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Fast Unsupervised Tensor Restoration via Low-rank Deconvolution

Background

$$\mathcal{J} \approx \sum_{r=1}^{R} \mu_r \mathbf{v}_r^{(1)} \circ \mathbf{v}_r^{(2)} \circ \ldots \circ \mathbf{v}_r^{(N)} \qquad j = 1 + \sum_{g=1}^{G} [(i_{c_g} - 1) \prod_{g'=1}^{G-1} I_{c_{g'}}]$$

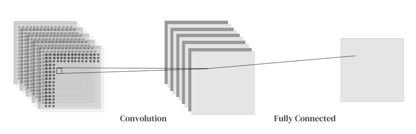
$$\mathbf{X}^{(n)} = [\mathbf{v}_1^{(n)}, \mathbf{v}_2^{(n)}, \ldots, \mathbf{v}_R^{(n)}] \qquad {}^{(n)}\mathbf{K} = \mathbf{X}^{(n)}(\mathbf{Q}^{(n)})^\top$$

$$[\![\mathbf{X}^{(1)}, \mathbf{X}^{(2)}, \ldots, \mathbf{X}^{(N)}]\!] = \sum_{r=1}^{R} \mathbf{v}_r^{(1)} \circ \mathbf{v}_r^{(2)} \circ \ldots \circ \mathbf{v}_r^{(N)} \qquad \mathbf{Q}^{(n)} = \mathbf{X}^{(N)} \odot \ldots \odot \mathbf{X}^{(n+1)} \odot \ldots \odot \mathbf{X}^{(n-1)} \odot \ldots \odot \mathbf{X}^{(1)}$$

$$\left({}^{(n)}\mathbf{K}\right)_{i_n, j} = \mathcal{K}_{i_1, i_2, \ldots, i_N} \qquad \mathrm{vec}\left({}^{(n)}\mathbf{K}\right) = [\mathbf{Q}^{(n)} \otimes \mathbf{I}_{I_n}] \mathrm{vec}(\mathbf{X}^{(n)})$$

Low-rank Deconvolution (LRD)

$$\mathcal{S} \approx \sum\nolimits_m \mathcal{D}_m * [\![\mathbf{X}_m^{(1)}, \ldots, \mathbf{X}_m^{(N)}]\!]$$



input : $\mathcal{S}$, $\{\mathcal{D}_m\}_{m=1}^{M}, \{\mathbf{X}_{0,m}^{(n)}\}_{n=1,m=1}^{N,M}, R > 0$
output: $\{\mathbf{X}_m^{(n)}\}_{n=1,m=1}^{N,M}$
`/* Initialize Kruskal Factors */`
$\{\mathbf{X}_m^{(n)}\} = \{\mathbf{X}_{0,m}^{(n)}\}$
`/* Main Loop, Eq.` (8) `*/`
while *not converged* **do**
  for $n = 1, \ldots, N$ **do**
    $\mathbf{X}_m^{(n)} = \arg\min \frac{1}{2} \left\| \sum_{m=1}^{M} \mathcal{D}_m * [\![\mathbf{X}_m^{(1)}, \ldots, \mathbf{X}_m^{(N)}]\!] - \mathcal{S} \right\|_2^2 + \Phi(\{\mathbf{X}_m^{(n)}\})$
  end
end

$$\underset{\{\mathbf{X}_m^{(n)}\}}{\arg\min} \frac{1}{2} \left\| \sum_{m=1}^{M} \mathcal{D}_m * [\![\mathbf{X}_m^{(1)}, \ldots, \mathbf{X}_m^{(N)}]\!] - \mathcal{S} \right\|_2^2 + \Phi(\{\hat{\mathbf{X}}_m^{(n)}\}),$$

$$\hat{\mathbf{W}}_m^{(n)} = \hat{\mathbf{D}}_m^{(n)} [\hat{\mathbf{Q}}_m^{(n)} \otimes \mathbf{I}_{I_n}],$$
$$\hat{\mathbf{W}}^{(n)} = [\hat{\mathbf{W}}_0^{(n)}, \hat{\mathbf{W}}_1^{(n)}, \ldots, \hat{\mathbf{W}}_M^{(n)}],$$
$$\hat{\mathbf{x}}^{(n)} = [(\hat{\mathbf{x}}_0^{(n)})^\top, (\hat{\mathbf{x}}_1^{(n)})^\top, \ldots, (\hat{\mathbf{x}}_M^{(n)})^\top]^\top,$$

