Estimating parameters in noisy low frequency exponentially damped sinusoids and exponentials

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Introduction

In [1] the model considered initially was

$$X_t = \mu + Ae^{-\gamma t}\cos(\omega t + \phi) + \varepsilon_t,\ t = 0, 1, \ldots, T-1 \tag{1}$$

where $\mu, A > 0, \gamma > 0, \omega$ and ϕ are unknown parameters, and $\{\varepsilon_t\}$ is some general 'noise' process, not necessarily Gaussian or white. Interest was in the estimation of these unknown parameters, and their asymptotic properties as $T \to \infty$. Since the amplitude $Ae^{-\gamma t}$ converges to 0 as $T \to \infty$, the Cramér-Rao lower bound does not converge to 0 as $T \to \infty$ and so the estimators are inconsistent. The model was reparametrized as

$$X_t = \mu + Ae^{-\gamma t/T}\cos(\omega t + \phi) + \varepsilon_t,\ t = 0, 1, \ldots, T-1, \tag{2}$$

as in [2] in order to avoid this problem. A review of estimation techniques was conducted and a generalization of [3] produced. Although the amplitude of the sinusoid does not converge to 0 as $T \to \infty$, the number of periods of the sinusoid is linear in T, and therefore diverges to ∞. In [4], a similar idea is used with model given by (1), but at the times $t = 0, 1/(T-1), 2/(T-1), \ldots, 1$, the number of periods of the sinusoid is fixed, and the stochastic properties of the noise process $\{\varepsilon_t\}$ thus become problematic.

In this paper, we propose the following model for the case of a damped sinusoid

$$X_t = \mu + Ae^{-\gamma t/T}\cos(at/T + \phi) + \varepsilon_t,\ t = 0, 1, \ldots, T-1 \tag{3}$$

for which there is a fixed number of sinusoidal periods. The same idea was used in [5], where limit theory was established for the least squares estimator of the frequency of a sinusoid, when the frequency was 'low'. We derive the asymptotic theory for the least squares estimators of the parameters. We then propose Fourier transform estimators of γ and a. A special case is that of $a = 0$, i.e. a purely exponential signal. The Fourier transform technique outperforms least squares from the computational point of view, and has very similar asymptotics. The technique is generalized to a broad class of nonlinear functions, using a more general class of transforms. Simulations are performed to evaluate the accuracy of the asymptotics in relatively small samples.

Least squares and the Gaussian CRLB

[5] examined (3) when $\gamma = 0$. The least squares procedure was defined and analyzed imposing only weak conditions on $\{\varepsilon_t\}$. In particular, Gaussianity and whiteness are not needed for the parameter estimators to satisfy a central limit theorem, which depends on $\{\varepsilon_t\}$ only through its spectral density $f(\omega)$ at 0 frequency. The derivation of the central limit theorem is complicated by the fact that (3) has *three* sinusoidal terms that 'interfere' with each other, at frequencies $-a/T, 0$ and a/T. In [6] it is shown that $T^{1/2}(\widehat{a}_T - a)$ is asymptotically normal with mean 0 and variance of the form

$$\frac{48\pi f(0)}{A^2}\left(\xi\cos^2\psi + \zeta\sin^2\psi\right),$$

where ξ and ζ depend only on a and $\psi = \phi + a/2$. Here we rewrite the model as

$$X_t = \nu + \alpha\left\{e^{-\gamma t/T}\cos(at/T) - c\right\} + \beta\left\{e^{-\gamma t/T}\sin(at/T) - s\right\} + \varepsilon_t,$$

$$\nu = \mu - \alpha c - \beta s, c + js = T^{-1}\sum_{t=0}^{T-1} e^{(ja-\gamma)t/T}.$$

We thus minimize with respect to ν, α, β and a,

$$S(\nu, \alpha, \beta, a, \gamma) = \sum_{t=0}^{T-1}\Big[X_t - \nu - \alpha\left\{e^{-\gamma t/T}\cos(at/T) - c\right\} - \beta\left\{e^{-\gamma t/T}\sin(at/T) - s\right\}\Big]^2. \tag{4}$$

