Covariance Matrix Recovery from One-Bit Data with Non-Zero Quantization Thresholds: Algorithm and Performance Analysis

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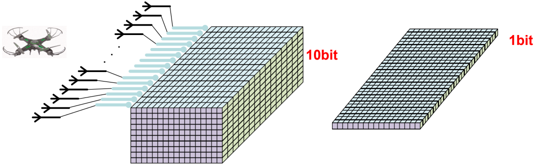
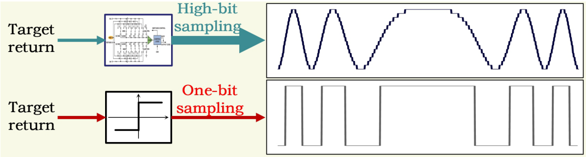
Background

One-bit sampling is especially well-suited for small platforms due to its reduced resource consumption and lower data volume.

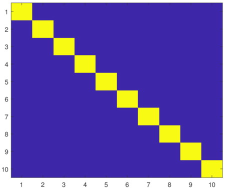




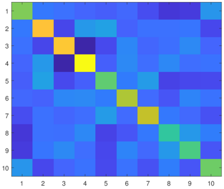
However, it poses great challenges for signal processing due to the absence of amplitude information.

Problem formulation

The covariance structure of different types of noise



(a) White noise



(b) Colored noise

Assumptions:

A1. The unquantized signal $\mathbf{y} \in \mathbb{R}^{M \times 1} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_{\mathbf{y}})$.

A2. We have N i.i.d. observations $\mathbf{x}(t) = \mathrm{sign}(\mathbf{y}(t))$, $t = 1, \cdots, N$.

The goal: To recover $\boldsymbol{\Sigma}_{\mathbf{y}}$ from the observations $\mathbf{x}(t)$.

Problem formulation

Conventional approach: Arcsine law(zero quantization thresholds):

$$\mathbf{\Sigma_x} = \frac{2}{\pi} \sin^{-1}(\mathbf{C_y}) \quad (1)$$

where $\mathbf{\Sigma}$ denotes covariance matrix and $\mathbf{C}$ denotes coherence matrix.
Limitation: We can only recover $\mathbf{C_y}$, but cannot recover the diagonal elements of $\mathbf{\Sigma_y}$.
Solution: Using non-zero quantization thresholds.

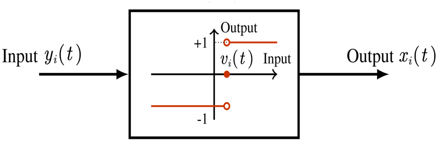


Figure: One-bit non-zero threshold quantization

Constant threshold approach [1]:
With a constant threshold ($\mathbf{v}(t) = v\mathbf{1}^{M\times 1}$), reconstruction can be accomplished based on the following probabilities:

$$p_i = \Pr\{x_i = +1\} = Q\left(\frac{v}{\sigma_i}\right), \quad i = 1, 2, \tag{2}$$

$$\begin{aligned} p_{12} &= \Pr\{x_1 = +1, x_2 = +1\} \\ &= \int_{\frac{v}{\sigma_1}}^{\infty} \int_{\frac{v}{\sigma_2}}^{\infty} f\left(y_1, y_2 \middle| \frac{\sigma_{12}}{\sigma_1 \sigma_2}\right) dy_1 dy_2, \end{aligned} \tag{3}$$

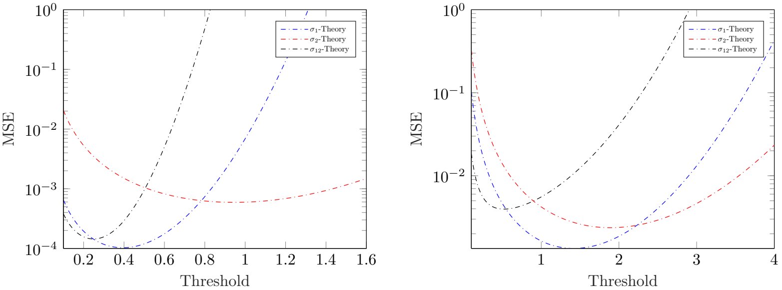
where $f(y_1, y_2|\rho)$ is the probability density function of bivariate Gaussian distribution with unit variances and correlation coefficient ρ, and

$$Q(a) = \int_a^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-t^2}{2}\right) dt.$$

[1] Liu et al, "One-bit autocorrelation estimation with nonzero thresholds," *Proc. IEEE Int. Conf. Acoust. Speech Signal Process.*, 2021.

