Exact classification of NMR spectra from NMR signals

Problem Formulation

In **Nuclear Magnetic Resonance (NMR)** a sample is plunged into an **external magnetic field** of magnitude B_0 and is excited with an external **radio-frequency (RF) wave**. The response produces the **NMR signal**:

$$u(t) = \sum_{k=1}^{p} c_k e^{\lambda_k t} + N(t)$$

where:

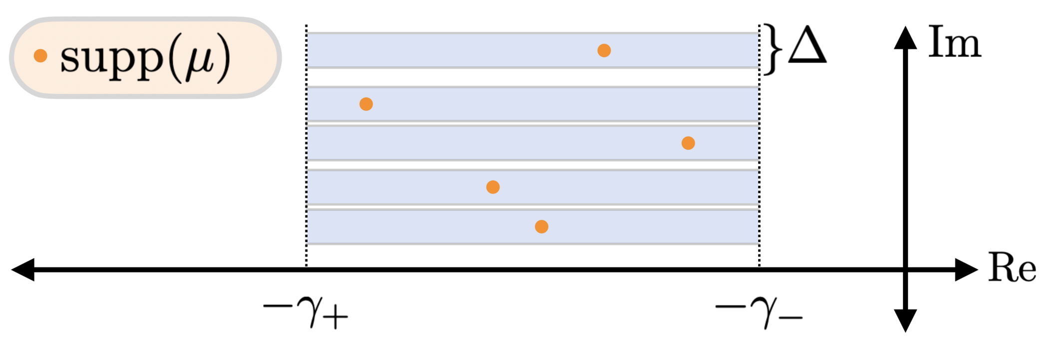
- $\{\mathrm{im}(\lambda_k)\}$ are the sample-dependent **frequencies**.
- $\{\mathrm{re}(\lambda_k)\}$ are the **decay rates** proportional to B_0^{-1}.
- $\{c_k\}$ are the **amplitudes** and **phases**.
- $N(t)$ is the **additive noise** on the signal.

The NMR spectrum and NMR signal are related as follows:

- The **NMR spectrum** is encoded in the **complex measure**

  $$\mu = \sum_{k=1}^{p} c_k \delta_{\lambda_k}$$

  for a known and fixed p. We assume that the decay rates are bounded and that the frequencies are separated, that is $-\mathrm{re}(\lambda_k) \in [\gamma_-, \gamma_+]$ and $|\mathrm{im}(\lambda_k - \lambda_\ell)| \geq \Delta$ for some $\gamma_+, \gamma_-, \Delta > 0$.



- The noiseless **NMR signal** is represented as the image of the spectrum under the injective linear operator S

