alpha < 1$.

Scale parameter: $\lambda > 0$. Unit mean ($\langle X \rangle = 1$) for $\lambda = 1/\Gamma(1 + 1/\alpha)$.

Clutter RV has product form $X = UI$, with speckle RV: I, texture RV: U.

Inclusion of thermal noise: $U \mapsto 1 - q + qU$

Clutter-to-interference ratio: $q \equiv C/(1 + C)$ for CNR $C \;\Rightarrow\; 0 \leq q \leq 1$.

Empirical texture distribution: PDF $f_\alpha(u)$, SF $\bar{F}_U(u;\alpha)$

$$f_\alpha(u) = \sum_{\ell=1}^{L} w_\ell \delta(u - u_\ell) \,, \quad \bar{F}_U(u;\alpha) = \sum_{\ell=1}^{L} w_\ell I(u_\ell > u) \,.$$

Non-Coherent Pulse Integration

An explicit clutter texture distribution $f_\alpha(u)$ simplifies the representation of many quantities of interest, *e.g.* Returned (normalized) interference power (due to Weibull clutter + thermal noise) averaged over N pulses:

$$Y = \frac{1}{N}\sum_{n=1}^{N} X_n \quad : \quad X_1, X_2, \dots, X_N \sim X \, ,$$

with single-pulse RV: $X = (1 - q + qU)I_{\text{n+c}}$

- Uncorrelated clutter speckle + thermal noise $I_{\text{n+c}}$ ($\langle I_{\text{n+c}} \rangle = 1$)
- Perfectly correlated clutter texture U ($\langle U \rangle = 1$)

Survival function, given the texture distribution and its quadrature form:

$$\bar{F}_Y(y) \;=\; \frac{1}{\Gamma(N)} \int_0^\infty du\, f_\alpha(u) \Gamma\left(N, \frac{Ny}{1-q+qu}\right) \;\simeq\; \frac{1}{\Gamma(N)} \sum_{\ell=1}^{L} w_\ell \Gamma\left(N, \frac{Ny}{1-q+qu_\ell}\right) \, ,$$

where $\Gamma(n, y)$ denotes the upper incomplete gamma function.

Saddle-Point Method

Since the heavy-tailed Weibull distribution ($0 < \alpha < 1$) is *completely monotonic*, there exists $\phi(t)$ such that

$$\bar{F}_X(x) = \int_0^\infty dt\, e^{-xt}\phi(t) \quad \Longrightarrow \quad f_\alpha(u) = u^{-2}\phi(1/u)\ .$$

The inverse Laplace transform yields $\phi(t)$ as

$$\phi(t) = \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i}\, e^{st}\bar{F}_X(s) = \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i}\, e^{\Phi(s)}\,, \quad \Phi(s) \equiv st - (s/\lambda)^\alpha\ .$$

Compute $\phi(t)$ by applying saddle-point (SP) methodology to the *phase function* $\Phi(s)$. Expanding about the SP $s = s_0 : \Phi'(s) = 0$ gives rise to the basic SP approximation (exact for $\alpha = 1/2$)

$$\phi_{\text{sp}}(t) = [2\pi\Phi''(s_0)]^{-1/2}\, e^{\Phi(s_0)}\,, \quad s_0(t) = \lambda\,(\lambda t/\alpha)^{1/(\alpha-1)}\ .$$

Steepest Descent Path

Let $s \in \mathbb{C}$ be the steepest descent path (SDP) of $\Phi(s)$ starting from the SP s_0, and

$$z \equiv (s_0 - s)t \ , \quad \tau(z) \equiv \Phi(s_0) - \Phi(s_0 - z/t) \in \mathbb{R}_0^+ \ .$$

Introduce residuum $T(c_0, \alpha)$, $c_0 \equiv 1/(s_0 t)$, as the multiplicative correction to the SP approximation:

$$\phi(t) = \phi_{\text{sp}}(t) \cdot T(c_0, \alpha) \ , \quad T(c_0, \alpha) = \sqrt{\frac{2r_2}{\pi}} \int_0^\infty d\tau \, e^{-\tau} \, \mathbf{Im} \, z(\tau) \ , \quad r_2 \equiv \Phi''(s_0)/t^2 \ .$$

Need to invert $\tau(z)$: Set $\bar{z} = c_0 z$, $\bar{\tau} = \alpha c_0 \tau$, and apply complex Newton-Raphson to

$$h(\bar{z}) - \bar{\tau} = 0 \ , \quad \text{with} \quad h(z) \equiv (1 - z)^\alpha + \alpha z - 1 \, .$$

Since $\mathbf{Im} \, z(\tau) \sim_{\tau \to 0} \sqrt{2\tau/r_2}$, the τ-integration for $T(c_0, \alpha)$ is best performed by means of a Gauss-Laguerre quadrature of index $1/2$.

