Non-uniform Frequency Spacing for Regularization-free Grideless DOA

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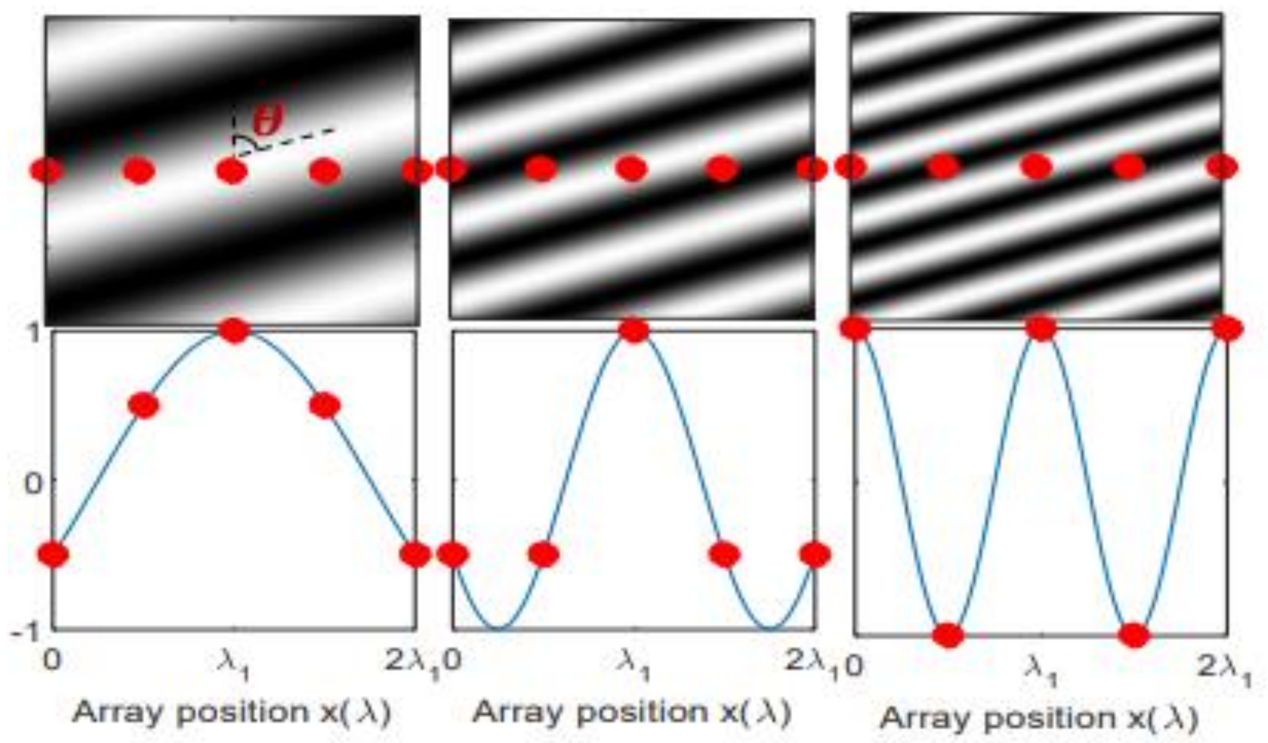
Problem Description



Frequency 1 (f_0) Frequency 2 ($2f_0$) Frequency 3 ($3f_0$)

How to estimate the Direction-of-Arrival (DOA) (θ) based on the array received data that consist of **multiple frequencies when the array spacing is not uniform or frequency spacing is not uniform**?

