Scaling Results for Robust Distributed Estimation in Sensor Networks using Order Statistics

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Introduction

- Robust modeling of estimation error in a distributed sensor network under random sensing environment is a challenging problem.
- We propose a novel methodology based on order statistics to statistically model scaling behavior of MSE for distributed estimation.
- By leveraging order statistics of the random SNRs over the entire network, we derive and compute CDFs of average MSE for distributed estimation.
- We also establish the scaling trend of the maximum of i.i.d. sensors' SNRs by deriving the distribution function of the estimation error.

System Model

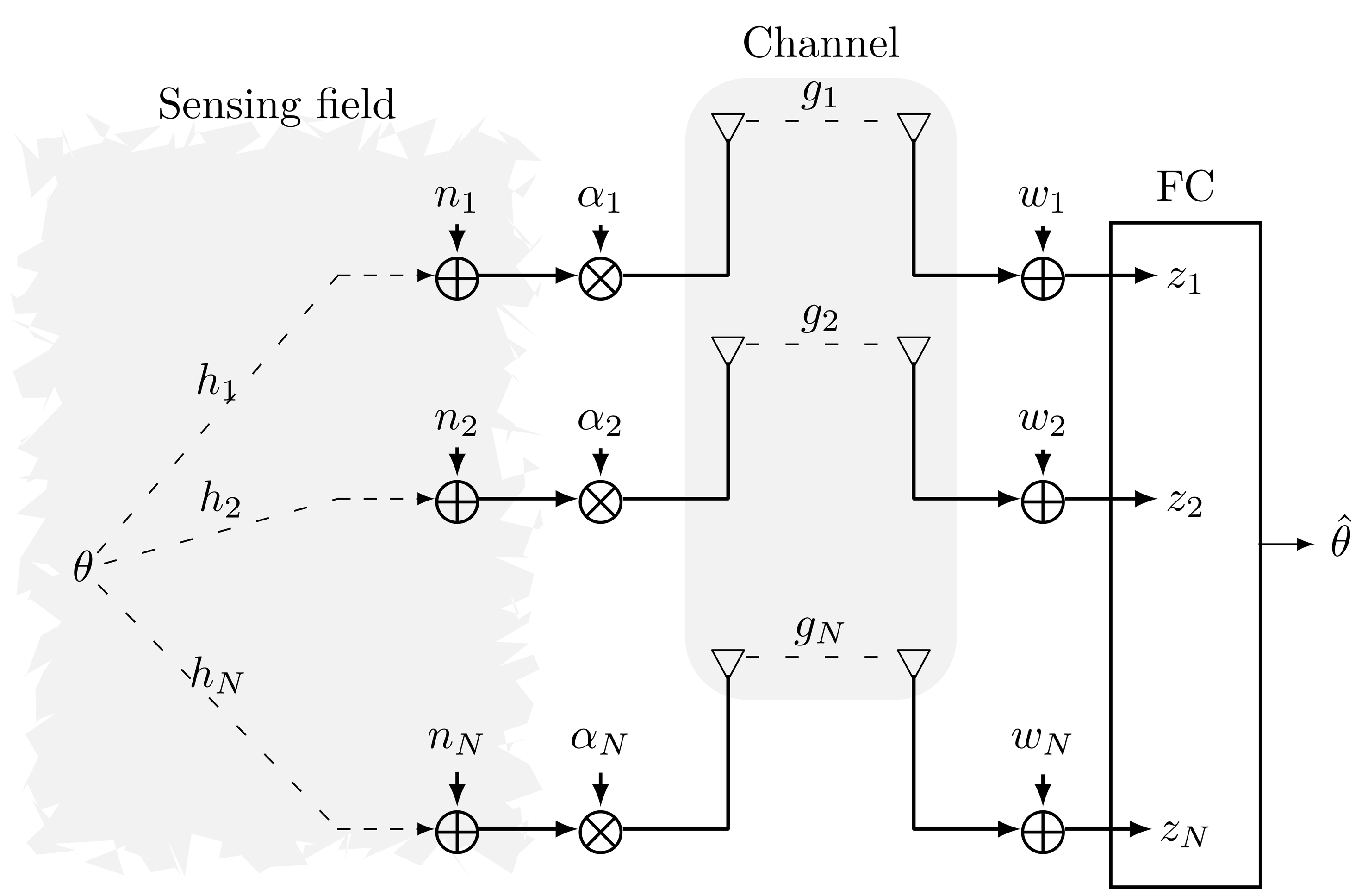


Figure 1: System model.

