Reconstructing Non-point Sources of Diffusion Fields From Sensor Measurements

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Outline

1. Introduction
   - PDE-driven Inverse Problems
2. Source Reconstruction Framework
   - Point Sources
   - Line and Polygonal Sources
3. Simulation Results
4. Conclusions
We consider *physics-driven* Inverse Problems

**Traditional Sampling Set-up:**

\[ f(t) \in \mathcal{V} \quad \rightarrow \quad \varphi(t) \quad \rightarrow \quad y_m \]

- The signal \( f(t) \) lies in a subspace, is sparse (e.g., CS), is parametric (e.g., FRI)
- The acquisition device given by the set-up or by design (e.g., random matrix)

**Sampling physical fields:**

sources: \( s(x, t) \in \mathcal{V} \quad \rightarrow \quad \text{PDE} \quad \rightarrow \quad f(x, t) \in \mathcal{W} \quad \rightarrow \quad \varphi(x, t) \quad \rightarrow \quad y_{m,n} \)

- No assumption on the field but on the sources,
- The acquisition device performs only temporal filtering, *no spatial filtering*
**Inverse Problems in Physics: Diffusion**

**Diffusion**

Stochastic movement of a collection of particles from regions of high concentration to regions of lower concentration (until an equilibrium is established).

Sensor networks measure:
- Leakages in/from factories,
- Temperature in server rooms,
- Nuclear fallouts (Fukushima).

The field \( u(x, t) \) induced by a source distribution \( f(x, t) \) satisfies:

\[
\frac{\partial}{\partial t} u(x, t) - \mu \nabla^2 u(x, t) = f(x, t). \tag{1}
\]
**Inverse Problems in Physics: Diffusion**

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The field $u(x, t)$ induced by a source distribution $f(x, t)$ satisfies:

$$\frac{\partial}{\partial t} u(x, t) - \mu \nabla^2 u(x, t) = f(x, t).$$  \hspace{1cm} (1)
Inverse Problems in Physics: Wave

Wave

A disturbance that travels through a medium from one location to another (transferring energy).

Such fields arise in acoustics, electromagnetics, fluid dynamics and so on. Sensor networks measure:

- Bioelectric neural currents in neurons of cerebral cortex (EEG/MEG),
- Pressure waves from a speaker/acoustic source.

\[
\frac{1}{c^2} \frac{\partial^2}{\partial t^2} u(x, t) - \nabla^2 u(x, t) = f(x, t).
\]
**Other PDEs:** Laplace’s Equation, Advection-/Convection-Diffusion Equation, Helmholtz and many more. Given these (spatiotemporal) measurements we may wish to find:

- source of factory leakage, detect plume sources
- find hot/cold spots in server clusters
- predict nuclear fallout concentration elsewhere
- center of mass of active regions
- acoustic source localization

Sources can be **localized** or **non-localized** → Parameterize sources $f$. 

**Sensor Networks and Inverse Problems**
## Problem Formulation: Field Sources

<table>
<thead>
<tr>
<th>Instantaneous</th>
<th>Non-Instantaneous</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Point</strong></td>
<td><strong>Line</strong></td>
</tr>
<tr>
<td>$f(x, t) = \sum_{m=1}^{M} c_m \delta(x - \xi_m, t - \tau_m)$</td>
<td>$f(x, t) = \sum_{m=1}^{M} c_m e^{\alpha_m(t - \tau_m)} \delta(x - \xi_m) H(t - \tau_m)$</td>
</tr>
<tr>
<td>$f(x, t) = c_L(x) \delta(t - \tau)$</td>
<td>$f(x, t) = c_L(x) e^{\alpha(t - \tau)} H(t - \tau)$</td>
</tr>
</tbody>
</table>

Where,

- $L(x) \in \Omega$ describes a line with endpoints $\{\xi_1, \xi_2\}$.
- $F(x) \in \Omega$ describes a convex polygon with vertices $\{\xi_1, \xi_2, \ldots, \xi_M\}$.
- $\alpha_m, c_m, \xi_m$ and $\tau_m$ is the release rate, intensity, location and activation time of $m$-th source.
**Problem Formulation: Field PDE Model**

Let $u(x, t)$ denote the field induced by a source distribution $f(x, t)$ then a physics-driven system, in general, has the Green’s function solution:

$$u(x, t) = (f \ast g)(x, t) = \int_{x' \in \mathbb{R}^2} \int_{t' \in \mathbb{R}} g(x', t') f(x - x', t - t') \, dt' \, dx' \quad (2)$$

where $g(x, t)$ is the Green function of the field.

