Fractional Fourier Transform in Time Series Prediction

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Introduction

- Thriving tendency towards integrating DNNs with classical signal processing tools; e.g. FT, STFT, WT.
- FrFT as a generalized version of FT, controlled by a fraction order a.
- No additional computational load; fast computation algorithm with $O(NlogN)$ time complexity [Ozaktaş et al., 1996].
- Allowing signal representation in a continuum between time and frequency.
- Different information flows and feature extractions in neural networks with FrFT [Sahinuç et al., 2022].
- Working with best signal representation in the underlying network.

Contributions

- **Introducing FrFT to machine learning by combining it with RNNs for time series prediction.**
- **Motivated by using a generalized transform for infinitely many transformations to enhance model performance.**
- **FrFT-based RNNs surpass those in time or frequency domain for prediction performance.**
- **Additionally, incorporating FrFT order a as a learnable parameter.**

Primer on Fractional Fourier Transform

- For $a \in \mathbb{R}$, the ath order FrFT $\mathcal{F}^a$ of a function or signal $f(t) \in \mathcal{L}^2(\mathbb{R})$ is defined as follows:

$$\mathcal{F}_a(u) = \mathcal{F}^a\{f(t)\}(u) = \int_{-\infty}^{\infty} K_a(u,t)f(t)dt,$$
$$K_a(u,t) = A_\phi e^{i\pi(u^2cot\phi - 2utcsc\phi + t^2cot\phi)},$$
$$A_\phi = \sqrt{1 - icot\phi},\ \ \phi = a\pi/2,$$
$$K_a(u,t) = \begin{cases} A_\phi e^{i\pi(u^2cot\phi - 2utcsc\phi + t^2cot\phi)} & \text{if } a \neq 2k \\ \delta(u-t) & \text{if } a = 4k-2 \\ \delta(u+t) & \text{if } a = 4k+2, \end{cases}$$

where k is an integer. FrFT operates with period 4 such that $\mathcal{F}^a = \mathcal{F}^b$ where $a \equiv b \mod 4$ [Ozaktaş et al., 2001, Tao et al., 2009].

- Possible to represent the discrete-time signals in FrFT domain
- Like DFT computed via matrix-vector multiplication, DFrFT can also be expressed similarly.
- Let $\mathcal{X}_n \in \mathbb{R}^L$ is a sequence of length L and the formal definition for DFrFT of $\mathcal{X}_n$ is given as a matrix multiplication:

$$X_{F^a} = W^a \mathcal{X}_n, \tag{1}$$

where $\mathcal{X}_n$ is manifested as a column vector, and W^a is the $L \times L$ DFrFT matrix with order a as given below:

$$W^a[m,n] = \sum_{k=0, k\neq(L-1+(L)_2)}^{L} u_k[m]e^{-i\frac{\pi}{2}ka}u_k[n], \tag{2}$$

where $(L)_2 \equiv L \mod 2$ and u_k is the kth discrete Hermite-Gaussian function [Candan et al., 2000].