Texture Integral

SP approximation via change of variable $t \to \xi$:

$$t(\xi) = \frac{\alpha}{\lambda}\left(\frac{\xi}{1-\alpha}\right)^{1-1/\alpha} \quad \Longrightarrow \quad \phi_{\mathrm{sp}}(t)dt = \frac{1}{\sqrt{2\pi\alpha}}\xi^{-1/2}e^{-\xi}d\xi\,.$$

Exact Weibull SF:

$$\bar{F}(x) = \int_0^\infty \frac{d\xi}{\Gamma(1-\alpha)}\,\xi^{-\alpha}e^{-\xi}.\tilde{T}(\xi)e^{-xt(\xi)}\,, \quad \tilde{T}(\xi) \equiv \frac{\Gamma(1-\alpha)}{\sqrt{2\pi\alpha}}.\xi^{\alpha-1/2}T(c_0(\xi);\alpha)\;.$$

Texture PDF as function of ξ:

$$\psi(\xi)d\xi \;=\; \phi(t)dt \;=\; f_\alpha(u)du \quad \Longrightarrow \quad \psi(\xi) \;=\; \frac{\xi^{-\alpha}e^{-\xi}}{\Gamma(1-\alpha)}\tilde{T}(\xi)\,.$$

Special exact cases for $\alpha = 1/2, 1/3$:

$$\psi_{1/2}(\xi) = \frac{\sqrt{2}}{\pi}K_{1/2}(\xi) = e^{-\xi}/\sqrt{\pi\xi}\,, \quad \psi_{1/3}(\xi) = \frac{\sqrt{3}}{\pi}K_{1/3}(\xi)\,.$$

Gaussian Quadrature

In terms of the standard Gauss-Laguerre quadrature nodes/weights $\xi_\ell^{\text{GL}}, w_\ell^{\text{GL}}, \ell = 1, 2, \ldots, L$:

$$\mathcal{J}[g] \equiv \int_0^\infty \frac{d\xi}{\Gamma(\nu)} \xi^{\nu-1} e^{-\xi} g(\xi) \simeq \sum_{\ell=1}^{L} w_\ell^{\text{GL}} g(\xi_\ell^{\text{GL}}) ,$$

for $\nu = 1 - \alpha$, the Weibull-texture quadrature nodes and weights of the quadrature sum are given, respectively, by

$$u_\ell = \frac{\lambda}{\alpha} \left(\frac{\xi_\ell^{\text{GL}}}{1-\alpha} \right)^{1/\alpha - 1} , \; w_\ell = w_\ell^{\text{GL}} \cdot \tilde{T}(\xi_\ell^{\text{GL}}) .$$

NB: Other quadrature schemes are possible. It is sometimes more efficient to use a split quadrature scheme comprising Gauss-Laguerre above a cut-off ξ_c and Gauss-Jacobi below it.

Weibull Texture ξ-PDF – SDP vs Exact

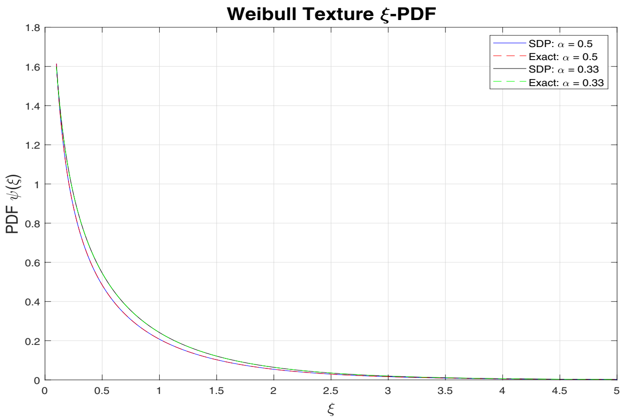

Figure: Weibull texture PDF as a function of ξ computed via the SDP for $\alpha = 1/2, 1/3$, and comparison with exact results.