$$\underset{\hat{\mathbf{x}}^{(n)}}{\arg\min} \frac{1}{2} \left\| \hat{\mathbf{W}}^{(n)} \hat{\mathbf{x}}^{(n)} - \hat{\mathbf{s}}^{(n)} \right\|_2^2 + \Phi(\{\hat{\mathbf{x}}_m^{(n)}\}).$$

with $\Psi(\{\mathbf{X}_m^{(n)}\}) = \sum_{m=1}^{M} \sum_{n=1}^{N} \frac{\alpha}{2} \left\| \mathbf{X}_m^{(n)} \right\|_2^2$

$$[(\hat{\mathbf{W}}^{(n)})^H \hat{\mathbf{W}}^{(n)} + \alpha \mathbf{I}_\beta] \hat{\mathbf{x}}^{(n)} = (\hat{\mathbf{W}}^{(n)})^H \hat{\mathbf{s}}^{(n)}$$

LRD With Differential Regularization

$$\underset{\{\mathbf{X}_m^{(n)}\}, \mathcal{U}}{\arg\min} \frac{1}{2} \|\mathcal{U} - \mathcal{S}\|_2^2 + \frac{\gamma}{2} \|\mathcal{U}\|_{TV}^2 + \frac{\zeta}{2} \|\mathcal{U}\|_{TI}^2 + \Psi(\{\mathbf{X}_m^{(n)}\}).$$

$$\text{subject to} \quad \mathcal{U} = \sum_{m=1}^{M} \mathcal{D}_m * [\![\mathbf{X}_m^{(1)}, \ldots, \mathbf{X}_m^{(N)}]\!]$$

Proposition 1 *The problem presented above with* $\Psi(\{\mathbf{X}_m^{(n)}\}) = \sum_{m=1}^{M} \sum_{n=1}^{N} \frac{\alpha}{2} \left\| \mathbf{X}_m^{(n)} \right\|_2^2$ *has a solution given by a linear expression in the DFT domain given by:*

$$[(\hat{\mathbf{W}}^{(n)})^H \hat{\mathbf{W}}^{(n)} + \gamma(\hat{\Theta}^{(n)})^H \hat{\Theta}^{(n)} + \zeta(\hat{\Omega}^{(n)})^H \hat{\Omega}^{(n)} + \alpha \mathbf{I}_\beta] \hat{\mathbf{x}}^{(n)} = (\hat{\mathbf{W}}^{(n)})^H \hat{\mathbf{s}}^{(n)},$$

where we have made use of $\hat{\mathbf{s}}^{(n)}$, $\hat{\mathbf{x}}^{(n)}$ *and* $\hat{\mathbf{W}}^{(n)}$ *defined in section 3. And defining:*

$$(\hat{\Theta}_i^{(n)})^T = 2\pi j \xi_i \oplus \hat{\mathbf{W}}^{(n)}$$
$$(\hat{\Omega}_i^{(n)})^T = (2\pi j \xi_i)^{-1} \oplus \hat{\mathbf{W}}^{(n)},$$
$$\hat{\Theta} = [\hat{\Theta}_0^{(n)}, \hat{\Theta}_1^{(n)}, \ldots, \hat{\Theta}_N^{(n)}],$$
$$\hat{\Omega} = [\hat{\Omega}_0^{(n)}, \hat{\Omega}_1^{(n)}, \ldots, \hat{\Omega}_N^{(n)}]$$

with ξ_i *being the vector of frequencies for the* i*-dimension, and* $\oplus$ *denoting element-wise product. The problem is equivalent to the LRD problem with:*

$$\Phi(\{\mathbf{x}_m^{(n)}\}) = \frac{\gamma}{2} \left\| (\hat{\Theta}^{(n)})^T \hat{\mathbf{x}}^{(n)} \right\|_2^2 + \frac{\zeta}{2} \left\| (\hat{\Omega}^{(n)})^T \hat{\mathbf{x}}^{(n)} \right\|_2^2 + \Psi(\{\mathbf{X}_m^{(n)}\}).$$

