Cyclic Misspecified Cramer-Rao Bound for Periodic Parameter Estimation

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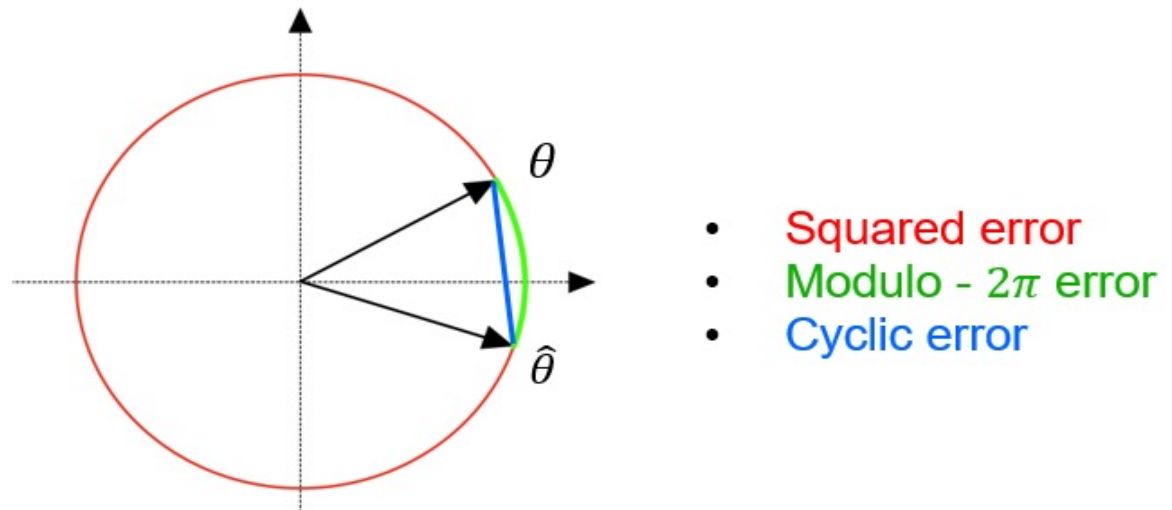
Introduction

- In many practical parameter estimation problems, the observation model is periodic w.r.t. the unknown parameters, thus only cyclic errors are relevant.
- Existing cyclic performance bounds do not account for model misspecification.
- The misspecified Cramér-Rao bound (MCRB) provides a lower bound on the MSE for estimation problems under model misspecification.
- The MCRB is not a valid lower bound for periodic problems.
- In this work, we close this gap by developing the cyclic MCRB.

Notations

- $\mathbf{x} \in \Omega_{\mathbf{x}}$ - Observation vector, $\Omega_{\mathbf{x}}$ - observation space.
- $\upsilon, \theta \in [-\pi, \pi) \triangleq \Theta$ - Deterministic unknown parameters.
- $p_{\mathbf{x}}(\mathbf{x};\upsilon)$ - True PDF of $\mathbf{x}$ and is 2π-periodic w.r.t. υ.
- $f_{\mathbf{x}}(\mathbf{x};\theta)$ - Assumed PDF of $\mathbf{x}$ and is 2π-periodic w.r.t. θ.
- $\hat{\theta}: \Omega_{\mathbf{x}} \to \Theta$ – Estimator based on $\mathbf{x}$ and the assumed PDF $f_{\mathbf{x}}(\mathbf{x};\theta)$.
- $\mathbb{E}_p[\cdot]$ - Expectation w.r.t. the true PDF $p_{\mathbf{x}}(\mathbf{x};\upsilon)$.
- The misspecified ML (MML) estimator: $\hat{\theta}_{MML} = \arg\max_{\theta \in \Theta} \log f_{\mathbf{x}}(\mathbf{x};\theta)$.
- The **pseudo-true** parameter $\theta_* \triangleq \arg\max_{\theta \in \Theta} \mathbb{E}_p[\log f_{\mathbf{x}}(\mathbf{x};\theta)]$, is the point that minimizes the Kullback-Leibler divergence (KLD) between the true and the assumed PDFs.
- ***Assumption:*** The pseudo-true parameter, θ_*, is unique.