Signal Model

Received signal from the sensors (N_m-- Number of sensors, N_l-- Number of snapshots, N_f-- **Number of frequencies**)

$$\mathcal{Y} = \mathcal{X} + \mathcal{N} \in C^{N_m \times N_l \times N_f}$$

$$\mathcal{X} = \sum_w c_w [\mathbf{a}(1,w)\mathbf{x}_w^T(1)|\ldots|\mathbf{a}(N_F,w)\mathbf{x}_w^T(N_F)] = \sum_w c_w \mathbf{A}(w) * \mathbf{X}_w^T$$

- $\mathbf{a}(f,w) := [1 \ldots e^{-j2\pi w f(N_m-1)}]^T := [1 \ldots z^{f(N_m-1)}]^T$ is the array manifold ($z = e^{-j2\pi w}$), $w = f_0 d\cos(\theta)/c$ is the **DOA that we need to estimate**, d is the separation between sensor, c is the propagation speed, f_0 is the fundamental frequency
- $\boldsymbol{x}_w$(f) is the source amplitude, N is the additive noise, $*$ is the reshaped Khatri-Rao product

Atomic Norm Minimization (ANM) for multiple frequencies

Atomic set for multiple frequencies

$$\mathcal{A} = \{\mathbf{A}(w) * \mathbf{X}_w^T \mid w \in [-1/2, 1/2], \|\mathbf{X}_w\|_F = 1\}.$$

ANM problem for the noise-free case

$$\min_{\mathcal{X}} \quad \|\mathcal{X}\|_{\mathcal{A}} \quad \text{s.t.} \quad \mathcal{Y} = \mathcal{X}.$$

ANM problem for the noisy case

$$\min_{\mathcal{X}} \quad \|\mathcal{X}\|_{\mathcal{A}} \quad \text{s.t.} \quad \|\mathcal{Y} - \mathcal{X}\|_{\text{HS}} \le \eta.$$

Equivalent SDP for ULA and UF Case

$$\min_{\mathcal{X}} \quad \|\mathcal{X}\|_{\mathcal{A}} \quad \text{s.t.} \quad \mathcal{Y} = \mathcal{X}.$$

$$\max_{\mathcal{Q},\mathbf{P}_0} \langle \mathcal{Q}, \mathcal{Y}\rangle_{\mathbb{R}} \quad \text{s.t.} \begin{bmatrix} \mathbf{P}_0 & \widetilde{\mathbf{Q}} \\ \widetilde{\mathbf{Q}}^H & \mathbf{I}_{N_l N_F} \end{bmatrix} \succeq 0,$$

$$\sum_{i=1}^{N-k} \mathbf{P}_0(i, i+k) = \delta_k, \widetilde{\mathbf{Q}} = [\mathcal{R}(\mathbf{Q}_1)\ldots\mathcal{R}(\mathbf{Q}_{N_F})],$$

$$\mathcal{Q} = [\mathbf{Q}_1|\ldots|\mathbf{Q}_{N_F}] \in \mathbb{C}^{N_M \times N_l \times N_F}$$

$$\widetilde{\mathbf{Q}} = [\widetilde{\mathbf{Q}}_1 \ldots \widetilde{\mathbf{Q}}_{N_F}] \in \mathbb{C}^{N \times N_l N_F}$$

$$\mathcal{R}(\mathbf{Q}_f)(i,l) = \begin{cases} \mathbf{Q}_f(m,l) & \text{for } (i,l) = (f(m-1)+1, l) \\ 0 & \text{otherwise.} \end{cases}$$

NUA and NUF Case

NUA Sensor Index Set $\mathcal{M} \subseteq \{0, 1, \ldots, N_M - 1\}$

NUF Frequency Index Set $\mathcal{F} \subseteq \{1, \ldots, N_F\}$

Define Spatial-frequency index set and its cardinality

$$\mathcal{U} = \{m \cdot f | m \in \mathcal{M}, f \in \mathcal{F}\}. \qquad N_u := |\mathcal{U}|$$

Key Observation: Every exponent in the array manifold vector involves a product of frequency index and sensor position index

Dual and Primal SDP for NUA and NUF Case

Dual

$$\max_{\mathcal{Q},\mathbf{P}_{r0}} \langle \mathcal{Q}, \mathcal{Y}\rangle_{\mathbb{R}} \quad \text{s.t.} \begin{bmatrix} \mathbf{P}_{r0} & \widetilde{\mathbf{Q}}_r \\ \widetilde{\mathbf{Q}}_r^H & \mathbf{I}_{N_l N_f} \end{bmatrix} \succeq 0,$$

$$\sum_{\mathcal{U}_j - \mathcal{U}_i = k} \mathbf{P}_{r0}(i,j) = \delta_k, \widetilde{\mathbf{Q}}_r = [\mathcal{R}_1(\mathbf{Q}_1)\ldots\mathcal{R}_1(\mathbf{Q}_{N_f})],$$

