Inferring Time-Varying Signals over Uncertain Graphs

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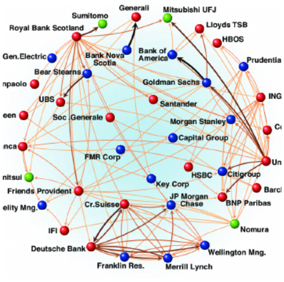
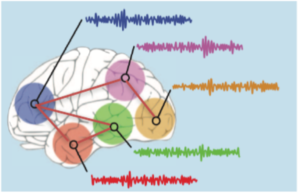
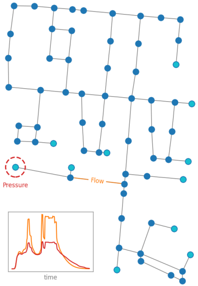
Multivariate Time Series over Networks



Financial networks



Brain recordings

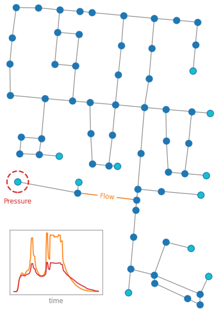


Water networks

- Data have now a spatial dependency and a temporal dependency

Graphs for Multivariate Time Series

- Graphs can be used to model multivariate time series
  - ⇒ The structure is a graph
  - ⇒ Time series are time-varying data assigned to the nodes



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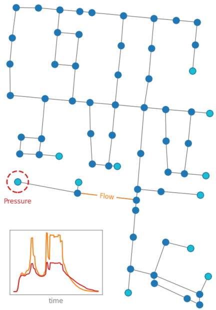
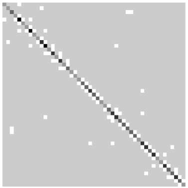


- Nodes: junctions/sources in the city $\mathcal{V}$
- Edges: pipes connecting the junctions $\mathcal{E}$
- Data: recorded pressure in each junction varying by time

⇒ $\mathbf{x}_t$ over the graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$

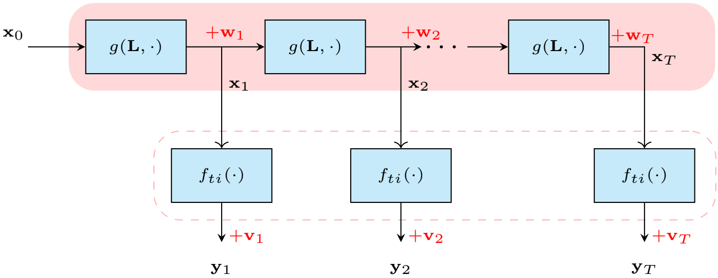
Shift op. ($\mathbf{A}$ or $\mathbf{L}$)



State Space Model for Time-Varying Graph Signals

equations

$$\begin{cases} \mathbf{x}_t = g(\mathbf{L}, \mathbf{x}_{t-1}) + \mathbf{w}_{t-1} & \rightarrow \text{state equation} \\ \mathbf{y}_t = f_t(\mathbf{L}, \mathbf{x}_t) + \mathbf{v}_t & \rightarrow \textbf{observation equation} \end{cases}$$



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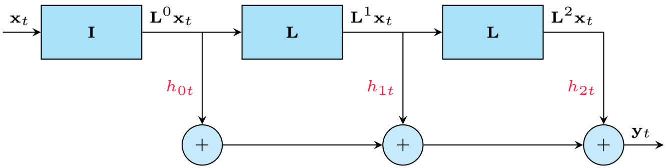
Observation Model: Graph Filter

- We consider a graph filter at each time t for observation model
- Graph filter: shift-and-sum over the graph

$$\mathbf{y}_t = \sum_{k=0}^{K} h_{kt} \mathbf{L}^k \mathbf{x}_t = \mathbf{H}_t(\mathbf{L})\mathbf{x}_t$$



- Goal: Learn h_{kt} to have a simple evolution in the state.

State Equation: Stochastic PDE over Graphs

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$$d\mathbf{x}_t = -c\mathbf{L}\mathbf{x}_t dt + \mathbf{S} d\boldsymbol{\beta}_t$$

- $\boldsymbol{\beta}_t \in \mathbb{R}^F$ is a standard Wiener process (a.k.a Brownian motion)
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- Goal: Learn $\mathbf{S}$ $\rightarrow$ has too many parameters

  $\Rightarrow$ We parameterize it by graph structure

$$\mathbf{S} = \mathbf{B}\mathrm{diag}(\boldsymbol{\alpha}) \longrightarrow d\mathbf{x}_t = -c\mathbf{L}\mathbf{x}_t dt + \mathbf{B}\mathrm{diag}(\boldsymbol{\alpha}) d\boldsymbol{\beta}_t$$

- $\mathbf{B}$ is the incidence matrix

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Problem Formulation

- Continuous-discrete state space model:

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- noise energy: $\mathbb{E}[\mathbf{v}_k \mathbf{v}_k^\top] = \sigma^2 \mathbf{I}$

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  $\Rightarrow$ given: temporal observations $\mathbf{y}_1, \ldots, \mathbf{y}_T$

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- Approach:

1. Discretization of state
2. Recovering the state
3. Estimating model parameters

Step1: Discretization of Continuous State

- Using the transition matrix of an LTI-SPDE:

$$\mathbf{x}_{t+\triangle t} = \tilde{\mathbf{L}}\mathbf{x}_t + \mathbf{q}_t$$

with:

$$\mathbf{q}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{Q})$$

$$\tilde{\mathbf{L}} = \exp(-c\triangle t\mathbf{L})$$

$$\mathbf{Q} = \int_0^{\triangle t} e^{-c\mathbf{L}(\triangle t - s)}\mathbf{B}\mathrm{diag}^2(\boldsymbol{\alpha})\mathbf{B}^\top e^{-c\mathbf{L}(\triangle t - s)} ds$$