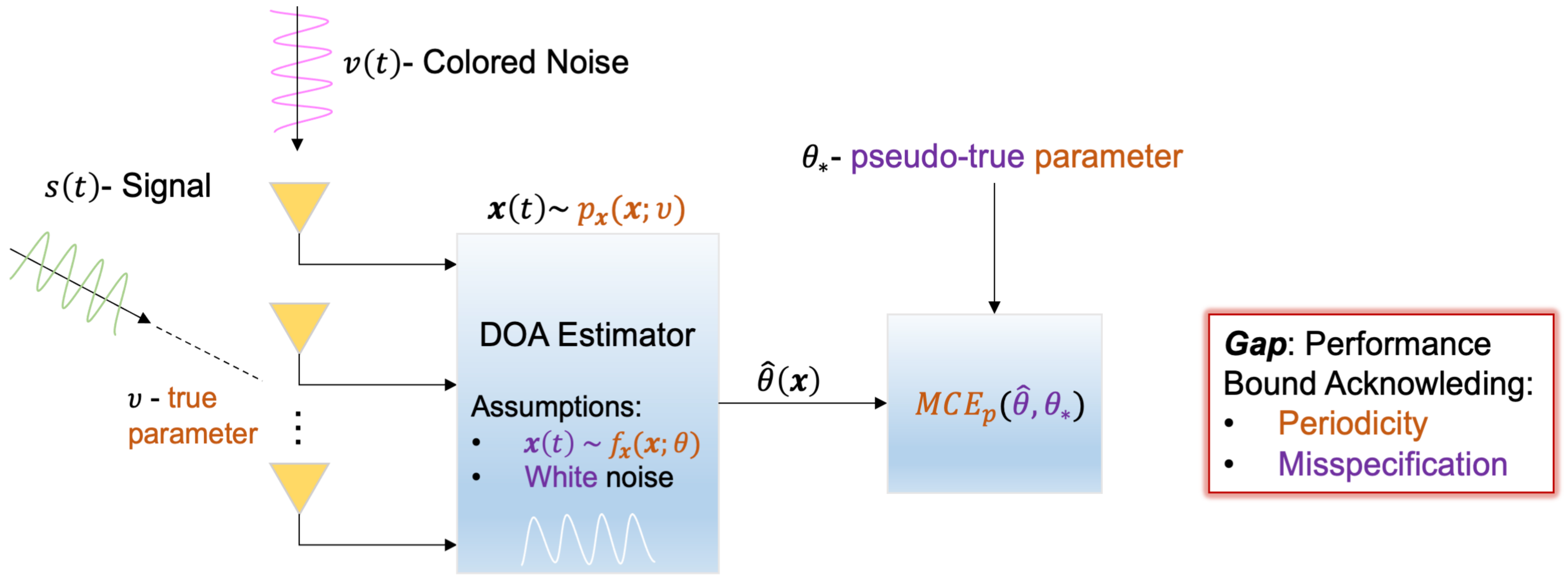


Cyclic Cost Function

- The conventional MSE criterion is inappropriate.
- No uniformly mean-unbiased estimator exists.
- The MCRB may not be valid at low SNRs.
- The mean-cyclic error (MCE) of the estimator $\hat{\theta}$ evaluated at $\theta \in \Theta$ is:

$$MCE_p(\hat{\theta},\theta) \triangleq \mathbb{E}_p[2(1-\cos(\hat{\theta}-\theta))] = \mathbb{E}_p\left[(\cos\hat{\theta}-\cos\theta)^2 + (\sin\hat{\theta}-\sin\theta)^2\right].$$

Motivation



Misspecified Cyclic Unbiasedness

Definition: Let the assumed PDF $f_{\mathbf{x}}(\mathbf{x};\theta)$ satisfy regularity conditions. Then, $\hat{\theta}$ is a misspecified cyclic-unbiased estimator of the pseudo-true parameter θ_* if

$$\mathbb{E}_p[\sin(\hat{\theta}-\theta_*)] = 0,$$
$$\mathbb{E}_p[\cos(\hat{\theta}-\theta_*)] > 0.$$

Theorem: The unbiasedness definition is the Lehmann unbiasedness definition w.r.t. the cyclic error cost function.

$\Rightarrow$ An estimator is Lehmann-unbiased if, on average, it is "closer" to the pseudo-true parameter, θ_*, than any other value in the parameter space, $\eta \in \Theta$.