Primal

$$\min_{\mathbf{W},\mathbf{u},\widetilde{\mathbf{Y}}} [\text{Tr}(\mathrm{T}(\mathbf{u})) + \text{Tr}(\mathbf{W})]$$

$$\text{s.t.} \begin{bmatrix} \mathrm{T}(\mathbf{u}) & \widetilde{\mathbf{Y}} \\ \widetilde{\mathbf{Y}}^H & \mathbf{W} \end{bmatrix} \succeq 0, \mathbf{Y}_f = \mathcal{R}_1^*(\widetilde{\mathbf{Y}}_f), f \in \mathcal{F},$$

$$\mathrm{T} \; : \; N \times 1 \to N_u \times N_u$$

IVD and DOA Extraction

If $\mathrm{Toep}(\boldsymbol{u})$ is an PSD matrix, then $\mathrm{T}(\boldsymbol{u})$ is an irregular Toeplitz matrix and it has the following irregular Vandermonde decomposition

$$\mathbf{T} = \mathbf{W}(\gamma, \mathbf{z})\mathbf{D}\mathbf{W}(\gamma, \mathbf{z})^H, |\mathbf{z}| = 1,$$

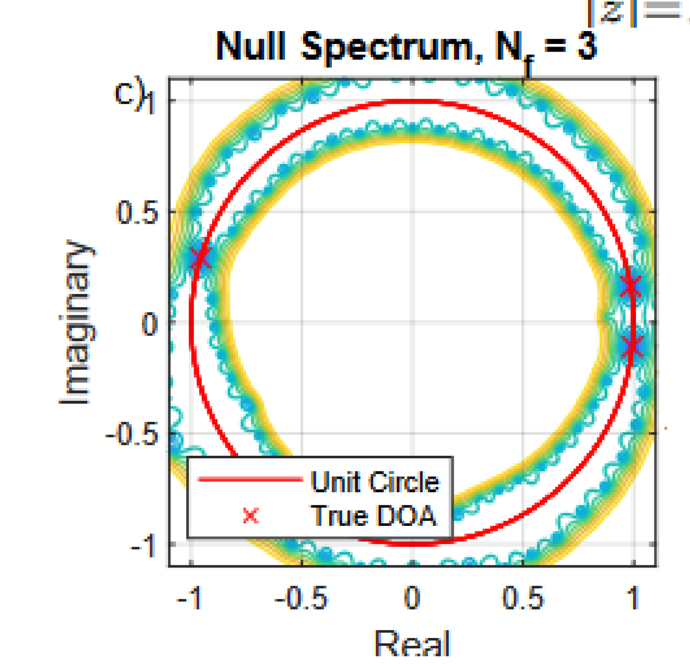
$$\mathbf{W} = \mathbf{W}(\gamma, \mathbf{z}) = [\mathbf{z}^{\gamma_1} \ldots \mathbf{z}^{\gamma_{N_\gamma}}]^T = [\mathbf{w}(\gamma, z_1) \ldots \mathbf{w}(\gamma, z_{N_z})].$$