Time Series Prediction Using Fractional Fourier Representations

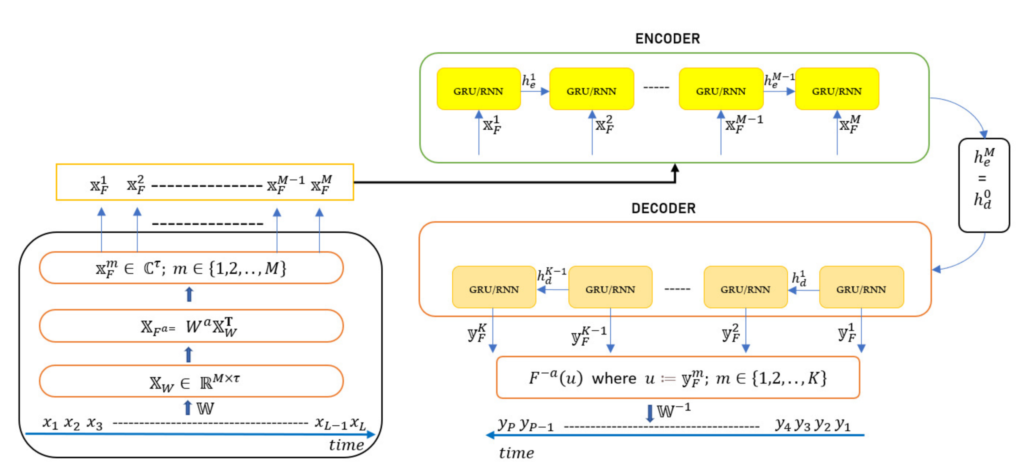

Figure 1:End-to-end FrFT based time series prediction with sequence models.

- The first stage is FrFT-based feature extraction:

$$X_n[m] = w_n[n - Sm]\mathcal{X}_n[n], \tag{3}$$
$$\mathbb{X}_W = \mathbb{W}(\mathcal{X}_n) \in \mathbb{R}^{M\times\tau}, \tag{4}$$
$$\mathbb{X}_{F^a} = W^a\mathbb{X}_W^\intercal, \tag{5}$$
$$\mathbb{X}_{F^a} \in \mathbb{C}^{\tau\times M} \mapsto \{\mathbf{x}_F^m\}_{m=1}^M \in \mathbb{C}^\tau, \tag{6}$$

where $\mathbb{W}$ is a mapping from a sequence to matrix of where each row is a windowed segment $X_n[m]$ and $W^a \in \mathbb{C}^{\tau\times\tau}$ is a DFrFT matrix of a

- **Second stage**: Many-to-many encoder-decoder with GRU or basic RNN cells, resulting in two variants.
- First stage features feed the second stage's encoder; decoder processes only time-domain data.

Model Training

- The information propagates through encoder cells.
- The final hidden state information is passed to the decoder.
- The output vectors of the decoder are multiplied with DFrFT of order $-a$ to calculate the inverse FrFT, then converted to one-dimensional sequences.
- In the backward pass, the mean-squared error (MSE) between the predicted sequence and the decoder training sequence is calculated.

Datasets

- **Mackey-Glass Chaotic Time Series:**
  - It is generated from a nonlinear, time delay differential system that is described by the following differential equation:

$$\frac{dx}{dt} = \frac{\beta x(t-T)}{1 + x^{10}(t-T)} - \gamma x(t), \tag{7}$$

where $\beta = 0.2$, $\gamma = 0.1$, $dt = 0.1$, and $T = 17$. Starting values up to T-th second are initialized as $1 + u[-0.1, +0.1]$ where u stands for the uniform distribution.

- **UCI Electricity Load Dataset**
  - Contains the electricity consumption of 370 customers from Jan. 1, 2011, to Jan. 1, 2015, in 15 min. resolution, and values are in kW.
  - Due to many missing values, data before 2011 is not utilized. Each value is divided by 4 (since 15 minutes is one-fourth of an hour) to convert energy consumption into kWh.
- **USD-TRY Currency Exchange Ratio:**
  - Data of the currency exchange ratio between USD and Turkish lira (TRY) from Jan. 1, 2007, to Jan. 1, 2020.

Results

- Normalized mean-squared percentage error (NMSPE) as our evaluation metric:
- Defined as $100\frac{\Sigma_{i=1}^N(y_i-\hat{y}_i)^2}{\Sigma_{i=1}^N(y_i)^2}$ and expressed as a percentage: y_i is the target value and $\hat{y}_i$ is the predicted value.

Table 1:Mackey-Glass for various architectures and orders a.