Problem formulation

It is difficult to use a single threshold to deal with all the parameters.



(a) $[\sigma_1, \sigma_2, \sigma_{12}] = [0.25, 0.6, -0.08]$ (b) $[\sigma_1, \sigma_2, \sigma_{12}] = [0.9, 1.2, 0.2]$

Figure: Mean squared error versus threshold.

Problem formulation

Random threshold approach [2]**:** the random threshold method ($\mathbf{v}(t) \sim \mathcal{N}(v\mathbf{1}_M, \boldsymbol{\Sigma}_t)$) is equivalent to adding a zero-mean dithering signal to the constant sampling threshold $v\mathbf{1}_M$.

The reconstruction can be accomplished based on **modified arcsine law**[2].

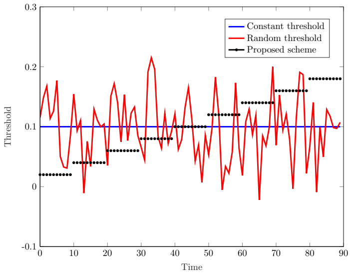


Figure: Thresholds for different quantization schemes.

[2] Eamaz et al, "Covariance recovery for one-bit sampled non-stationary signals with time-varying sampling thresholds," *IEEE Trans. Signal Process.*, 2022.

Problem formulation

Proposed scheme: $\mathbf{v}(t)$ is known and time-varying.

The reconstruction can be accomplished based on the following probabilities:

$$p_{i,t} = \Pr\{x_i(t) = +1\} = Q\left(\frac{v_i(t)}{\sigma_i}\right), \quad i = 1, 2, \tag{4}$$

$$\begin{aligned} p_{12,t} &= \Pr\{x_1(t) = +1, x_2(t) = +1\} \\ &= \int_{\frac{v_1(t)}{\sigma_1}}^{\infty} \int_{\frac{v_2(t)}{\sigma_2}}^{\infty} f\left(y_1, y_2 \middle| \frac{\sigma_{12}}{\sigma_1 \sigma_2}\right) dy_1 dy_2, \end{aligned} \tag{5}$$

The covariance recovery algorithm

General steps:

1. Set the time-varying, known sampling threshold.
2. Estimate diagonal entries by the following Newton's iteration:

$$\hat{\sigma}_i^{(u+1)} = \hat{\sigma}_i^{(u)} - \frac{\partial \mathcal{L}(\mathbf{x}_i; \sigma_i)}{\partial \sigma_i} \Big/ \frac{\partial^2 \mathcal{L}(\mathbf{x}_i; \sigma_i)}{\partial \sigma_i^2} \bigg|_{\sigma_i = \hat{\sigma}_i^{(u)}}, \tag{6}$$

3. Estimate off-diagonal entry by the following Newton's iteration:

$$\hat{\sigma}_{12}^{(u+1)} = \hat{\sigma}_{12}^{(u)} - \frac{\partial^2 \mathcal{L}(\mathbf{X}; \tilde{\boldsymbol{\theta}})}{\partial \sigma_{12}} \Big/ \frac{\partial^2 \mathcal{L}(\mathbf{X}; \tilde{\boldsymbol{\theta}})}{\partial \sigma_{12}^2} \bigg|_{\sigma_{12} = \hat{\sigma}_{12}^{(u)}}. \tag{7}$$

where $\tilde{\boldsymbol{\theta}} = [\hat{\sigma}_1, \hat{\sigma}_2, \sigma_{12}]^T$

4. Seek the joint MLE of σ_1, σ_2, and σ_{12} by using the gradient descent approach:

$$\hat{\boldsymbol{\theta}}^{(u+1)} = \hat{\boldsymbol{\theta}}^{(u)} + \gamma^{(u)} \frac{\partial \mathcal{L}(\mathbf{X}; \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \bigg|_{\boldsymbol{\theta} = \hat{\boldsymbol{\theta}}^{(u)}}, \tag{8}$$

where $\gamma^{(u)}$ is the learning rate at the uth iteration.