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- With first-order Taylor approximation:

$$\tilde{\mathbf{L}} \approx \mathbf{I} - c\triangle t\mathbf{L}$$
$$\mathbf{Q} \approx \triangle t\mathbf{B}\mathrm{diag}^2(\boldsymbol{\alpha})\mathbf{B}^\top$$

Step2: Recovering the State

Kalman-Bucy filtering/smoothing

- Given model parameters $\boldsymbol{\alpha}$ and $\mathbf{H}_t$: state recovery via Kalman filtering
  - $\Rightarrow$ optimum Bayesian solution (recursive closed form $\rightarrow$ linear complexity in time)

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prediction step:

$$\mathbf{x}_t^{t-1} = \tilde{\mathbf{L}}\mathbf{x}_{t-1}^{t-1};$$

$$\mathbf{P}_t^{t-1} = \tilde{\mathbf{L}}\mathbf{P}_{t-1}^{t-1}\tilde{\mathbf{L}} + \mathbf{B}\mathrm{diag}^2(\boldsymbol{\alpha})\mathbf{B}^\top;$$

correction step:

$$\mathbf{K}_t = \mathbf{P}_t^{t-1}\mathbf{H}_t{}^\top(\mathbf{H}_t\mathbf{P}_t^{t-1}\mathbf{H}_t{}^\top + \sigma^2\mathbf{I})^{-1}$$

$$\mathbf{x}_t^t = \mathbf{x}_t^{t-1} + \mathbf{K}_t(\mathbf{y}_t - \mathbf{H}_t\mathbf{x}_t^{t-1});$$

$$\mathbf{P}_t^t = \mathbf{P}_t^{t-1} - \mathbf{K}_t\mathbf{P}_t^{t-1}\mathbf{K}_t^\top;$$

- $\mathbf{K}_t$ is the Kalman gain
- $(\mathbf{y}_t - \mathbf{H}_t\mathbf{x}_t^{t-1})$ is the prediction error
- $\mathbf{P}_t$ is the covariance matrix of the state at time t

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Step3: Estimating Model Parameters

a maximum a priori approach

- Having the state via Kalman filter → likelihood on the observation
  - ⇒ can be even computed recursively!

$$\mathcal{L}_t(\boldsymbol{\alpha}, \mathbf{h}_t) = \mathcal{L}_{t-1}(\boldsymbol{\alpha}, \mathbf{h}_t) + \frac{1}{2}\log|\mathbf{S}_t| + \frac{1}{2}(\mathbf{y}_t - \mathbf{H}_t\mathbf{x}_t^{t-1})^\top \mathbf{S}_t^{-1}(\mathbf{y}_t - \mathbf{H}_t\mathbf{x}_t^{t-1})$$

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where: $\mathbf{S}_t = \mathbf{H}_t\mathbf{P}_t^{t-1}\mathbf{H}_t{}^\top + \sigma^2\mathbf{I}$

- error matters more when confidence is high!
- no l_2-norm risk anymore $\rightarrow$ covariance-based norm replaced
- gradient descent and inference is over!

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Experiments

- **Datasets:**
  - ⇒ Synthetic data: based on presented state space model (200 nodes, 10000 samples)
  - ⇒ Weather data: NOAA (109 nodes, ∼8500 samples)
  - ⇒ Traffic data: METR-LA (207 nodes, ∼28000 samples)

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- **Tasks:**
  - ⇒ Interpolation: randomly removing data over nodes at each time
  - ⇒ Extrapolation: forecasting time series, using state equation

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Table: Interpolation task performance for both synthetic and weather temperature dataset. The experiments are performed based on different portions of unobserved data.

rNMSE	Synthetic			Weather			Traffic		
	10%	20%	30%	10%	20%	30%	10%	20%	30%
LMS	0.40	0.46	0.46	0.42	0.43	0.49	0.41	0.45	0.48
StarGP	0.31	0.31	0.36	0.25	0.24	0.31	0.21	0.25	0.29
G-SPDE	**0.12**	**0.14**	**0.16**	**0.13**	**0.14**	**0.17**	**0.16**	**0.15**	**0.27**
No-$\boldsymbol{\alpha}$	0.24	0.27	0.30	0.23	0.28	0.31	0.27	0.33	0.37
Fixed-$\boldsymbol{\alpha}$	0.24	0.26	0.29	0.22	0.26	0.29	0.24	0.27	0.36
Learn-$\mathbf{S}$	0.23	0.20	0.20	0.21	0.21	0.27	0.19	0.22	0.34

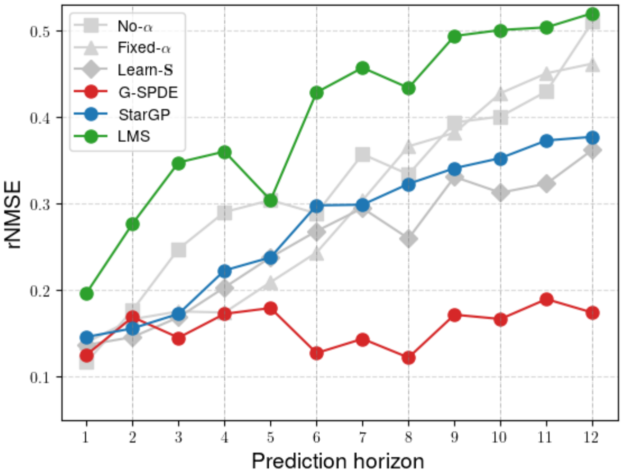

Figure. Traffic forecasting performance in rNMSE for proposed models with different prediction horizons.

Thanks for your attention!

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