Now for fixed a and γ, S is minimized with respect to ν, α and β when $\nu = \overline{X} = T^{-1}\sum_{t=0}^{T-1} X_t$ and

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \begin{bmatrix}D_{11} & D_{12}\\ D_{12} & D_{22}\end{bmatrix}^{-1}\begin{bmatrix}C_1\\ C_2\end{bmatrix}$$

where

$$\begin{bmatrix}C_1\\ C_2\end{bmatrix} = \begin{bmatrix}\sum_{t=0}^{T-1}\left(X_t - \overline{X}\right)e^{-\gamma t/T}\cos(at/T)\\ \sum_{t=0}^{T-1}\left(X_t - \overline{X}\right)e^{-\gamma t/T}\sin(at/T)\end{bmatrix},$$

$$\begin{bmatrix}D_{11}\\ D_{12}\\ D_{22}\end{bmatrix} = \begin{bmatrix}\sum_{t=0}^{T-1} e^{-2\gamma t/T}\cos^2(at/T) - Tc^2\\ \sum_{t=0}^{T-1} e^{-2\gamma t/T}\cos(at/T)\sin(at/T) - Tsc\\ \sum_{t=0}^{T-1} e^{-2\gamma t/T}\sin^2(at/T) - Ts^2\end{bmatrix}$$

The least squares procedure is then the same as maximizing

$$P(a, \gamma) = \begin{bmatrix}C_1 & C_2\end{bmatrix}\begin{bmatrix}D_{11} & D_{12}\\ D_{12} & D_{22}\end{bmatrix}^{-1}\begin{bmatrix}C_1\\ C_2\end{bmatrix}.$$

The elements of D may be asymptotically approximated. The (asymptotic) Cramér-Rao bounds under Gaussian assumptions are computed in the appendix of the paper. In fact, these are also the asymptotic variances in the central limit theorem even under non-Gaussian and colored noise assumptions. The fixed-frequency case has been discussed in [2, 7, 1].

Fourier coefficient technique

Let

$$Y_k = \sum_{t=0}^{T-1} X_t e^{-j2\pi kt/T}, U_k = \sum_{t=0}^{T-1}\varepsilon_t e^{-j2\pi kt/T}.$$

$$Y_k = T\mu\delta_{0k} + D\frac{1 - e^{-\gamma+ja}}{1 - e^{-(\gamma-ja+2\pi jk)/T}} + D^*\frac{1 - e^{-\gamma-ja}}{1 - e^{-(\gamma+ja+2\pi jk)/T}} + U_k,$$

where $D = Ae^{j\phi}/2$ and δ_{ij} is Kronecker's delta. Unlike the fixed frequency case, D^* is of the same order as D. As in [3], suppose that $a = 2\pi(n + \delta)$, where $\delta \in (-1/2, 1/2)$. Then, although n is unknown, it may be shown that, if $n > 0$,

$$\operatorname{argmax}_{1 \le k \le \lfloor (T-1)/2 \rfloor} |Y_k|^2 \to n,$$

a.s. as $T \to \infty$, and be used to estimate n. If $|\delta| = 1/2$, the limit points are the set $\{n-1, n, n+1\}$, but this will not matter, for the same reason as in [6]. Assume first that $a > 3\pi$. Then for $k = -1, 0, 1$ and $n \ge 2$,

$$Y_{n+k} = D\frac{1 - e^{-\gamma+2\pi j\delta}}{1 - e^{-(\gamma-2\pi j\delta+2\pi jk)/T}} + D^*\frac{1 - e^{-\gamma-2\pi j\delta}}{1 - e^{-(\gamma+2\pi j\delta+4\pi jk)/T}} + U_{n+k}.$$

