# Phaseless Super-Resolution Using Masks

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ICASSP 2016
Shanghai, China, March 23, 2016

# Phase Retrieval

- In several measurement systems, the magnitude-square of the Fourier transform is the measurable quantity

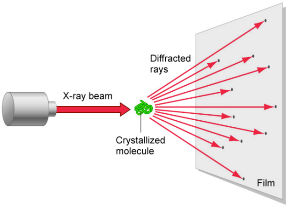


Figure: X-ray imaging

- Phase retrieval: Recovering a signal from its Fourier magnitude

- Classic algorithms use alternating projections; convex programs proposed recently [Candes'11, Eldar'11, Jaganathan'12]

# Super-Resolution

- It is very difficult to obtain high-frequency measurements in general, due to physical limitations (e.g., diffraction limit)
- Super-resolution: Recovering a sparse signal from low-frequency Fourier measurements
- Classic algorithms like MUSIC, ESPRIT; convex program proposed recently [Fernandez-Granda'14, Recht'13]

# Phaseless Super-Resolution

- Recovering a signal from its low-frequency Fourier magnitude measurements
- Combination of phase retrieval and super-resolution

# Phaseless Super-Resolution

- Let $\mathbf{x} = (x[0], x[1], \ldots, x[N-1])^T$ be a complex-valued signal of sparsity $k$ (where $k \ll N$)
- Phaseless super-resolution:

$$\begin{aligned} &\text{find} && \mathbf{x} \\ &\text{subject to} && z[m] = \left|\langle \mathbf{f}_m, \mathbf{x} \rangle\right|^2 \quad \text{for} \quad 0 \leq m \leq K-1 \end{aligned} \tag{1}$$

- $\mathbf{z} = (z[0], z[1], \ldots, z[K-1])^T$ is the $K \times 1$ observed vector corresponding to the $K$ low-frequency Fourier magnitude-square measurements (where $K \ll N$)

# Phaseless Super-Resolution

- Question: Is phaseless super-resolution well-posed? No...
- In fact, phase retrieval, even with high-frequency magnitude measurements, is not well-posed
  - Time shift
  - Conjugate flip (time-reversal for real signals)
  - Global phase (global sign for real signals)
  - In 1D, many non-trivial ambiguities exist
- We use "masks" to obtain additional information

# Masking a signal

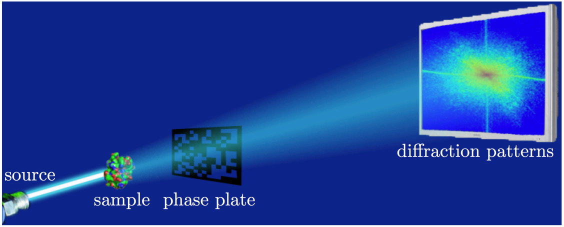


Figure: X-ray imaging (picture courtesy: [Candes'13])

- Mathematically, multiply the signal by a diagonal matrix **D**

$$\mathbf{Dx} = \begin{bmatrix} d[0] & 0 & \dots & 0 \\ 0 & d[1] & \dots & 0 \\ & & & \\ 0 & 0 & \dots & d[N-1] \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ \\ x[N-1] \end{bmatrix} = \begin{bmatrix} d[0]x[0] \\ d[1]x[1] \\ \\ d[N-1]x[N-1] \end{bmatrix}$$

# Phaseless Super-Resolution using Masks

- Measurements using $R$ masks, defined by diagonal matrices $\mathbf{D}_r$, for $1 \leq r \leq R$

$$
\begin{aligned}
&\text{find} && \mathbf{x} \\
&\text{subject to} && Z[m,r] = \left|\langle \mathbf{f}_m, \mathbf{D}_r \mathbf{x} \rangle\right|^2 \\
&\quad \text{for} \quad 0 \leq m \leq K-1 \quad \text{and} \quad 1 \leq r \leq R
\end{aligned}
\tag{2}
$$

Natural questions:

- How many, and what masks to choose? (easy to implement in practice)
- How to reconstruct the signal? (efficient and robust algorithm)

# Convex Program

- Quadratic-constrained problem $\Rightarrow$ lifting/ semidefinite relaxation

(i) Use the transformation $\mathbf{X} = \mathbf{x}\mathbf{x}^\star$ to obtain a problem of recovering a rank-one matrix with affine constraints

(ii) Relax the rank-one constraint

$$\begin{aligned} \text{minimize} \quad & \|\mathbf{X}\|_1 \\ \text{subject to} \quad & Z[m,r] = \operatorname{trace}(\ \mathbf{D}_r^\star \mathbf{f}_m \mathbf{f}_m^\star \mathbf{D}_r \mathbf{X}\ ) \\ \text{for} \quad 0 \le m \le K-1 \quad & \text{and} \quad 1 \le r \le R, \\ & \mathbf{X} \succcurlyeq 0. \end{aligned} \tag{3}$$