	Model					
a	GRU64	RNN64	GRU64$_{LP_{16}}$	RNN64$_{LP_{16}}$	cgGRU64	cgGRU32
0.5	0.030	**0.022**	71.84	71.63	**0.005**	**0.022**
0.6	**0.021**	0.034	64.14	64.20	**0.010**	**0.020**
0.7	0.026	**0.026**	47.14	47.55	**0.011**	**0.032**
0.8	**0.020**	**0.017**	2.256	3.614	**0.006**	**0.054**
0.9	0.024	**0.013**	**0.039**	0.027	**0.010**	**0.039**
1.0	0.023	0.034	0.044	0.012	0.015	**0.175**
1.1	0.027	**0.031**	**0.040**	0.041	**0.004**	**0.078**
1.2	0.033	**0.023**	0.942	2.556	**0.011**	**0.029**
1.3	**0.018**	**0.019**	46.85	46.91	**0.009**	**0.013**
1.4	0.024	**0.016**	64.13	64.19	**0.009**	**0.014**
1.5	**0.016**	**0.017**	71.71	71.81	**0.013**	**0.035**

Table 2:Try/Usd for various architectures and orders a.

	Model					
a	GRU64	RNN64	GRU64$_{LP_{16}}$	RNN64$_{LP_{16}}$	cgGRU64	cgGRU32
0.5	**9.33**	9.78	63.22	63.31	9.07	**8.81**
0.6	**9.72**	9.32	49.01	48.95	8.95	**7.65**
0.7	**9.61**	10.17	43.43	43.05	9.03	**8.48**
0.8	9.84	9.28	**10.08**	**10.40**	8.84	**8.49**
0.9	9.99	8.72	**9.91**	**10.39**	9.23	**9.10**
1.0	9.74	8.15	10.74	10.92	8.57	9.73
1.1	**9.54**	9.03	**10.08**	**10.39**	9.33	**8.94**
1.2	**9.43**	9.89	10.83	10.92	8.84	**8.48**
1.3	**9.70**	10.44	44.55	44.58	**8.06**	**8.15**
1.4	10.13	10.05	49.08	48.89	9.38	**8.23**
1.5	**9.05**	10.17	62.70	62.96	9.22	**8.68**

Table 3:Electric consumption for various architectures and orders a.

	Model					
a	GRU64	RNN64	GRU64$_{LP_{16}}$	RNN64$_{LP_{16}}$	cgGRU64	cgGRU32
0.5	28.54	11.98	79.52	71.43	17.25	22.30
0.6	21.26	10.92	59.73	55.93	19.39	**20.30**
0.7	20.62	11.58	56.78	47.2	**16.27**	**19.78**
0.8	26.65	**10.34**	12.98	14.33	**15.14**	21.71
0.9	19.95	**10.36**	12.47	**7.51**	18.41	**19.00**
1.0	19.00	10.54	10.12	8.38	16.98	20.63
1.1	**17.24**	11.52	12.94	**8.03**	**16.86**	**20.62**
1.2	**18.58**	**10.40**	11.89	8.65	17.18	**19.70**
1.3	25.38	10.64	53.55	47.26	18.68	25.65
1.4	**18.82**	**10.42**	61.86	56.41	19.72	**18.38**
1.5	21.54	12.19	85.77	71.43	18.41	**20.31**

Table 4:Best model performances of the proposed method compared to baselines that use GRU/RNN.

Dataset	**Model**	**NMSPE**(%)
Mackey-Glass	RNN64 (a_l=0.944)	**0.0035**
	cgGRU64 (a=1.1)	0.004
	GRU64$_{\text{baseline}}$	0.23
	RNN64$_{\text{baseline}}$	5.48
TRY-USD	cgGRU32 (a_l=1.277)	**6.59**
	cgGRU64 (a=1.3)	8.06
	GRU64$_{\text{baseline}}$	10.30
	RNN64$_{\text{baseline}}$	12.01
Electric Consumption	GRU64 (a_l=0.858)	**6.20**
	RNN64$_{\text{LP}_{16}}$ (a=0.9)	7.51
	GRU64$_{\text{baseline}}$	10.84
	RNN64$_{\text{baseline}}$	14.76

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