Cyclic MCRB

Theorem: Let the assumed PDF $f_{\mathbf{x}}(\mathbf{x};\theta)$ satisfy regularity conditions. Let $\hat{\theta}$ be a cyclic-unbiased estimator under the assumed PDF $f_{\mathbf{x}}(\mathbf{x};\theta)$. Then,

$$MCE_p(\hat{\theta},\theta_*) \geq 2 - \frac{2}{(LB_p(\theta_*)+1)^{0.5}} \triangleq LB_p^{cyc}(\theta_*),$$

where LB_p^{cyc} is the cyclic MCRB, and LB_p is the MCRB, and is given by:

$$LB_p(\theta_*) \triangleq \left(A_p^f(\theta_*)\right)^{-1} B_p^f(\theta_*) \left(A_p^f(\theta_*)\right)^{-1},$$

where $B_p^f(\theta_*) \triangleq \mathbb{E}_p\left[\left(\frac{\partial \log f_{\mathbf{x}}(\mathbf{x};\theta)}{\partial \theta}|_{\theta=\theta_*}\right)^2\right]$, and $A_p^f(\theta_*) \triangleq \mathbb{E}_p\left[\frac{\partial^2 \log f_{\mathbf{x}}(\mathbf{x};\theta)}{\partial \theta^2}|_{\theta=\theta_*}\right] \neq 0.$

Properties:

1) In the case of a correctly specified model, the cyclic MCRB coincides with the cyclic CRB.
2) The cyclic MCRB and the MCRB satisfy: $LB_p(\theta_*) \geq LB_p^{cyc}(\theta_*)$.
3) Unlike the conventional MCRB, which neglects the inherent periodicity present in directional quantities, the cyclic MCRB provides a valid bound for periodic parameter estimation.

Example: Seismic Azimuth Estimation

$$\mathbf{x}(t_n) = s(t_n)\mathbf{a}(\upsilon) + \mathbf{v}(t_n), \quad n = 0, \ldots, N-1.$$