As in [3], solving the equations

$$Y_{n+1} = D\frac{1 - e^{-\gamma+2\pi j\delta}}{1 - e^{-(\gamma-2\pi j\delta+2\pi j)/T}} + D^*\frac{1 - e^{-\gamma-2\pi j\delta}}{1 - e^{-(\gamma+2\pi j\delta+4\pi j)/T}}$$

$$Y_n = D\frac{1 - e^{-\gamma+2\pi j\delta}}{1 - e^{-(\gamma-2\pi j\delta)/T}} + D^*\frac{1 - e^{-\gamma-2\pi j\delta}}{1 - e^{-(\gamma+2\pi j\delta)/T}}$$

yields one set of estimators of D, γ and δ, since the equations above represent four (real) equations in four (real) unknowns. Solving

$$Y_{n-1} = D\frac{1 - e^{-\gamma+2\pi j\delta}}{1 - e^{-(\gamma-2\pi j\delta-2\pi j)/T}} + D^*\frac{1 - e^{-\gamma-2\pi j\delta}}{1 - e^{-(\gamma+2\pi j\delta-4\pi j)/T}}$$

$$Y_n = D\frac{1 - e^{-\gamma+2\pi j\delta}}{1 - e^{-(\gamma-2\pi j\delta)/T}} + D^*\frac{1 - e^{-\gamma-2\pi j\delta}}{1 - e^{-(\gamma+2\pi j\delta)/T}}$$

gives another. There appear to be no closed-form formulae for solving the equations, or choosing between the two sets of solutions, even if asymptotic versions of the equations are used. Moreover, when $a \le 3\pi$, Y_0 cannot be used, as it involves μ, and is also real. Thus Y_1 and Y_2 need to be used when $a < 5\pi$.

A special case: $a = 0$

When $a = \phi = 0$, we have

$$Y_k = T\mu\delta_{0k} + A\frac{1 - e^{-\gamma}}{1 - e^{-(\gamma+2\pi jk)/T}} + U_k.$$

We may thus estimate γ by solving

$$Y_1 = A\frac{1 - e^{-\gamma}}{1 - e^{-(\gamma+2\pi j)/T}},$$

which reduces to

$$\frac{\operatorname{Re}(Y_1)}{\operatorname{Im}(Y_1)} = \frac{\operatorname{Re}\left(\frac{1-e^{-(\gamma-j2\pi)/T}}{\left|1-e^{-(\gamma+j2\pi)/T}\right|^2}\right)}{\operatorname{Im}\left(\frac{1-e^{-(\gamma-j2\pi)/T}}{\left|1-e^{-(\gamma+j2\pi)/T}\right|^2}\right)} = \frac{1 - e^{-\gamma/T}\cos(2\pi/T)}{e^{-\gamma/T}\sin(2\pi/T)},$$

for which the solution is

$$\gamma = \widehat{\gamma}_T = T\log\left(\cos(2\pi/T) - \frac{\operatorname{Re}(Y_1)}{\operatorname{Im}(Y_1)}\sin(2\pi/T)\right) \sim -2\pi\operatorname{Re}(Y_1)/\operatorname{Im}(Y_1). \tag{5}$$

The estimator $\widehat{\gamma}_T$ is remarkably simple, and certainly much faster to compute than the nonlinear least squares estimator, found by minimizing with respect to μ, A and γ,

$$\sum_{t=0}^{T-1}\left\{X_t - \mu - Ae^{-\gamma t/T}\right\}^2,$$

or equivalently by maximizing with respect to γ

$$\frac{\left\{\sum_{t=0}^{T-1}\left(X_t - \overline{X}\right)e^{-\gamma t/T}\right\}^2}{\sum_{t=0}^{T-1} e^{-2\gamma t/T} - T^{-1}\left(\sum_{t=0}^{T-1} e^{-\gamma t/T}\right)^2}$$