# Choice of Masks

- Three masks $\mathbf{D}_0, \mathbf{D}_1, \mathbf{D}_2$, defined as follows:

$$\mathbf{D}_0 = \mathbf{I}$$
$$\mathbf{D}_1 = \mathbf{I} + \mathrm{Diag}(\mathbf{f}_1)$$
$$\mathbf{D}_2 = \mathbf{I} - j\mathrm{Diag}(\mathbf{f}_1)$$

- Practical motivation: easy to implement (e.g., optics)

# Theoretical motivation #1

- Use $\mathbf{D}_0$ to infer $\left|y[m]\right|^2$ (Fourier magnitude-square)
- Use $\mathbf{D}_1$ to infer

$$\left|\mathbf{f}_m^{\star}(\mathbf{I} + \mathrm{Diag}(\mathbf{f}_1))\mathbf{x}\right|^2 = \left|\mathbf{f}_m^{\star}\mathbf{x} + \mathbf{f}_{m-1}^{\star}\mathbf{x}\right|^2 = \left|y[m] + y[m-1]\right|^2$$

- Similarly, $\left|y[m] - jy[m-1]\right|^2$ can be inferred from $\mathbf{D}_2$
- Phases of $y[m]$ can be established (up to a global factor)!

# Theoretical motivation #2

- Inspiration from recent work by Bahmani and Romberg on phase retrieval from random measurements

$$b[m] = \mathrm{trace}(\ \mathbf{c}_m \mathbf{c}_m^\star \mathbf{A} \mathbf{X} \mathbf{A}^\star\ )$$

- If $\mathbf{A}$ is any compressed-sensing type matrix and $\mathbf{c}_m$ is a Gaussian random vector, then provable recovery using orderwise optimal measurements via two convex programs:
  - First recover low rank matrix $\mathbf{AXA}^\star$ from $\mathbf{b}$
  - Then recover sparse matrix $\mathbf{X}$ from $\mathbf{AXA}^\star$

# Theoretical motivation #2

- $\mathrm{trace}(\ \mathbf{f}_m\mathbf{f}_m^\star\mathbf{D}_r\mathbf{X}\mathbf{D}_r^\star\ )$ looks similar to $\mathrm{trace}(\ \mathbf{c}_m\mathbf{c}_m^\star\mathbf{A}\mathbf{X}\mathbf{A}^\star\ )$, but none of the conditions are satisfied

- For the chosen masks, after some algebra, we show equivalence to measurements of the form $\mathrm{trace}(\ \mathbf{s}_{mr}\mathbf{s}_{mr}^\star\mathbf{F}_K\mathbf{X}\mathbf{F}_K^\star\ )$
  - $\mathbf{s}_{mr}\mathbf{s}_{mr}^\star$ such that knowledge of the diagonal and the first off-diagonal values of $\mathbf{F}_K\mathbf{X}\mathbf{F}_K^\star$ available, enough to do rank-one reconstruction via convex program
  - $\mathbf{F}_K$ is not compressed-sensing type, but can still be used to do sparse recovery (super-resolution)

# Main Result

**Theorem**

*The convex program succeeds in recovering $\mathbf{x}_0\mathbf{x}_0^\star$ uniquely, when measurements obtained using $\mathbf{D}_0, \mathbf{D}_1, \mathbf{D}_2$ are used, if*

1. $K \geq \frac{2N}{\Delta(\mathbf{x}_0)}$
2. *The first $K$ values of the $N$-point DFT of $\mathbf{x}_0$ are non-zero.*

Remarks:

- Generalizes super-resolution results of [Fernandez-Granda'14, Recht'13] to phaseless super-resolution using masks
- $\Delta(\mathbf{x}_0)$: minimum separation between two non-zero locations in $\mathbf{x}_0$

# Numerical Simulations

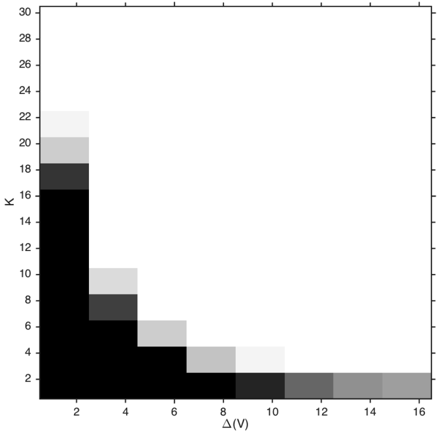


Figure: Probability of successful reconstruction of convex program for $N = 32$ and various choices of $K$ and $\Delta(\mathbf{x}_0)$, using masks $\{\mathbf{D}_0, \mathbf{D}_1, \mathbf{D}_2\}$.

# Conclusions and Future Directions

- Considered the problem of phaseless super-resolution, suggested using masks to make the problem well-posed
- Three masks are enough for provable convex-programming based recovery, more such masks would improve stability constant
- Further directions: generalize to other class of masks

Thank you!