Usefulness of Exact Threshold Values

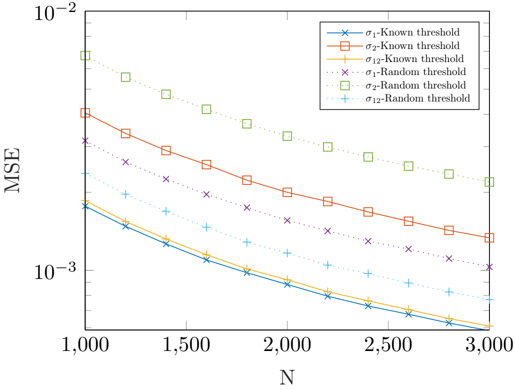


Figure: Mean squared error versus number of samples

Comparison of Mean Squared Errors

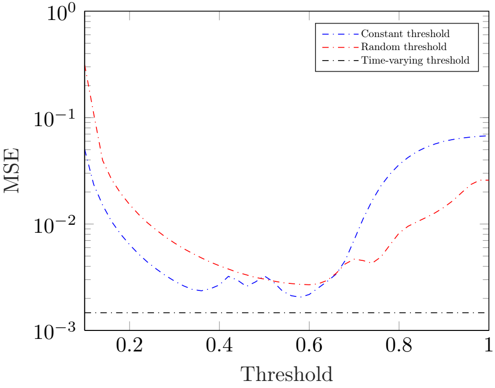


Figure: Mean squared error versus threshold

Performance Analysis

Theorem

The MSE matrix of the MLE can be approximated asymptotically ($N \to \infty$) by

$$\mathbf{Q} = \mathbf{F}^{-1}(\boldsymbol{\theta}_0).$$

Here, $\mathbf{F}(\boldsymbol{\theta})$ denotes the Fisher information matrix (FIM) defined as:

$$\mathbf{F}(\boldsymbol{\theta}) = \mathbb{E}\left[\frac{\partial \mathcal{L}(\mathbf{X};\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \frac{\partial \mathcal{L}(\mathbf{X};\boldsymbol{\theta})}{\partial \boldsymbol{\theta}^T}\right].$$

Furthermore, $\boldsymbol{\theta}_0 = [\sigma_1, \sigma_2, \sigma_{12}]^T$ represents the genuine parameter vector.

Performance Analysis

Since the samples are mutually independent, we can compute the Fisher information contributed by each sample separately.

$$\mathbf{F}(\boldsymbol{\theta}) = \sum_{t=1}^{N} \sum_{\mathbf{x}(t)\in\{\pm 1,\pm 1\}} o_t(\boldsymbol{\theta}) \left[\frac{\partial \mathcal{L}(\mathbf{x}(t))}{\partial \boldsymbol{\theta}} \frac{\partial \mathcal{L}(\mathbf{x}(t))}{\partial \boldsymbol{\theta}^T} \right]. \quad (9)$$

where $o_t(\boldsymbol{\theta})$ is the probability density function of the sample $\mathbf{x}(t)$.

Building upon Theorem, the asymptotic MSE for the individual components can be gleaned from the diagonal entries of $\mathbf{F}^{-1}(\boldsymbol{\theta}_0)$.

Theoretical Mean Squared Error

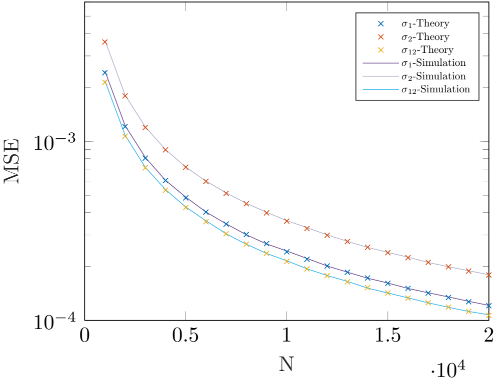
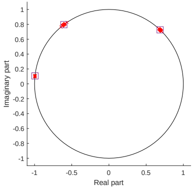


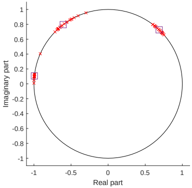
Figure: Mean squared error versus number of samples

Simulations

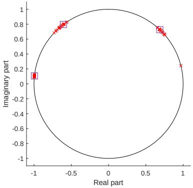
DOA Estimation of Coherent sources



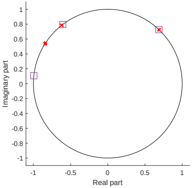
(a) Time-varying threshold



(b) Constant threshold



(c) Random threshold



(d) Zero threshold

Figure: Comparison of estimated DOA

Conclusion

Summary:

In this work, we present a novel approach based on a known and time-varying threshold to recover the the covariance matrix of the unquantized signal from one-bit quantized observations, Moreover, we study the performance of the proposed method.

Advantages:

1. It offers higher estimation accuracy.
2. It demonstrates improved robustness against parameter unevenness and high correlation coefficients.