For e.g.,

- **2D diffusion field:** \( \frac{\partial}{\partial t} u(x, t) - \mu \nabla^2 u(x, t) = f(x, t) \), has

  $$g(x, t) = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{\|x\|^2}{4\mu t}} H(t),$$

  where $H(t)$ is the step function.
**Problem Formulation: Field Measurements**

**Aim**

Estimate $f(x, t)$ from spatiotemporal samples $\{\varphi_{n,l} = u(x_n, t_l)\}_{n,l}$ for $n = 1, \ldots, N$ and $l = 0, \ldots, L$, of the measured field.
Recall that

$$u(x, t) = \int_{x' \in \mathbb{R}^2} \int_{t' \in \mathbb{R}} g(x', t') f(x - x', t - t') \, dt' \, dx'$$

$$= \langle f(x', t'), g(x - x', t - t') \rangle_{x', t'}.$$ 

Mathematically the spatiotemporal sample $\varphi_{n,l}$ is

$$\varphi_{n,l} = u(x_n, t_l)$$

$$= \langle f(x, t), g(x_n - x, t_l - t) \rangle_{x, t}$$ \hspace{1cm} (3)
Consider a weighted-sum of the samples \( \{ \varphi_{n,l} \}_{n,l} \):

\[
\sum_{n=1}^{N} \sum_{l=0}^{L} w_{n,l} \varphi_{n,l} = \sum_{n=1}^{N} \sum_{l=0}^{L} w_{n,l} \langle f(x, t), g(x_n - x, t_l - t) \rangle_{x,t}
\]

\[
= \left\langle f(x, t), \sum_{n=1}^{N} \sum_{l=0}^{L} w_{n,l} g(x_n - x, t_l - t) \right\rangle,
\]

where \( w_{n,l} \in \mathbb{C} \) are some arbitrary weights (to be determined).
We wish to find \( f(x, t) \):

- For our source types, can we choose functions \( \Psi_k(x) \) and \( \Gamma(t) \) that makes this problem tractable? — YES!

Let these (new) generalized measurements be

\[ R(k) = \sum_{n=1}^{N} \sum_{l=0}^{L} w_{n,l} \varphi_{n,l} = \langle f(x, t), \psi_k(x) \Gamma(t) \rangle \]

\[ = \int_{\Omega} \int_{t \in [0, T]} \psi_k(x) \Gamma(t) f(x, t) dt dV, \]

where \( \psi_k(x) \) for \( k \in \mathbb{Z}^d, \ d = \{1, 2\} \), and \( \Gamma(t) \) a family of properly chosen spatial and temporal sensing functions, respectively. Proper choice \( \Rightarrow \) solvability & stability of new problem.

- As an example, take the instantaneous source distribution

\[ f(x, t) = \sum_{m=1}^{M} c_m \delta(x - \xi_m, t - \tau_m), \]

then:

\[ R(k) = \sum_{m=1}^{M} c_m \psi_k(\xi_m) \Gamma(\tau_m). \]
Choice of Sensing Functions: 2D Case

For $x \in \mathbb{R}^2$, we may choose

- $\Gamma(t) = e^{-jt/T}$, and
- $\Psi_k(x) = e^{-k(x_1+jx_2)}$, for $k = 0, 1, \ldots, K$.

Then,

$$\mathcal{R}(k) = \sum_{m=1}^{M} c_m e^{-j\tau_m/T} e^{-k(\xi_1,m+j\xi_2,m)}$$

$$= \sum_{m=1}^{M} c'_m v_m^k.$$

Can be solved to jointly recover $c'_m = c_m e^{-j\tau_m/T}$ and $v_m = e^{-k(\xi_1,m+j\xi_2,m)}$ using Prony’s method for $m = 1, \ldots, M$ providing $K \geq 2M - 1$. 
**Line Source**

**Instantaneous** case: \( f(x, t) = cL(x)\delta(t - \tau) \), thus \( R(k) \) reduces to:

\[
R(k) = \int_{\Omega} \int_{t} \Psi_k(x)\Gamma(t)f(x, t)dt \, dV
\]

\[= c\Gamma(\tau) \int_{\Omega} \Psi_k(x)L(x) \, dV \]

\[= c\Gamma(\tau) \int_{L(x)} \Psi_k(x) \, dS \]

\[= \frac{1}{k} c\ell(\xi_1, \xi_2)\Gamma(\tau) \sum_{m=1}^{2} (-1)^m \Psi_k(\xi_m) \]
Line source