Generalization

Suppose we wish to fit

$$X_t = \mu + \beta f(\gamma t/T) + \varepsilon_t, t = 0, 1, \ldots, T-1$$

where $\{\varepsilon_t\}$ is 'noise' and f is known. Let $\{g_k(x)\}$ be a family of functions whose domains are $[0,1]$, and put $Y_k = \sum_{t=0}^{T-1} X_t g_k(t/T)$. As long as $\{g_k(x)\}$ is suitably well-behaved,

$$\operatorname{var}\left\{T^{-1/2}\sum_{t=0}^{T-1}\varepsilon_t g_k(t/T)\right\} \to 2\pi f(0)\int_0^1 g_k^2(x)\,dx.$$

Thus, at least in probability as $T \to \infty$,

$$T^{-1}Y_k \to \mu\int_0^1 g_k(x)\,dx + \beta\int_0^1 g_k(x) f(\gamma x)\,dx = \mu G_k + \beta H_k(\gamma),$$

say. For fixed γ, we might thus estimate μ and β by solving the above equation for $k = 0, 1$, i.e. by finding zeros of

$$\kappa(\gamma) = (G_0 Y_1 - G_1 Y_0) H_2(\gamma) + (G_2 Y_0 - G_0 Y_2) H_1(\gamma) + (G_1 Y_2 - G_2 Y_1) H_0(\gamma). \tag{6}$$

For example, suppose $f(x) = e^{-x}$, $g_0(x) = 1$ and

$$g_k(x) = \begin{cases}\cos(ax) & ;\ k = 1\\ \sin(ax) & ;\ k = 2.\end{cases}$$

Then $G_0 = 1$,

$$G_k = \begin{cases}\sin a/a & ;\ k = 1\\ (1 - \cos a)/a & ;\ k = 2,\end{cases}$$

$$H_0(\gamma) = \left(1 - e^{-\gamma}\right)/\gamma$$

$$H_1(\gamma) = \left(\gamma - \gamma\cos a\, e^{-\gamma} + a\sin a\, e^{-\gamma}\right)/\left(a^2 + \gamma^2\right)$$

$$H_2(\gamma) = \left(a - a\cos a\, e^{-\gamma} - \gamma\sin a\, e^{-\gamma}\right)/\left(a^2 + \gamma^2\right).$$

When $a = 2n\pi$, n an integer, $G_k = \delta_{0k}$,

$$H_0(\gamma) = \left(1 - e^{-\gamma}\right)/\gamma$$

$$H_1(\gamma) = \gamma\left(1 - e^{-\gamma}\right)/\left(4n^2\pi^2 + \gamma^2\right)$$

$$H_2(\gamma) = 2n\pi\left(1 - e^{-\gamma}\right)/\left(4n^2\pi^2 + \gamma^2\right)$$

$$\kappa(\gamma) = (\gamma Y_1 - 2n\pi Y_2)\left(1 - e^{-\gamma}\right)/\left(4n^2\pi^2 + \gamma^2\right).$$

$\widehat{\gamma}_T$ is thus $2n\pi Y_2/Y_1$, agreeing with (5) when $n = 1$. Generally zeros of $\kappa(\gamma)$ must be found by an iterative procedure. In any case, $\widehat{\gamma}_T$ converges a.s. to γ, and $T^{1/2}(\widehat{\gamma}_T - \gamma)$ is asymptotically normal with mean 0. When $a = 2n\pi$, n an integer, the asymptotic variance is

$$\pi f(0)\frac{1 + \left\{\frac{H_2(\gamma)}{H_1(\gamma)}\right\}^2}{\left\{-\frac{d}{d\gamma}H_2(\gamma) + \frac{H_2(\gamma)}{H_1(\gamma)}\frac{d}{d\gamma}H_1(\gamma)\right\}^2}.$$

Simulations

Only a few results for the $a = 0$ case are reported. There were 5000 replications for each combination of parameters, and the noise was simulated Gaussian and white. Figure 1 shows that the theoretical and simulated, least squares and Fourier estimates are all in close agreement. The mean square errors initially decrease as γ increases, but then increase, the least squares estimates better at low and high values of γ. Figures 2 and 3 show that there is a threshold effect for fixed γ with decreasing SNR.