Prony's method

The texture nodes and weights for Weibull clutter can also be derived by directly fitting to an exponential mixture via use of a modern (least squares) (LS) implementation of Prony's method.

- Weibull SF is equidistantly sampled with $N = 150$ points down to probability level 10^{-12}.
- To cover a large dynamic range, a segmented approach is used, splitting into *body* and *tail* intervals,
- Separate Prony application to the two intervals, with $L_{1,2} = 10$ quadrature nodes assigned to each.
- The nodes u_ℓ from both regions are combined after thinning out the overlapping ones.
- These are used in a constrained LS regression to recompute the combined set of weights w_ℓ.

NB: Some limitations

- The cut point x_{c} that separates the two intervals is currently tuned by trial and error.
- The number of nodes possible is limited by ill-conditioned matrices, emergence of complex values, *etc*.

Weibull Texture Survival Function – Prony Method

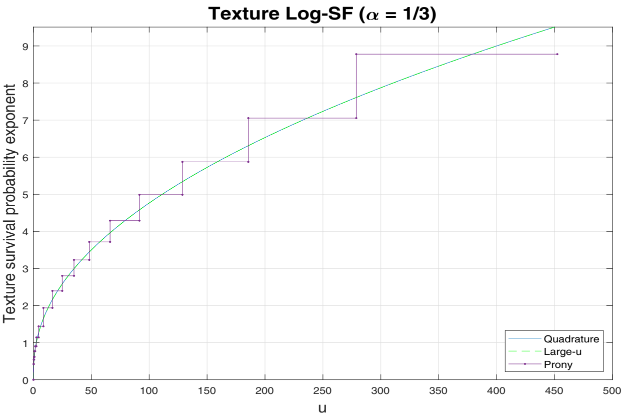

Figure: Weibull texture empirical SF on a log scale as a function of u for $\alpha = 1/3$, constructed from the Prony weights and nodes.

Scan-to-Scan Feedback Integration

Scan-to-scan feedback integration ($0 \leq \beta \leq 1$) corresponds to the auto-regressive Markov process

$$Y_k = \beta Y_{k-1} + X_k\,, \quad Y_0 \equiv 0\,, \quad k = 1, 2, \dots, M,$$

where Y_M is the returned interference power after M samples (scans) of the innovation X_k – iid Weibull + thermal noise ($\langle X_k \rangle = 1$).

Renormalize for fixed M, so that uniform sum ($\beta = 1$) is well-defined as $M \to \infty$ and $\langle Z_M \rangle = 1$:

$$Z_k \equiv \eta_M(\beta) Y_k\,, \quad 1/\eta_M(\beta) = \sum_{m=0}^{M-1} \beta^m = \frac{1-\beta^M}{1-\beta}\,.$$

Detection threshold z_{b}, y_{b} from the steady-state ($M \to \infty$) RV Z_∞:

$$P_{\mathsf{FA}} = \bar{F}_{Z_\infty}(z_{\mathsf{b}}) \equiv \mathbf{Pr}(Z_\infty > z_{\mathsf{b}})\,, \quad y_{\mathsf{b}} = z_{\mathsf{b}}/(1-\beta)\,.$$

Application to Discounted Sums

Equivalence, in distribution, with a discounted sum:

$$Y_M \sim \sum_{m=0}^{M-1} \beta^m X_m \ , \quad Z_M \sim \sum_{m=0}^{M-1} \omega_m X_m \ , \quad \omega_m(\beta, M) \equiv \eta_M(\beta) \cdot \beta^m \ .$$

MGFs:

$$\mathcal{M}_X(s) \equiv \langle e^{-sX} \rangle_X \simeq \sum_{\ell=1}^{L} \frac{w_\ell}{1 + u_\ell s} \ , \quad \mathcal{M}_Z(s) \equiv \langle e^{-sZ} \rangle_Z = \prod_{m=0}^{M-1} \mathcal{M}_X(\omega_m s) \ .$$

Rational form:

$$\mathcal{M}_Z(s) \simeq \prod_{\ell=1}^{L} \prod_{m=0}^{M-1} \frac{1 + b_{\ell,m} s}{1 + a_{\ell,m} s} \ ,$$

$\longrightarrow$ SF recovered via generic saddle-point methods.