From \( \mathcal{R}(k) = \frac{1}{k} c \ell(\xi_1, \xi_2) \Gamma(\tau) \sum_{m=1}^{2} (-1)^m \Psi_k(\xi_m) \) and the usual choice for sensing functions \( \Gamma(t) = e^{-j t / T} \) and \( \Psi_k(x) = e^{-k(x_1 + j x_2)} \), then:

\[
\mathcal{R}'(k) \triangleq k\mathcal{R}(k) = c \ell(\xi_1, \xi_2) \Gamma(\tau) \sum_{m=1}^{2} (-1)^m \Psi_k(\xi_m) \\
= c \ell(\xi_1, \xi_2) e^{-j \tau / T} \sum_{m=1}^{2} (-1)^m e^{-k(\xi_1, m + j \xi_2, m)}
\]

Can again recover \( c, \tau \) and the endpoints \( \xi_1 \) and \( \xi_2 \) of the line source using Prony’s method (providing \( K \geq 3 \)).

- For polygonal sources:
  - surface integral \( \rightarrow \) line integral \( \rightarrow \) \( \Psi_k \) evaluated at vertices
Computing $\mathcal{R}(k)$ reliably from sensor measurements?

Recall that,

$$\mathcal{R}(k) = \sum_{n=1}^{N} \sum_{l=0}^{L} w_{n,l} \phi_{n,l}$$

Thus computing $\mathcal{R}(k)$ is equivalent to finding the weights $w_{n,l}$. These weights may be found:

1. Using Green’s second identity
   - For 2D diffusion field.
2. Formulating and solving a linear system (explicitly)
   - Inversion of large matrices.
   - Conditioning and stability considerations.
3. Results from non-uniform/universal sampling theory?
   - Stable iterative/non-iterative algorithms.
Explicit computation of weights \( \{ w_{n,l} \}_n, l \)

We desire \( \{ w_{n,l} \}_n, l \), so that \( \sum_{n=1}^{N} \sum_{l=0}^{L} w_{n,l} g(x_n - x, t_l - t) = \psi_k(x) \Gamma(t) \), where \( g, \psi_k \) and \( \Gamma \) are known.

For e.g. the 2D heat problem \( g(x, t) = \frac{1}{4\pi t} e^{-\frac{\|x\|^2}{4\mu t}} H(t) \), and we may choose, \( \psi_k(x) = e^{-k(x_1 + jx_2)} \) and \( \Gamma(t) = e^{-jt/T} \).

Can formulate a linear system as follows:

\[
\begin{bmatrix}
g(x_1 - x'_1, t_l - t_j) & \cdots & g(x_N - x'_1, t_l - t_j) \\
\vdots & \ddots & \vdots \\
g(x_1 - x'_I, t_l - t_j) & \cdots & g(x_N - x'_I, t_l - t_j)
\end{bmatrix}
\begin{bmatrix}
w_{1,l} \\
\vdots \\
w_{N,l}
\end{bmatrix}
= \begin{bmatrix}
\psi_k(x'_1) \Gamma(t_j) \\
\vdots \\
\psi_k(x'_I) \Gamma(t_j)
\end{bmatrix}
\]

\[
G_{l,j} w_j = p_j
\]

\[
\Rightarrow \begin{bmatrix}
G_{0,1} & \cdots & G_{0,J} \\
\vdots & \ddots & \vdots \\
G_{L,1} & \cdots & G_{L,J}
\end{bmatrix}^T
\begin{bmatrix}
p_1 \\
\vdots \\
p_J
\end{bmatrix}
= \begin{bmatrix}
\psi_k(x'_1) \Gamma(t_1) \\
\vdots \\
\psi_k(x'_J) \Gamma(t_J)
\end{bmatrix}.
\]

\[
G w = p
\]

Solve \( G w = p \), where \( G \in \mathbb{R}^{N(L+1) \times IJ} \), \( w \in \mathbb{R}^{N(L+1)} \) and \( p \in \mathbb{R}^{IJ} \).
Implicit computation of weights \( \{W_{n,l}\}_{n,l} \)

1. **Green’s second identity:** Let \( u(x, t) \) and \( \Psi_k(x) \) be scalar functions in \( C^2 \), over \( \Omega \in \mathbb{R}^2 \), then:

\[
\oint_{\partial \Omega} \left( \Psi_k \nabla u - u \nabla \Psi_k \right) \cdot \hat{n}_{\partial \Omega} \, dS = \int_{\Omega} \left( \Psi_k \nabla^2 u - u \nabla^2 \Psi_k \right) \, dV,
\]

where \( \hat{n}_{\partial \Omega} \) is the outward pointing unit normal to the boundary \( \partial \Omega \).