Consider a sensor network consisting of N nodes make independent measurements of an unknown parameter θ, with unknown statistics, Under a noisy environment over orthogonal channels to the FC

$$\boldsymbol{z} = \boldsymbol{D}_h \boldsymbol{g}\theta + \boldsymbol{w} \tag{1}$$

where $\boldsymbol{D}_h$ = diag$[h_1\alpha_1, ..., h_N\alpha_N]$, $\boldsymbol{g} = [g_1, g_2, ..., g_N]^T$ is the channel gain from sensor network to the FC, and $\boldsymbol{w} = [g_1\alpha_1 n_1 + v_1, ..., g_N\alpha_N n_N + v_N]^T$ normally distributed as $\boldsymbol{w} \sim \mathcal{N}(0, \boldsymbol{R}_w)$ with $\boldsymbol{R}_w$ = diag$[g_i^2\alpha_i^2\sigma_{i,n}^2 + \zeta_i^2]$.

According to the BLUE estimator, the expression of the source signal estimate conditioned on random channel gains is given by

$$\hat{\theta} = \left(\boldsymbol{g}^T\boldsymbol{D}_h^T\boldsymbol{R}_w^{-1}\boldsymbol{D}_h\boldsymbol{g}\right)^{-1}\boldsymbol{g}^T\boldsymbol{D}_h^T\boldsymbol{R}_w^{-1}\boldsymbol{y} \tag{2}$$

The corresponding MSE of this estimator is given as [?]

$$E[(\theta-\hat{\theta})^2] = \left(\boldsymbol{g}^T\boldsymbol{D}_h^T\boldsymbol{R}_w^{-1}\boldsymbol{D}_h\boldsymbol{g}\right)^{-1} = \left(\frac{1}{\|\boldsymbol{h}\|_{\boldsymbol{B}}^2}\right) \tag{3}$$

Distribution of Estimation Error

For a channel $\boldsymbol{h}$ which is characterized by its statistical information, we can express the corresponding cdf and pdf of the error estimate $e := \theta - \hat{\theta}$ as follows

$$p_e(x) = \sum_{n=1}^{N} \frac{\beta_n^{N-1}}{x^2|\boldsymbol{B}|\prod_{i=1,i\neq n}^{N}(\beta_n/\beta_i - 1)} e^{\frac{-1}{\beta_n x}} \tag{4a}$$

$$F_e(x) = \sum_{n=1}^{N} \frac{\beta_n^{N}}{|\boldsymbol{B}|\prod_{i=1,i\neq n}^{N}(\beta_n/\beta_i - 1)} e^{\frac{-1}{\beta_n x}}\tilde{u}(x) \tag{4b}$$

where $\boldsymbol{B}$ = diag$[\beta_1, \beta_2, ..., \beta_N]$ and $\beta_i = \frac{g_i^2\alpha_i^2}{g_i^2\alpha_i^2\sigma_{n,i}^2 + \zeta_i^2}$. If $g_i \sim \mathcal{N}(0, \sigma_g^2)$

$$f_Y(y_i) = \frac{\zeta_i}{\sqrt{\pi}\sigma_g(2y_i)^{3/2}} \exp\left(\frac{-\zeta_i^2}{2\sigma_g^2 y_i}\right) \tag{5a}$$

$$F_Y(y_i) = 1 - 2Q\left(\frac{\zeta_i}{\alpha_i\sqrt{y_i}\sigma_g}\right) \tag{5b}$$

where $Y_i = \Theta(x_i) = \zeta_i^2/g_i^2$

Order Statistics & Scaling

Consider the ordering of the diagonal values of $\boldsymbol{B}$ = diag$[\beta_i]$ by noting that $\beta_1, \beta_2, ..., \beta_N$ are i.i.d. random variables with continuous distribution of $Y_i = 1/\beta_i$. Noting that the $\max(\beta)$, or $\min(y)$, corresponds to the dominant exponential term, we proceed to determine an extremal distribution of y

$$\lim_{N\to\infty} P\left[\min_{n=1,...,N} Y_n > y\right] = e^{-\xi} \tag{6}$$

where $\lim_{N\to\infty} NF_Y(y) = \xi$.