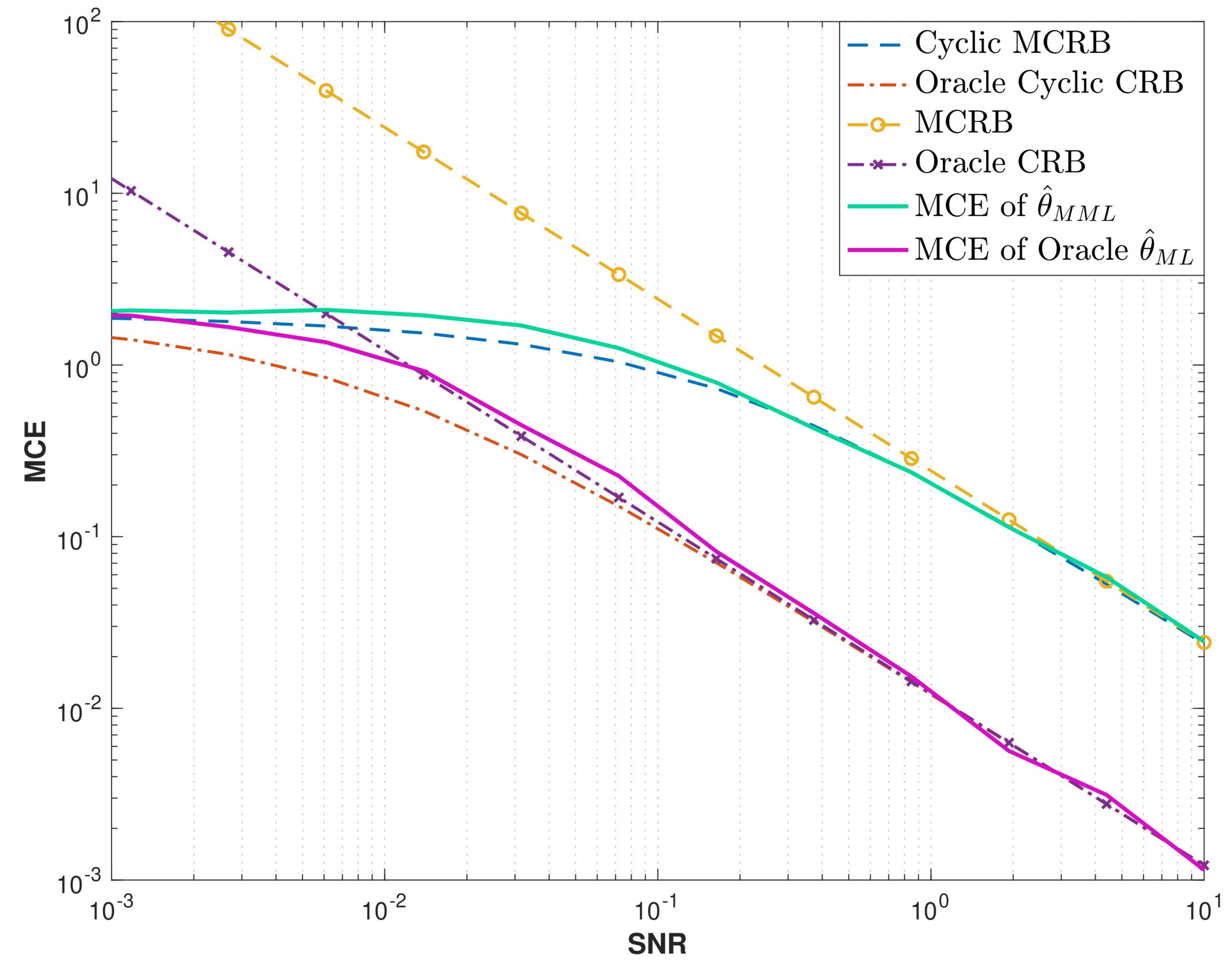
- The steering vector of the k-th sensor is defined as: $[\mathbf{a}(\upsilon)]_k = \exp\left\{-j\frac{2\pi}{\lambda}\left(z_{k_x}\sin(\upsilon) + z_{k_y}\cos(\upsilon)\right)\right\}$.
- The signal $\{s(t_n)\}_{n=0}^{N-1}$ and the wavelength λ are known.
- The *true* additive noise, $\mathbf{v}(t_n)$, is a **colored** complex Gaussian, i.e., $\mathbf{v}(t_n) \sim \mathcal{CN}(\mathbf{0},\mathbf{\Sigma_v})$, where $\mathbf{\Sigma_v}$ is a full-rank matrix.
- The *true* PDF of $\mathbf{x}$ is given by:

$$p_{\mathbf{x}}(\mathbf{x};\upsilon) = \mathcal{CN}\left(\sum_{n=0}^{N-1} s(t_n)\mathbf{a}(\upsilon), \mathbf{I}_N \otimes \mathbf{\Sigma_v}\right).$$

- The *assumed* additive noise, $\mathbf{w}(t_n)$, is a **white** complex Gaussian, i.e., $\mathbf{w}(t_n) \sim \mathcal{CN}(\mathbf{0},\sigma_v^2 \mathbf{I}_K)$, where σ_v^2 is the noise variance.
- The *assumed* PDF of $\mathbf{x}$ is given by:

$$f_{\mathbf{x}}(\mathbf{x};\theta) = \mathcal{CN}\left(\sum_{n=0}^{N-1} s(t_n)\mathbf{a}(\theta), \sigma_v^2\mathbf{I}_{KN}\right).$$

- The true covariance matrix is set to: $[\mathbf{\Sigma_v}]_{i,j} = \sigma_v^2(1-|i-j|/K), \quad i,j = 1,\ldots,K.$
- We show that the pseudo-true parameter in this case is $\theta_* = \upsilon$.



	Valid	Periodic	Misspecified
Cyclic MCRB	✓	✓	✓
MCRB	✗	✓	✓
Oracle CRB	✗	✗	✗
Oracle Cyclic CRB	✓	✓	✗

Conclusions

- A new lower bound on the MCE under model misspecification was proposed.
- The concept of misspecified cyclic unbiasedness in the Lehmann sense was introduced.
- The cyclic MCRB provides a valid lower bound for periodic parameter estimation under model misspecification.
- The cyclic MCRB is always lower than or equal to the MCRB.
- The cyclic MCRB reduces to the cyclic CRB for perfectly specified models.

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