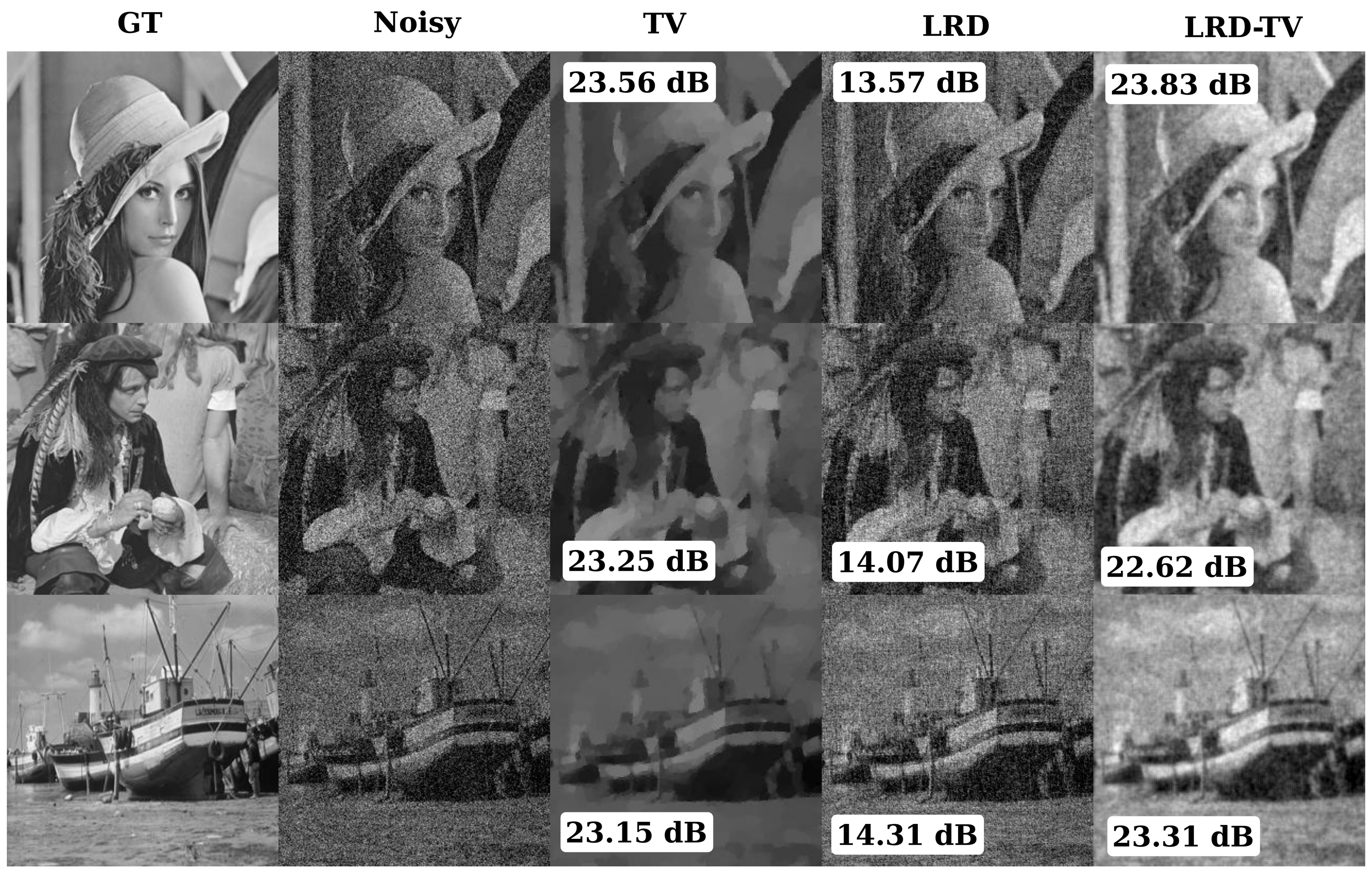
Detail Enhancement

$$\underset{\{\mathbf{X}_m^{(n)}\}, \{\mathcal{U}_m\}}{\arg\min} \frac{1}{2} \left\| \sum_{m=1}^{M} \mathcal{U}_m - \mathcal{S} \right\|_2^2 + \sum_{m=1}^{M} \frac{\gamma_m}{2} \|\mathcal{U}_m\|_{TV}^2 + \sum_{m=1}^{M} \frac{\zeta_m}{2} \|\mathcal{U}_m\|_{TI}^2 + \Psi(\{\mathbf{X}_m^{(n)}\}),$$