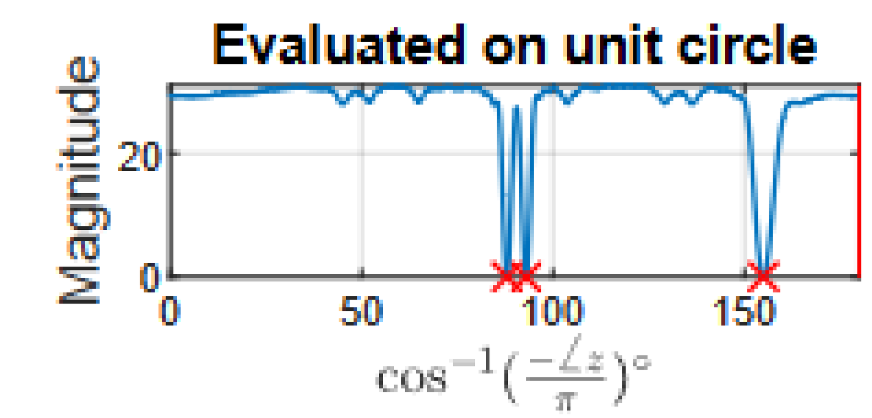
Goal: Estimate $\boldsymbol{z}$

$$\mathbf{T}(\mathbf{u}) = \mathbf{U}_S \mathbf{\Lambda}_S \mathbf{U}_S^H + \mathbf{U}_N \mathbf{\Lambda}_N \mathbf{U}_N^H$$

$$\tilde{D}(z) = \mathbf{w}(\gamma, z)^H \mathbf{U}_N \mathbf{U}_N^H \mathbf{w}(\gamma, z) = \mathbf{w}(\gamma, z)^H \mathbf{G} \mathbf{w}(\gamma, z)$$

$$\hat{\mathbf{z}} = \arg \overset{k}{\min_{|z|=1}} \tilde{D}(z), \quad k = 1, \ldots, K$$





Rank Minimization and Atomic l_0 Minimization

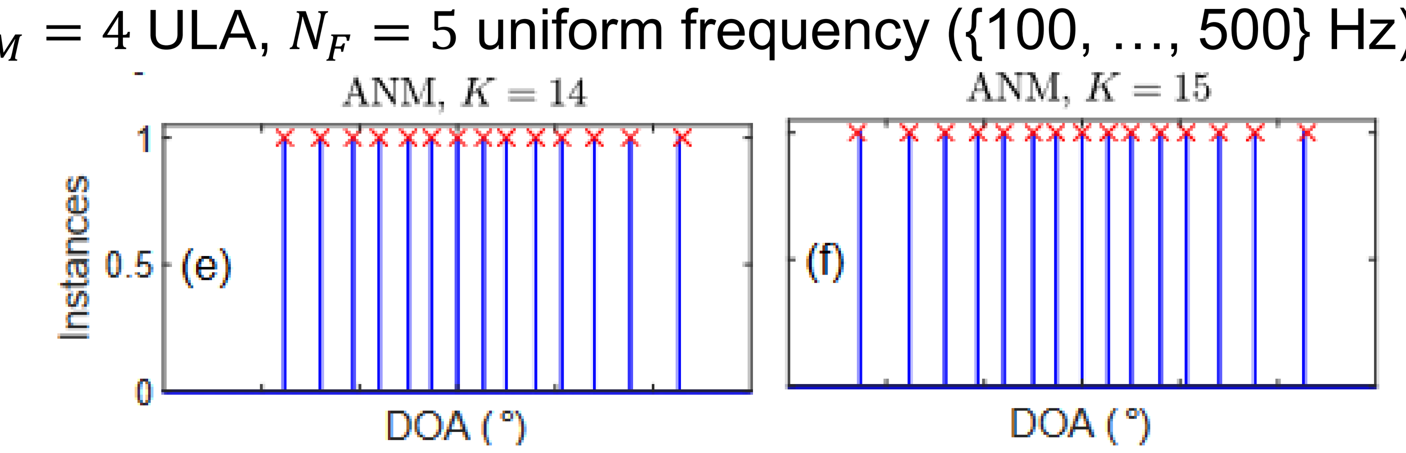
Atomic l_0 Norm $\|\widetilde{\mathbf{Y}}\|_{\mathcal{A},0} := \inf \left\{ K \middle| \widetilde{\mathbf{Y}} = \sum_{k=1}^{K} c_k \mathbf{z}_k \mathbf{x}_k^H, c_k > 0 \right\}$