Furthermore, the condition for the maximum is $2NQ(\frac{1}{\sigma\sqrt{y}}) \to \xi$ or, equivalently, $y = \left(\frac{1}{\sigma Q^{-1}(\xi/2N)}\right)$

$$\lim_{N\to\infty} P\left[\max_{1\leq i\leq N} Y_i \leq \left(\frac{1}{\sigma Q^{-1}(y/2N)}\right)\right] = e^{-y} \tag{7}$$

and

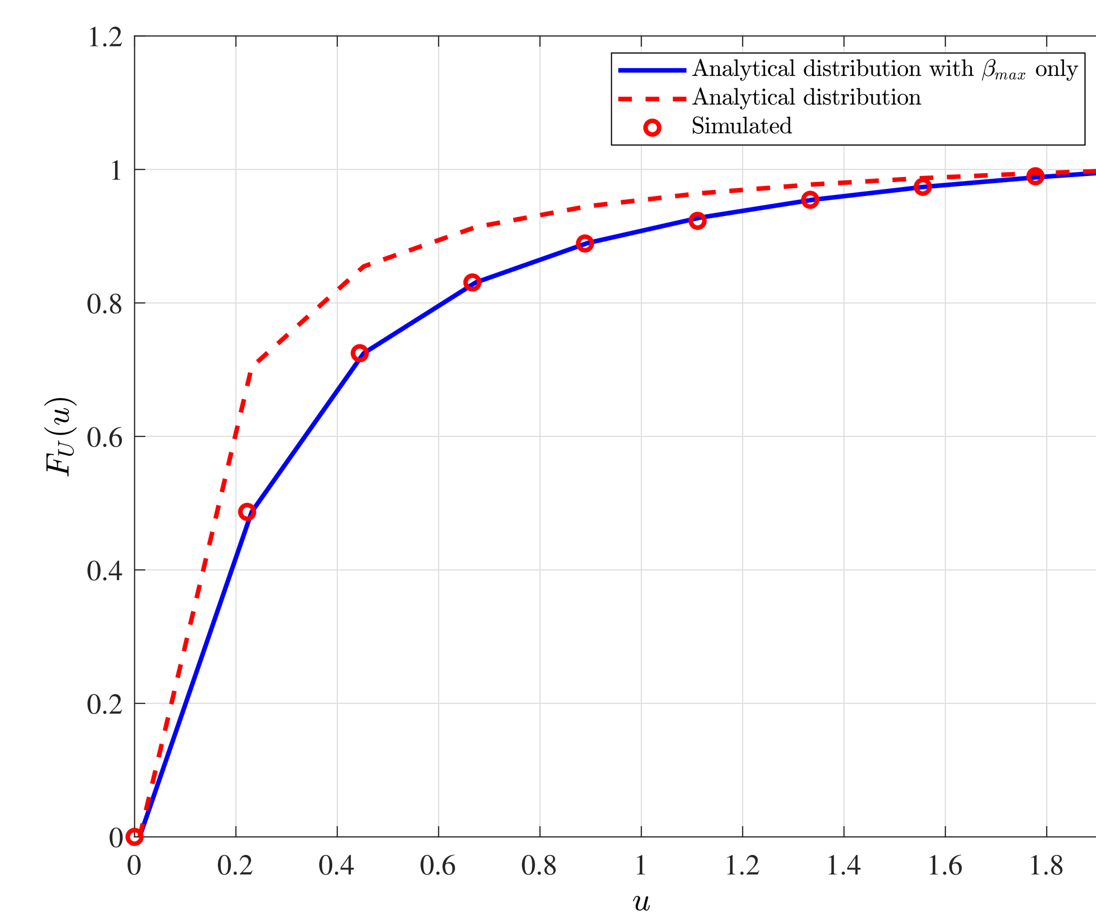
$$\begin{aligned} f_U(u) &= \frac{d}{du}[F_Y(u)]^N = N[F_Y(u)]^{N-1}f_Y(u) \\ &= N\left[G\left(\frac{\zeta_i}{\alpha_i\sigma_g\sqrt{u}}\right)\right]^{N-1} \frac{\zeta_i}{\sqrt{\pi}\sigma_g(2u)^{3/2}} \exp\left(-\frac{\zeta_i^2}{2\sigma_g^2 u}\right) \end{aligned}$$

Numerical Example

consider a distributed setting for collecting observations by sensors deployed over a square field with side length of 100 m. Furthermore, the location of the source of the unknown signal is assumed to be at the center of the square region. To capture the random variations of the channel coefficients, all numerical results are averaged over 1000 independent Monte Carlo runs and then compared with the analytic results.

Results

We plot the CDF of the distortion when all sensor nodes participate as well as the CDF corresponding to the maximum of all $\beta_n, n = 1, ..., N$.



Next figure illustrates the scaling behavior of $\beta_{\max}$ with the increasing number of sensor nodes.