Conjugate gradient acceleration of non-linear smoothing filters
Iterated edge-preserving smoothing

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Outline:

- Bilateral (BF) filter
- Guided (GF) filter
- Preconditioned conjugate gradient iteration
- Numerical results

Announcement of numerical results:
For the test 1D signal of length 4730, the PCG acceleration provides the 9 times speedup for BF and 4 times speedup for GF filters.
Standard references for BF and GF filters


Bilateral filter (BF)

A discrete function \(x[j], j \in \{1, 2, \ldots, N\}\), is an input signal for the bilateral filter. The output signal \(y[i]\) is a weighted average of the signal values \(x[j]\):

\[
y[i] = \sum_j \frac{w_{ij}}{\sum_j w_{ij}} x[j].
\]

Every index \(i\) has a spatial position \(p_i\), and a spatial distance \(\|p_i - p_j\|\) is determined for all pairs \(i\) and \(j\).

The weights \(w_{ij}\) are defined by means of a guidance signal \(g[i]\):

\[
w_{ij} = \exp \left( -\frac{\|p_i - p_j\|^2}{2\sigma_d^2} \right) \exp \left( -\frac{(g[i] - g[j])^2}{2\sigma_r^2} \right),
\]

where \(\sigma_d\) and \(\sigma_r\) are the filter parameters. When \(g = x\), the bilateral filter is nonlinear and called self-guided. Numerical cost can be \(O(N)\).
Iterated bilateral filter

- The nonnegative weights $w_{ij}$ are the entries of the symmetric matrix $W$. Let us denote by $D$ the diagonal matrix with the positive diagonal entries $d_i = \sum_j w_{ij}$. Then BF is the vector transform $y = D^{-1}Wx$. The spectrum of $D^{-1}W$ is real, and the eigenvalues corresponding to highly oscillating eigenvectors lie near 0.

The symmetric nonnegative definite matrix $L = D - W \geq 0$ is called the Laplacian. The normalized Laplacian is $D^{-1}L = I - D^{-1}W$.

- The BF transform $y = D^{-1}Wx$ can be applied iteratively,
  1. changing the weights $w_{ij}$ at each iteration using the result of the previous iteration as a guidance signal $g$, or
  2. using the fixed weights, calculated from the initial signal as a guidance signal, for all iterations.

The former results in a nonlinear filter, the latter generates a linear filter, which may be faster to evaluate, since the BF weights are computed only once at the very beginning.
Guided filter (GF)

Algorithm 1 Guided Filter (GF)

**Input:** \(x, g, w, \epsilon\)

**Output:** \(y\)

\[
\begin{align*}
\text{mean}_g &= f_{\text{mean}}(g, w) \\
\text{mean}_x &= f_{\text{mean}}(x, w) \\
\text{corr}_g &= f_{\text{mean}}(g \ast g, w) \\
\text{corr}_{gx} &= f_{\text{mean}}(g \ast x, w) \\
\text{var}_g &= \text{corr}_g - \text{mean}_g \ast \text{mean}_g \\
\text{cov}_{gx} &= \text{corr}_{gx} - \text{mean}_g \ast \text{mean}_x \\
a &= \text{cov}_{gx} / (\text{var}_g + \epsilon) \\
b &= \text{mean}_x - a \ast \text{mean}_g \\
\text{mean}_a &= f_{\text{mean}}(a, w) \\
\text{mean}_b &= f_{\text{mean}}(b, w) \\
y &= \text{mean}_a \ast g + \text{mean}_b
\end{align*}
\]
Guided filter (GF)

$f_{\text{mean}}(\cdot,w)$ is a mean filter with the window width $w$. The constant $\epsilon$ determines the smoothness degree: the larger $\epsilon$ the larger smoothing effect. The dot preceded operations $\cdot \ast$ and $\cdot /$ denote the componentwise multiplication and division. Numerical complexity of the GF algorithm can be $O(N)$.

GF is $y = Wx$, where the entries of the symmetric matrix $W(g)$ are

$$W_{ij}(g) = \frac{1}{|\omega|^2} \sum_{k: (i,j) \in \omega_k} \left( 1 + \frac{(g_i - \mu_k)(g_j - \mu_k)}{\sigma_k^2 + \epsilon} \right).$$

The windows $\omega_k$ of width $w$ around all $k$ have the number of pixels $|\omega|$. The values $\mu_k$ and $\sigma_k^2$ are the mean and variance of $g$ over $\omega_k$. Since $d_i = \sum_j w_{ij} = 1$, the graph Laplacian matrix is automatically normalized, i.e. $L = I - W$. The eigenvalues of $L(g)$ are real nonnegative with the low frequencies accumulated near 0 and high frequencies near 1. Similar to BF, the guided filter can be applied iteratively.
Preconditioned Conjugate Gradient acceleration of a smoothing filter

**Algorithm** \( \text{PCG}(k_{\max}) \) with \( l_{\max} \) restarts

**Input:** \( x_0, k_{\max}, l_{\max} \)
**Output:** \( x \)

\[
x = x_0
\]

**for** \( l = 1, \ldots, l_{\max} \) **do**

\[
r = W(x)x - D(x)x
\]

**for** \( k = 1, \ldots, k_{\max} - 1 \) **do**

\[
s = D^{-1}(x)r; \quad \gamma = s^T r
\]

**if** \( k = 1 \) **then** \( p = s \) **else** \( \beta = \gamma/\gamma_{old}; \ p = s + \beta p \) **endif**

\[
q = D(x)p - W(x)p; \quad \alpha = \gamma/(p^T q)
\]

\[
x = x + \alpha p; \quad r = r - \alpha q; \quad \gamma_{old} = \gamma
\]

**endfor**

**endfor**
noisy = clean + randn(size(clean))*0.1

PSNR = 20.0494, SNR = 13.0753
Bilateral filter: 390 iter. vs. $7 \times PCG(6)$

![Graph showing clean signal and signal error comparison]

- Clean signal
- Error BF: SNR = 25.9119
- Error CG: SNR = 25.7187

Comparing the bilateral filter (BF) and PCG methods, the bilateral filter shows a slightly better PSNR (32.8859) compared to PCG (32.6927). However, the error signal for the bilateral filter has a higher SNR (25.9119) compared to PCG (25.7187).
Guided filter: 66 iterations vs. $5 \times PCG(3)$
Krylov subspace acceleration of iterated smoothing filters

Conclusion:
- We present smoothing filters in the guided form.
- We can apply the preconditioned conjugate gradient method, with the zero right-hand side and restarts, to iterated smoothing filters in the self-guided form.
- The PCG acceleration may give dramatical speedup without quality degradation.

Numerical results in 1D:
For the test 1D signal of length 4730, the PCG acceleration provides the 9 times speedup for BF and 4 times speedup for GF.

Future work:
- Evaluation for 2D imaging.
- Total Variation Filter.
Related publications by the authors