  $$t > 0: \quad S\mu(t) := \int_{\mathbb{C}} e^{\lambda t}\, d\mu(\lambda)$$

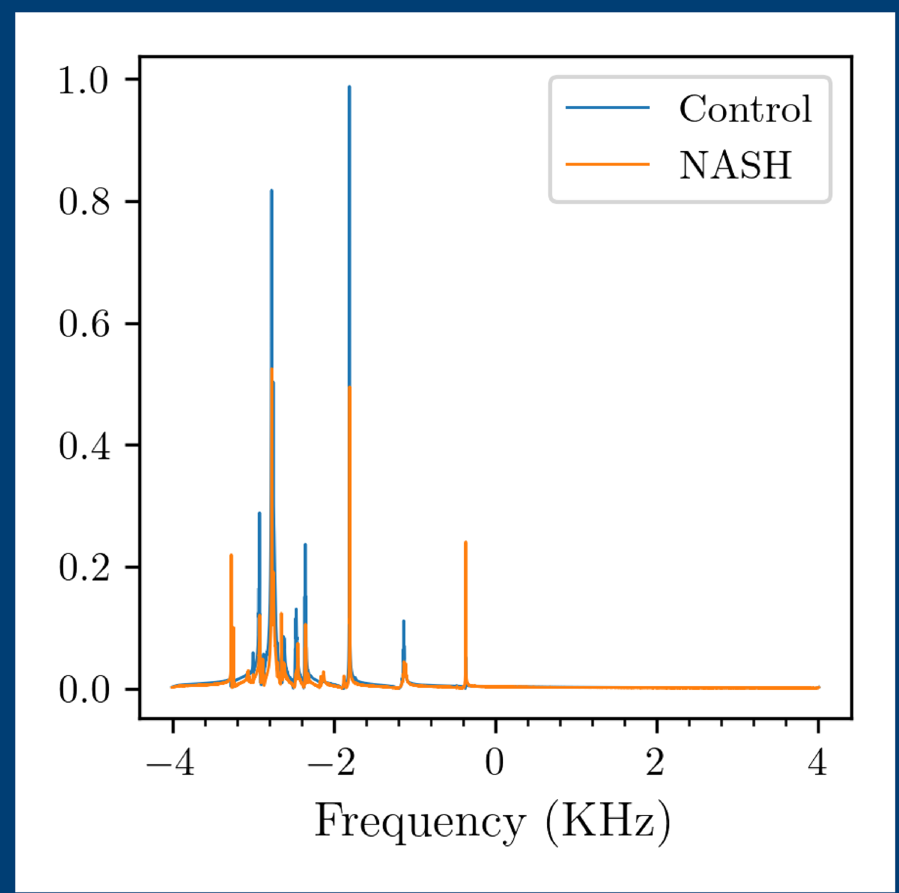
- The noisy NMR signal becomes $u(t) = S\mu(t) + N(t)$.

To compare samples belonging to two classes we quantify the difference between:

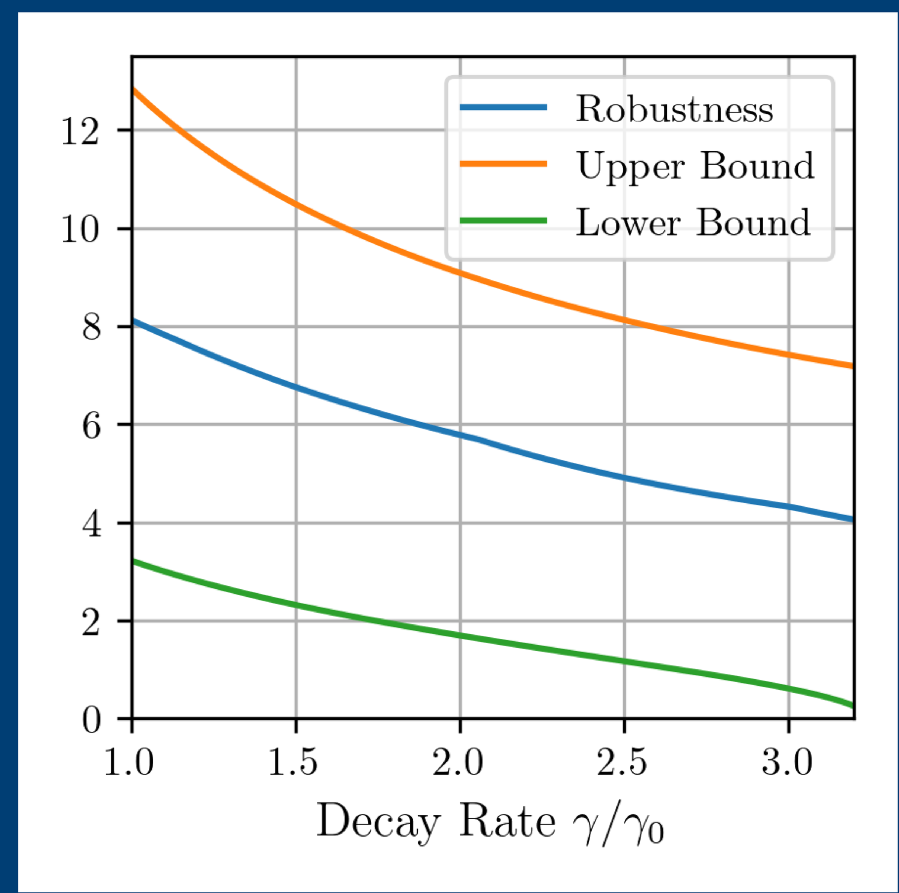
- NMR spectra using the **total variation distance** d_{TV}.
- NMR signals using the L^2 **distance** d_{L^2}.