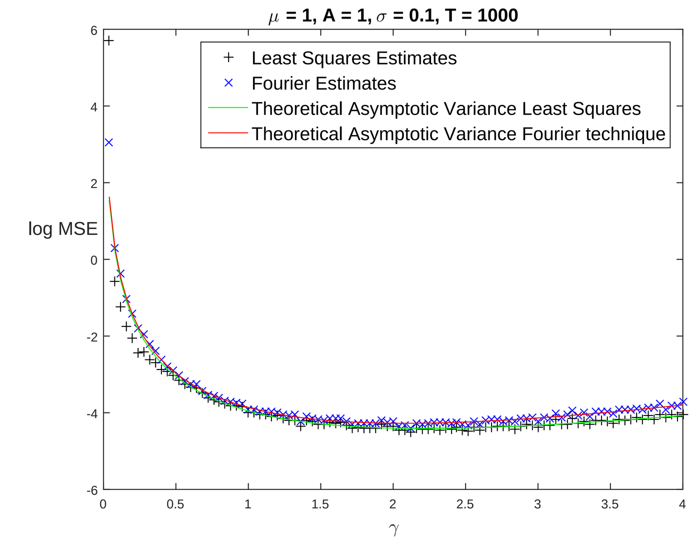

Fig 1. MSE for fixed σ as function of γ

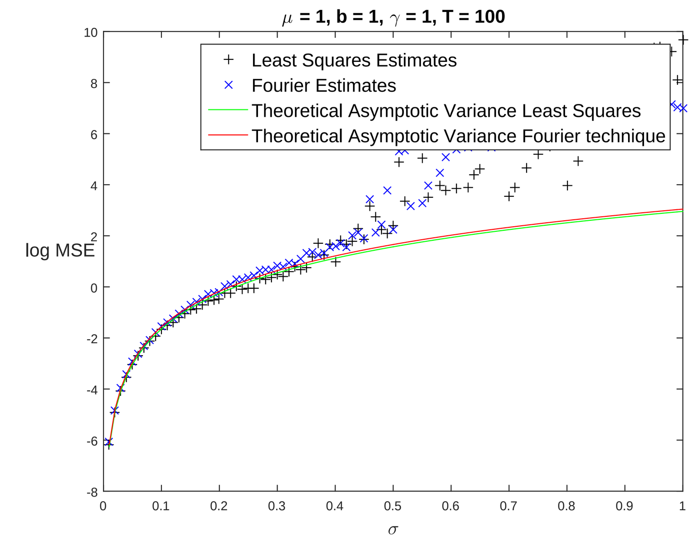

Fig 2. MSE for fixed γ as a function of σ

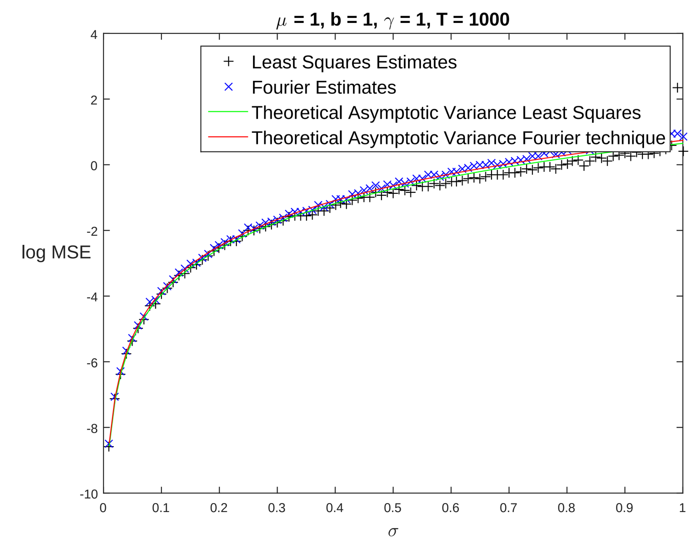

Fig 3. MSE for fixed γ as a function of σ

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