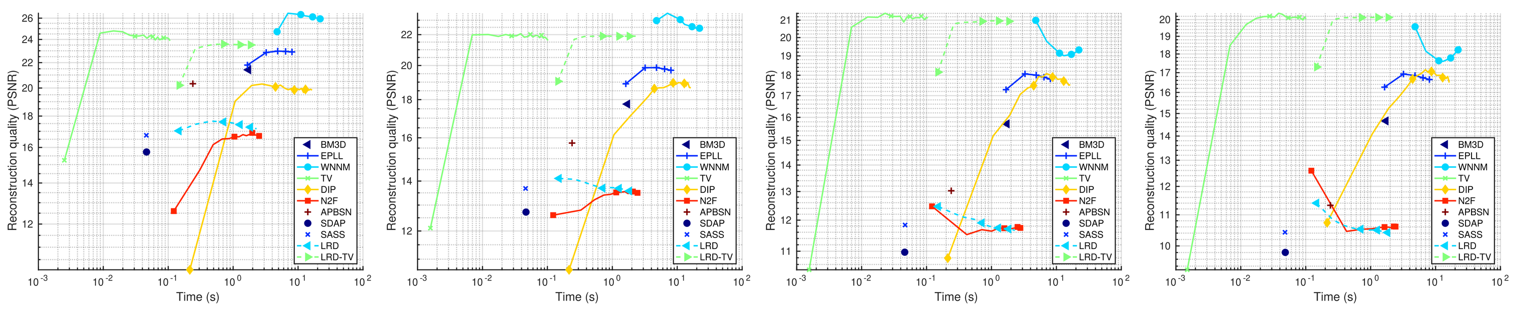
$$\text{subject to} \quad \mathcal{U}_m = \mathcal{D}_m * [\![\mathbf{X}_m^{(1)}, \ldots, \mathbf{X}_m^{(N)}]\!]$$

$$\tilde{\mathcal{U}} = \sum_{m=1}^{M} \delta_m \mathcal{D}_m * [\![\mathbf{X}_m^{(1)}, \ldots, \mathbf{X}_m^{(N)}]\!] + \mathcal{S}$$

Results



Qualitative evaluation on image denoising. We display ground truth (GT), noisy image and recovered images for three chosen methods including ours for images 9, 10 and 11.



Quality of reconstruction (PSNR) vs. execution time (s). We display overall results on the whole dataset for the four levels of input noise respectively. PSNR evolution as a function of time for a single image denoising for the chosen methods.