Uniform Weibull Sum – Compare with REMC

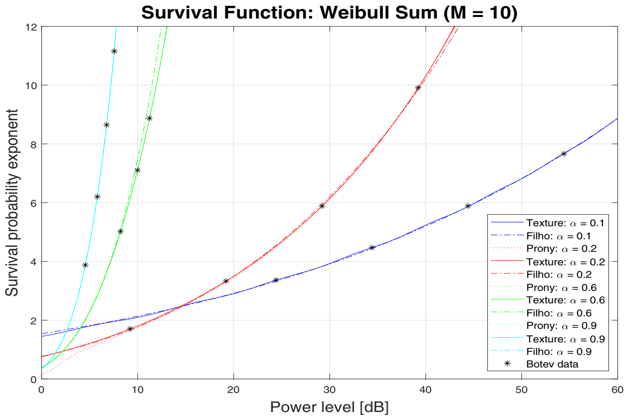

Figure: Log-survival function for the uniform sum ($\beta = 1$) compared with published results from rare-event MC simulation.

Discounted Weibull Sum – Compare With CMC

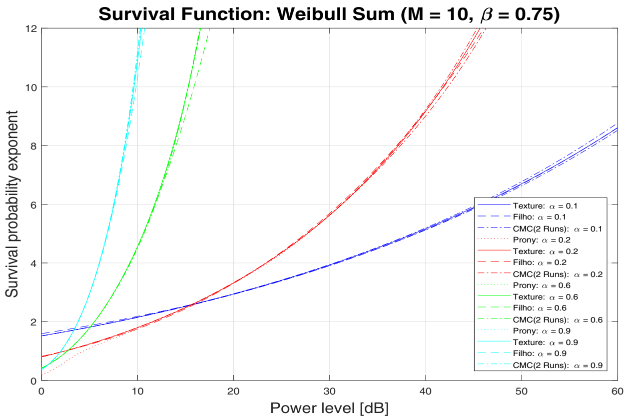

Figure: Log-survival function for the discounted sum ($\beta = 3/4$) compared with results from rare-event conditional MC simulation.

Discounted Sum – Weibull Clutter + Thermal Noise

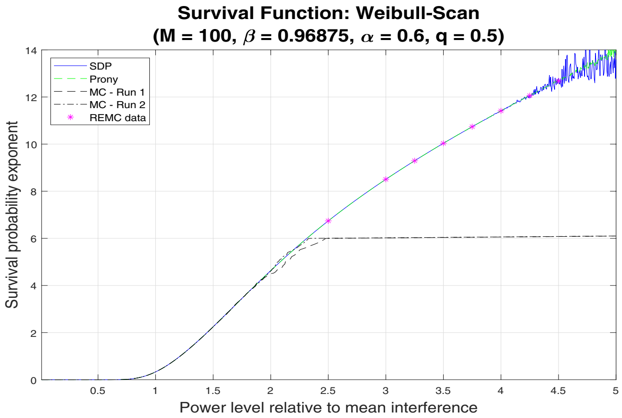

Figure: Log-survival function for the discounted sum ($\beta = 31/32$) compared with results from crude MC simulation.

Conclusions

An explicit characterization of the heavy-tailed Weibull distribution as an example of a compound clutter model has been developed.

A tractable representation of the Weibull texture distribution has several benefits:

- **For non-coherent pulse-to-pulse integration:** Simplifies the computational aspects of Weibull clutter in the presence of thermal noise. It also allows additional complexities to be introduced, such as clutter speckle correlation.
- **For coherent pulse-to-pulse integration:** Enables Weibull clutter to be explicitly implemented as a SIRP.
- **For scan-to-scan feedback integration:** Facilitates the computation of the ensuing distribution that arises from a discounted sum of RVs for Weibull clutter in thermal noise. Importantly, it allows accurate tail estimation, required for computing the threshold from the desired PFA.
- **More generally:** Enables efficient computation of the distribution of sums of uniformly or arbitrarily positively weighted iid Weibull RVs in a way that remains accurate into the tail of the distribution. This has general applicability in many diverse areas.