Channel Estimation in Underdetermined Systems Utilizing Variational Autoencoders

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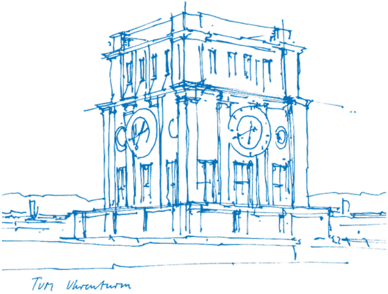

Learning the Underlying Distribution

$\Rightarrow$ Knowledge of the ambient channel distribution is a strong prior

$\Rightarrow$ But: Underlying distribution is generally high-dimensional and complex

$\Rightarrow$ Solution: Learning the distribution via a variational autoencoder (VAE)

GENERATIVE PRIOR

$\Downarrow$

How to deal with underdetermined systems?

System Model

Linear Inverse Problem

$$y = Ah + n$$

- Noisy observation $\boldsymbol{y} \in \mathbb{C}^M$
- Observation matrix $\boldsymbol{A} \in \mathbb{C}^{M \times N}, M < N$
  - Hybrid: phase-shift matrix
  - Wideband: selection matrix
- Channel realization $\boldsymbol{h} \sim p(\boldsymbol{h})$ (unknown prior)
- AWGN $\boldsymbol{n} \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \varsigma^2 \mathbf{I})$
- Further work on MIMO systems[1] and measured data[2]

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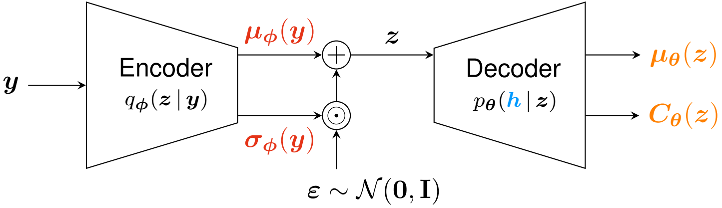
[1] Baur, Fesl, Utschick, ”Leveraging Variational Autoencoders for Parameterized MMSE Channel Estimation,” *submitted to IEEE T-SP, arXiv:2307.05352*, 2023.

[2] Baur, Böck, Turan, Utschick, ”Variational Autoencoder for Channel Estimation: Real-World Measurement Insights,” *WSA*, 2024, *to be published*.

Variational Autoencoder – VAE-noisy

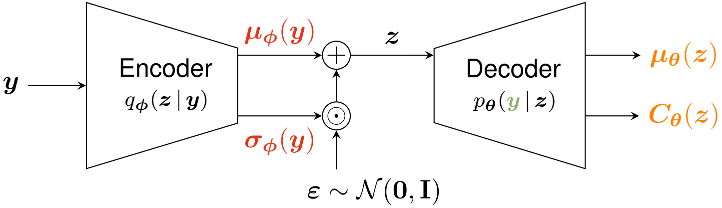


- Training: $\mathcal{L}_{\boldsymbol{\theta},\boldsymbol{\phi}}(\boldsymbol{h}) = \mathrm{E}_{q_{\boldsymbol{\phi}}(\boldsymbol{z} \mid \boldsymbol{y})}\left[\log p_{\boldsymbol{\theta}}(\boldsymbol{h} \mid \boldsymbol{z})\right] - \mathrm{D}_{\mathrm{KL}}(q_{\boldsymbol{\phi}}(\boldsymbol{z} \mid \boldsymbol{y}) \,\|\, p(\boldsymbol{z}))$
- Trained with groud-truth channels $\boldsymbol{h}$!
- Conditionally Gaussian (CG) probability distributions are defined as:

$$p_{\boldsymbol{\theta}}(\boldsymbol{h} \mid \boldsymbol{z}) = \mathcal{N}_{\mathbb{C}}(\boldsymbol{\mu}_{\boldsymbol{\theta}}(\boldsymbol{z}), \boldsymbol{C}_{\boldsymbol{\theta}}(\boldsymbol{z}))$$
$$q_{\boldsymbol{\phi}}(\boldsymbol{z} \mid \boldsymbol{y}) = \mathcal{N}(\boldsymbol{\mu}_{\boldsymbol{\phi}}(\boldsymbol{y}), \mathrm{diag}(\boldsymbol{\sigma}^2_{\boldsymbol{\phi}}(\boldsymbol{y})))$$

- Fixed matrix $\boldsymbol{A}$ during the training phase.