Images	1	2	3	4	5	6	7	8	9	10	11	12
Input PSNR = 15.36 dB / $\sigma = 30$												
BM3D	17.96	22.59	24.68	18.64	21.32	22.86	22.59	18.58	22.76	22.48	23.14	19.37
EPLL	20.76	24.36	24.67	**21.80**	24.13	23.84	22.38	20.98	24.34	22.72	24.33	20.52
TV	21.33	**27.81**	**26.47**	19.99	23.58	25.49	**24.48**	**21.39**	**26.66**	**25.34**	24.10	19.96
WNNM	21.52	**29.67**	**28.34**	**23.54**	**26.12**	**28.65**	**26.44**	**24.30**	**26.29**	**24.91**	**24.73**	**25.78**
DIP	**25.40**	24.78	23.53	18.91	19.16	20.30	20.08	13.46	18.15	18.11	19.09	16.66
N2F	20.19	17.72	19.59	17.78	15.78	16.79	17.82	15.65	13.55	10.74	18.22	16.62
AP-BSN	19.51	21.84	21.72	17.62	22.13	23.80	20.13	15.15	22.02	20.37	22.11	17.52
SDAP	15.17	17.96	10.75	16.78	16.52	13.00	15.66	16.68	17.66	17.65	14.26	16.65
SASS	17.82	15.99	18.66	15.76	18.00	19.60	17.46	11.04	17.80	16.98	18.10	13.74
LRD	16.86	15.69	18.53	16.55	17.95	18.61	18.10	13.09	18.48	17.43	18.60	16.11
LRD-TV (Ours)	**21.56**	25.57	23.00	20.56	**24.69**	**25.68**	23.74	20.33	25.80	24.73	**24.42**	**22.13**
Input PSNR = 12.18 dB / $\sigma = 50$												
BM3D	15.69	15.32	21.00	16.57	18.47	19.68	19.44	14.28	19.20	17.53	20.35	15.51
EPLL	19.17	21.50	20.68	18.52	20.48	19.81	18.75	18.52	20.30	20.02	19.93	18.82
TV	**20.41**	24.23	21.30	18.17	**23.41**	**23.80**	**22.25**	**18.86**	**23.48**	**22.69**	**21.32**	19.45
WNNM	**21.45**	**26.92**	**22.09**	**20.86**	21.26	**25.63**	**21.99**	**21.92**	22.41	21.73	18.00	**24.45**
DIP	12.62	20.82	19.56	17.38	20.88	22.10	20.13	15.97	19.72	19.27	19.35	16.79
N2F	17.83	13.71	15.17	12.76	13.19	13.28	14.04	12.99	12.12	8.52	14.45	13.99
AP-BSN	17.53	16.42	17.11	14.69	16.23	17.67	16.08	11.13	15.81	16.35	15.23	14.65
SDAP	12.47	14.24	8.37	11.95	13.34	11.50	13.25	12.93	13.80	15.96	12.16	12.69
SASS	15.78	13.63	14.83	13.49	14.19	15.46	14.01	8.95	13.83	14.55	13.08	12.39
LRD	14.08	13.21	14.49	13.68	13.75	14.91	14.35	9.00	13.61	14.10	14.17	12.98
LRD-TV (Ours)	19.60	**24.35**	**22.07**	**19.16**	**23.32**	23.75	**21.99**	17.87	**23.77**	**22.73**	**23.07**	**21.04**
Input PSNR = 10.49 dB / $\sigma = 70$												
BM3D	14.84	14.64	18.11	14.50	16.21	16.63	15.96	11.63	17.73	16.73	17.53	14.00
EPLL	17.32	19.37	17.92	16.82	18.97	18.46	16.38	16.90	18.04	17.30	18.62	17.65
TV	**19.12**	**24.87**	**20.16**	17.88	**22.91**	**23.37**	**20.82**	**19.72**	**22.86**	**21.66**	**20.43**	19.69
WNNM	15.02	21.82	15.51	**19.06**	18.65	**23.50**	**20.91**	**20.78**	19.11	19.46	16.49	**22.99**
DIP	13.19	19.74	19.39	17.08	19.47	21.16	18.41	14.16	18.31	17.52	16.87	15.40
N2F	16.47	12.13	14.03	11.11	12.28	10.90	12.05	10.76	10.96	7.35	11.76	10.88
AP-BSN	16.03	12.78	13.18	13.17	13.71	15.87	12.90	8.85	12.30	11.82	12.90	12.83
SDAP	10.36	13.37	6.94	10.58	11.15	10.50	10.13	11.39	11.16	14.44	11.22	10.41
SASS	15.08	11.48	12.01	12.22	12.49	14.17	11.61	7.66	11.32	10.94	11.68	11.39
LRD	12.53	11.02	12.91	11.92	12.25	12.79	12.13	7.96	11.81	11.62	12.01	11.00
LRD-TV (Ours)	**18.27**	**23.46**	**20.98**	**18.42**	**22.28**	23.13	20.58	17.43	**22.75**	**21.90**	**21.99**	**19.84**
Input PSNR = 9.49 dB / $\sigma = 90$												
BM3D	14.50	13.02	17.69	13.74	15.82	16.29	15.26	11.14	16.28	13.60	15.75	12.87
EPLL	17.01	18.16	16.18	15.46	17.83	16.61	15.26	16.00	16.80	16.68	17.44	16.32
TV	**18.67**	**22.92**	**20.04**	16.98	**22.62**	18.77	**20.77**	17.05	**21.85**	**21.33**	**20.58**	18.95
WNNM	14.26	19.23	15.11	**17.96**	18.16	**21.91**	**20.43**	**19.53**	19.06	17.74	15.68	**22.19**
DIP	10.32	17.00	15.74	16.21	20.74	21.01	17.34	13.17	18.25	16.75	16.40	15.50
N2F	15.95	10.76	12.69	10.03	10.76	9.74	11.12	9.81	9.52	6.19	10.05	10.33
AP-BSN	15.76	11.02	10.64	11.45	11.76	13.50	11.61	7.23	10.45	11.05	10.74	10.68
SDAP	9.21	11.76	5.80	9.19	10.28	9.09	9.65	10.10	9.34	14.08	10.02	9.12
SASS	15.07	10.11	9.86	10.51	10.81	12.12	10.44	6.56	9.72	10.19	10.01	9.73
LRD	12.55	9.90	10.95	10.88	10.60	11.97	10.70	7.03	10.11	10.09	9.92	9.96
LRD-TV (Ours)	**17.77**	**22.41**	**20.57**	**17.75**	**21.66**	**21.45**	19.79	**17.17**	**21.58**	**21.17**	**21.13**	**19.38**

Quantitative evaluation on image denoising. The table reports the PSNR in dB (higher is better) using ten state-of-the-art approaches and our method for the task of image denoising. Results are reported for different levels of input noise. We have marked in blue and red color the best and second-best achievers.

Conclusion

LRD is a framework for learning compressed representations for multi-dimensional data. Facilitating the inclusion of priors like TV, allowing for tensor restoration tasks at the same time.

Benefiting from its analytical formulation, the solution is very fast and does not require an extensive training stage.

Our results in image denoising and video enhancement (see paper) verify our claims.

Paper

arXiv