How **robust to adversarial additive noise** is the **end-to-end classification** of two disjoint sets O_1 and O_2 of NMR **spectra** from their NMR **signals** SO_1 and SO_2?

The **NMR spectrum** can be **exactly classified** from the **NMR signal** as long as its **SNR** is, at most, proportional to the **square root of the external magnetic field.**



NMR signals of a control group mouse u_1 and of a mouse with NASH u_{13}.



Theoretical upper and lower bounds, and empirical robustness.

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Main Result

Let $R_p = \lceil (p-1)/2 \rceil$ and $H_p = 2(1 + 1/2 + \ldots + 1/R_p)$. If there exists $\Delta > 0$ such that

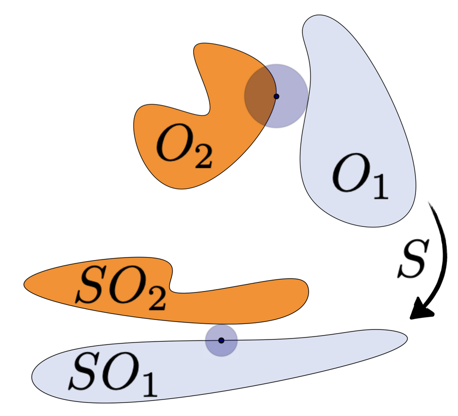
$$|\mathrm{im}(\lambda_k - \lambda_\ell)| \geq \Delta > 2\sqrt{H_p^2 \gamma_+^2 - p^{-2}(\gamma_+ + \gamma_-)^2}$$

for every $\mu \in O_1 - O_2$, then

$$\sqrt{\frac{\eta}{2p}} \leq \frac{\mathrm{dist}_{L^2}(SO_1, SO_2)}{\mathrm{dist}_{\mathrm{TV}}(O_1, O_2)} \leq \sqrt{\frac{1}{2\gamma_+}}$$

where

$$\eta = \frac{1}{2\gamma_+} - \frac{H_p R_p}{\sqrt{(\gamma_+ + \gamma_-)^2 + \Delta^2 R_p}}$$



Numerical Experiments

We used 9.4T [1]H NMR spectroscopy data from the liver of 12 control mice (C) and 12 mice with NASH (N). The high-field data allows us to estimate $p = 9$, $\Delta = 100$ Hz, the average decay rate $\gamma_0 = 3.36$ Hz, and the groups

$$O_C = \{\mu_1, \ldots, \mu_{12}\} \quad \text{and} \quad O_N = \{\mu_{13}, \ldots, \mu_{24}\}.$$

To simulate decay rates, we synthetize the signals

$$t > 0: \quad u_k^\gamma(t) = e^{-\gamma t} S\mu_k(t) \quad i \in \{1, \ldots, 24\}$$

to then define the classes of NMR signals

$$SO_C^\gamma = \{u_1^\gamma, \ldots, u_{12}^\gamma\} \quad \text{and} \quad SO_N^\gamma = \{u_{13}^\gamma, \ldots, u_{24}^\gamma\}.$$

We validate our theoretical findings by computing

$$\mathrm{dist}_{L^2}(SO_C^\gamma, SO_N^\gamma)/\mathrm{dist}_{\mathrm{TV}}(O_C, O_N)$$

and the upper and lower bounds with $\gamma_+ = 1.1\gamma_0 + \gamma$ and $\gamma_- = 0.9\gamma_0 + \gamma$.

Future work

We will analyze the classification of spectra that have an atomic and an absolutely continuous part, modelling spectral leakage, and the effect of distortions due to the push-forward of a measure, modelling spectral shifts.

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