Variational Autoencoder – VAE-real



- Training: $\mathcal{L}_{\boldsymbol{\theta},\boldsymbol{\phi}}(\boldsymbol{y}) = \mathrm{E}_{q_{\boldsymbol{\phi}}(\boldsymbol{z} \,|\, \boldsymbol{y})}\left[\log p_{\boldsymbol{\theta}}(\boldsymbol{y} \mid \boldsymbol{z})\right] - \mathrm{D}_{\mathrm{KL}}(q_{\boldsymbol{\phi}}(\boldsymbol{z} \mid \boldsymbol{y}) \,\|\, p(\boldsymbol{z}))$

- Trained solely with noisy observations $\boldsymbol{y}$!

- CG probability distributions are defined as (exploit $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{h} + \boldsymbol{n}$):

$$p_{\boldsymbol{\theta}}(\boldsymbol{y} \,|\, \boldsymbol{z}) = \mathcal{N}_{\mathbb{C}}(\boldsymbol{A}\boldsymbol{\mu}_{\boldsymbol{\theta}}(\boldsymbol{z}), \boldsymbol{A}\boldsymbol{C}_{\boldsymbol{\theta}}(\boldsymbol{z})\boldsymbol{A}^{\mathrm{H}} + \varsigma^2\mathbf{I})$$
$$q_{\boldsymbol{\phi}}(\boldsymbol{z} \,|\, \boldsymbol{y}) = \mathcal{N}(\boldsymbol{\mu}_{\boldsymbol{\phi}}(\boldsymbol{y}), \mathrm{diag}(\boldsymbol{\sigma}^2_{\boldsymbol{\phi}}(\boldsymbol{y})))$$

- Matrix $\boldsymbol{A}$ must be **varied** during the training phase!

VAE-based Channel Estimation

- Conditional mean estimation is MSE-optimal

$$\mathrm{E}[\boldsymbol{h} \,|\, \boldsymbol{y}] = \arg\min_{\hat{\boldsymbol{h}}} \mathrm{E}\left[\|\boldsymbol{h} - \hat{\boldsymbol{h}}\|_2^2\right] = \mathrm{E}_{\boldsymbol{z}}\left[\mathrm{E}[\boldsymbol{h} \,|\, \boldsymbol{z}, \boldsymbol{y}] \,|\, \boldsymbol{y}\right]$$

- $\mathrm{E}[\boldsymbol{h} \,|\, \boldsymbol{z}, \boldsymbol{y}] = t_{\boldsymbol{\theta}}(\boldsymbol{z}, \boldsymbol{y})$ in closed-form due to $\boldsymbol{h} \,|\, \boldsymbol{z}$ being CG:

$$t_{\boldsymbol{\theta}}(\boldsymbol{z}, \boldsymbol{y}) = \boldsymbol{\mu}_{\boldsymbol{\theta}}(\boldsymbol{z}) + \boldsymbol{C}_{\boldsymbol{\theta}}(\boldsymbol{z})\boldsymbol{A}^{\mathrm{H}}(\boldsymbol{A}\boldsymbol{C}_{\boldsymbol{\theta}}(\boldsymbol{z})\boldsymbol{A}^{\mathrm{H}} + \varsigma^2\mathbf{I})^{-1}(\boldsymbol{y} - \boldsymbol{A}\boldsymbol{\mu}_{\boldsymbol{\theta}}(\boldsymbol{z}))$$

- Replace $p_{\boldsymbol{\theta}}(\boldsymbol{z} \,|\, \boldsymbol{y})$ with approximation $q_{\boldsymbol{\phi}}(\boldsymbol{z} \,|\, \boldsymbol{y})$ above