2. Substitute (inhomogenous) PDE and choose \( \Psi_k \) to satisfy \( \frac{\partial \Psi_k}{\partial t} + \mu \nabla^2 \Psi_k = 0 \), thus:

\[
\int_{\Omega} \frac{\partial}{\partial t} (u \Psi_k) \, dV - \mu \oint_{\partial \Omega} \left( \Psi_k \nabla u - u \nabla \Psi_k \right) \cdot \hat{n}_{\partial \Omega} \, dS = \int_{\Omega} \Psi_k f \, dV.
\]

3. Multiply through by \( \Gamma(t) \) and integrate over \( t = [0, T] \):

\[
\int_{0}^{T} \Gamma \int_{\Omega} \Psi_k \frac{\partial u}{\partial t} + u \frac{\partial \Psi_k}{\partial t} \, dV - \mu \oint_{\partial \Omega} \left( \Psi_k \nabla u - u \nabla \Psi_k \right) \cdot \hat{n}_{\partial \Omega} \, dS \, dt = \int_{\Omega} \int_{0}^{T} \Psi_k f \, dt \, dV
\]

\[
= \mathcal{R}(k)
\]
Implicit computation of weights \( \{W_{n,l}\}_{n,l} \)

From:

\[
\int_{0}^{T} \int_{\Omega} \psi_{k} \frac{\partial u}{\partial t} + u \frac{\partial \psi_{k}}{\partial t} \, dV - \mu \int_{\partial \Omega} \left( \psi_{k} \nabla u - u \nabla \psi_{k} \right) \cdot \hat{n} \partial_{\Omega} \, dS \, dt = \int_{\Omega} \int_{0}^{T} \psi_{k} \Gamma f \, dt \, dV
\]

\[= \mathcal{R}(k)\]

\[\Rightarrow \mathcal{R}(k) = \langle f(x, t), \psi_{k}(x)\Gamma(t) \rangle\]

As such we can obtain \( \{\mathcal{R}(k)\} \) by approximating the integrals from the spatiotemporal samples using standard quadrature schemes.

- Mesh required.
- Integral simply a linear combination of field samples.
- Distributed computation (consensus-based estimation).
1 **Introduction**
- Motivation
- Problem Formulation

2 **Source Reconstruction Framework**
- Point Sources
- Line Source
- Computing $\mathcal{R}(k)$

3 **Simulation Results**

4 **Conclusion**
Distributed estimation for $M = 1$ source using 45 sensors, field is sampled for $T_{end} = 10s$ at $\frac{1}{\Delta t} = 1Hz$. $K = 1$. 
Synthetic data: Line Diffusion Source

\[ N = 45 \] arbitrarily placed sensors, field sampled at 10Hz for \( T = 10s \) with measurement SNR= 20dB. \( K = 6 \) and \( R = 5 \).
Synthetic data: Triangular Diffusion Source

$N = 90$ arbitrarily placed sensors, field sampled at $10Hz$ for $T = 10s$ with measurement $\text{SNR}= 35\text{dB}$. $K = 6$ and $R = 5$. 
Simulation Results: Real Diffusion Data

(a) Thermal distribution (immediately after activation) and location estimates.

(b) Real field (left) and its reconstruction (right) at $t = 7.1s$.

(c) Real field (left) and its reconstruction (right) at $t = 8.2s$. 

0 0.02 0.04 0.06 0.08 0.1
0 0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
x1
x2
True Thermal Field (t=7.1s)
Reconstructed Thermal Field (t=7.1s)

0 0.02 0.04 0.06 0.08 0.1
0 0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
x1
x2
True Thermal Field (t=8.2s)
Reconstructed Thermal Field (t=8.2s)
**Simulation Results: Laplace - Synthetic data**

**Figure 1:** Single point source recovery in 3D using samples obtained by $N = 57$ sensors with $K_1 = K_2 = 1$ for spatial sensing function family. Results for 20 independent trials are given.
Reconstructing non-localized sources: line and (convex) polygons.

- Compute generalized measurements.
- Use tools from complex analysis to modify $\mathcal{R}(k)$.
- Recover endpoints (vertices) of line (polygonal) source.
Further Extensions

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   - Compute generalized measurements.
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2. Further extensions
   - Reconstructing localized sources in bounded regions (rooms).
   - 3D source recovery.
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   - Compute generalized measurements.
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2. Further extensions
   - Reconstructing localized sources in bounded regions (rooms).
   - 3D source recovery.

3. Generalisation Possible?
   - Same principle can be generalized to PDE-driven fields: wave, Poisson etc.
   - How to compute the field analysis coefficients $\{w_{n,l}\}$?
   - Turn to FRI theory: exponential reproduction.
Thank You.