Atomic l_0 Norm is equal to the optimal value of the following rank minimization problem $\min_{\mathbf{W},\mathbf{u}} \; \text{rank}(\text{Toep}(\mathbf{u}))$

$$\text{s.t.} \begin{bmatrix} \text{Toep}(\mathbf{u}) & \widetilde{\mathbf{Y}} \\ \widetilde{\mathbf{Y}}^H & \mathbf{W} \end{bmatrix} \succeq 0$$

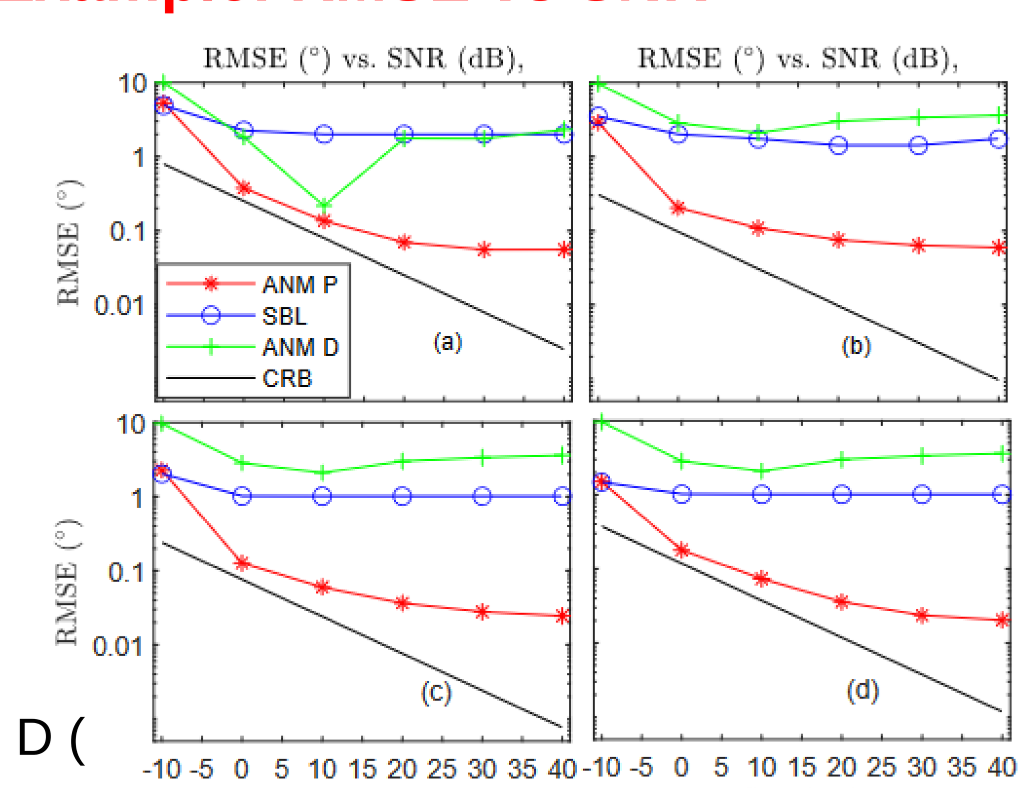
Example: More Sources than Sensors for ULA

- $N_M = 4$ ULA, $N_F = 5$ uniform frequency ({100, …, 500} Hz)



Numerical Example: RMSE vs SNR

- $N_M = 16$ ULA,
- $d = \lambda_{100}/2$
- $N_l = 1$
- Frequency: (a) $N_F = 2$, {100, 200} Hz, (b)-(d) $N_F = 4$. {100,…, 400} Hz for (b); and {100, 200, 300, 500} Hz for (c); {200, 300, 400, 500} Hz for (d).
- ANM P (Proposed); ANM D (ANM proposed in [1])



References

[1] Y. Wu et al, 2023. Gridless DOA Estimation With Multiple Frequences. *IEEE Transactions on Signal Processing, vol. 71, 417-432.*

[2] M. Wagner et al, 2021. Gridless DOA Estimation and Root-MUSIC for Non-uniform Linear Arrays. *IEEE Transactions on Signal Processing, vol. 69, 2144-2157.*