- Evaluate $t_{\boldsymbol{\theta}}(\boldsymbol{z}, \boldsymbol{y})$ with MAP estimate $\boldsymbol{\mu}_{\boldsymbol{\phi}}(\boldsymbol{y})$ of $q_{\boldsymbol{\phi}}(\boldsymbol{z} \,|\, \boldsymbol{y})$:

$$\hat{\boldsymbol{h}}_{\text{VAE}}(\boldsymbol{y}) = t_{\boldsymbol{\theta}}(\boldsymbol{z} = \boldsymbol{\mu}_{\boldsymbol{\phi}}(\boldsymbol{y}), \boldsymbol{y})$$

Covariance Matrix Parameterization

Hybrid System

- SIMO channel $\boldsymbol{h} \in \mathbb{C}^N$
- ULA at BS induces a Toeplitz channel covariance
- For large arrays, circulant approximation is well motivated:

$$\boldsymbol{C}_{\boldsymbol{\theta}}(\boldsymbol{z}) = \boldsymbol{F}^{\mathrm{H}} \operatorname{diag}(\boldsymbol{c}_{\boldsymbol{\theta}}(\boldsymbol{z}))\boldsymbol{F},\ \boldsymbol{c}_{\boldsymbol{\theta}}(\boldsymbol{z}) \in \mathbb{R}_+^N$$

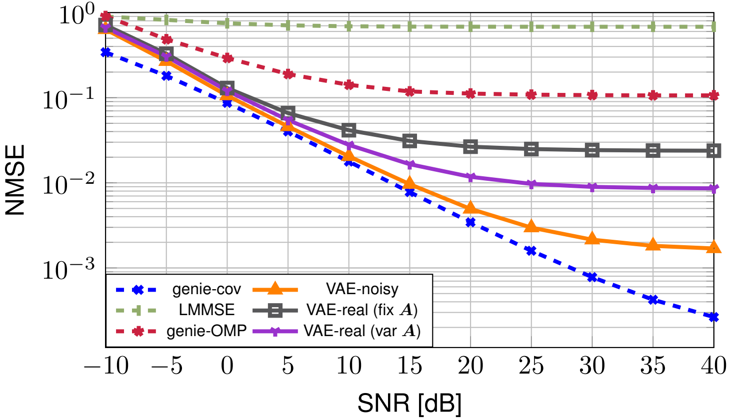
Wideband System

- Doubly-selective fading SISO channel $\boldsymbol{H} \in \mathbb{C}^{N_c \times N_t}$
- Toeplitz channel covariance along time and frequency
- Utilize block-Toeplitz matrix:

$$\boldsymbol{C}_{\boldsymbol{\theta}}(\boldsymbol{z}) = \boldsymbol{C}_{\boldsymbol{\theta},t}(\boldsymbol{z}) \otimes \boldsymbol{C}_{\boldsymbol{\theta},c}(\boldsymbol{z}) = \boldsymbol{Q}^{\mathrm{H}} \operatorname{diag}(\boldsymbol{c}_{\boldsymbol{\theta}}(\boldsymbol{z}))\boldsymbol{Q}, \quad \boldsymbol{c}_{\boldsymbol{\theta}}(\boldsymbol{z}) \in \mathbb{R}_+^{4N_cN_t}$$

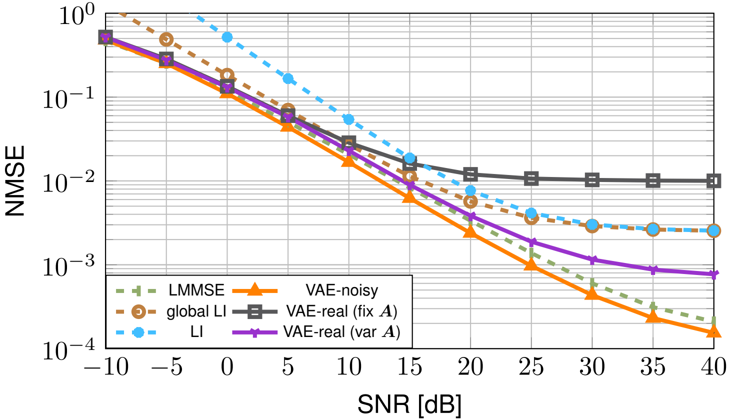
Simulations – Hybrid System

- Channel covariance according to 3GPP specification
- $N = 128$ antennas at BS, $N_r = 32$ RF chains
- Phase-shift matrix $\boldsymbol{A} \in \mathbb{C}^{N_r \times N}$ with $A_{i,k} = \frac{1}{\sqrt{M}} \exp(\mathrm{j}\varphi), \varphi \sim \mathcal{U}([0, 2\pi])$



Simulations – Wideband System

- Doubly-selective fading QuaDRiGa channel $\boldsymbol{H} \in \mathbb{C}^{N_c \times N_t}$
- 2.1 GHz center frequency, 180 kHz bandwidth, $N_c = 12$, $N_t = 14$
- $N_p = 20$ pilots in lattice layout with selection matrix $\boldsymbol{A} \in \{0,1\}^{N_p \times N_c N_t}$



Thank You!

Github:

`https://github.com/baurmichael/vae-est-ud/`

Appendix

Variational Autoencoder – Further Details

- The log-likelihood can be decomposed as

$$\log p_{\boldsymbol{\theta}}(\boldsymbol{h}) = \mathcal{L}_{\boldsymbol{\theta},\boldsymbol{\phi}}(\boldsymbol{h}) + \mathrm{D}_{\mathrm{KL}}\left(q_{\boldsymbol{\phi}}(\boldsymbol{z} \mid \boldsymbol{y}) \,\|\, p_{\boldsymbol{\theta}}(\boldsymbol{z} \mid \boldsymbol{h})\right)$$

  with

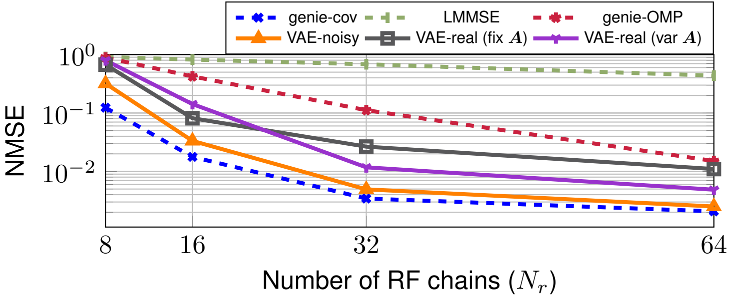
$$\mathcal{L}_{\boldsymbol{\theta},\boldsymbol{\phi}}(\boldsymbol{h}) = \mathrm{E}_{q_{\boldsymbol{\phi}}(\boldsymbol{z} \mid \boldsymbol{y})}\left[\log p_{\boldsymbol{\theta}}(\boldsymbol{h} \mid \boldsymbol{z})\right] - \mathrm{D}_{\mathrm{KL}}(q_{\boldsymbol{\phi}}(\boldsymbol{z} \mid \boldsymbol{y}) \,\|\, p(\boldsymbol{z})).$$

- $\mathcal{L}_{\boldsymbol{\theta},\boldsymbol{\phi}}(\boldsymbol{h})$ is the evidence lower bound (ELBO), a lower bound to $\log p_{\boldsymbol{\theta}}(\boldsymbol{h})$.

- $q_{\boldsymbol{\phi}}(\boldsymbol{z} \mid \boldsymbol{y})$ is supposed to approximate the intractable $p_{\boldsymbol{\theta}}(\boldsymbol{z} \mid \boldsymbol{h})$.

- A maximization of the evidence lower bound (ELBO) maximizes the log-likelihood $p_{\boldsymbol{\theta}}(\boldsymbol{h})$ as well as minimizes $\mathrm{D}_{\mathrm{KL}}\left(q_{\boldsymbol{\phi}}(\boldsymbol{z} \mid \boldsymbol{y}) \,\|\, p_{\boldsymbol{\theta}}(\boldsymbol{z} \mid \boldsymbol{h})\right)$.

Simulations – RF Chain Number



VAE Estimator – Architecture