Binary Rate Distortion With Side Information: The Asymmetric Correlation Channel Case

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Overview

- Motivation: DSC principles
  - Asymmetric Correlation Models in DVC
- Coding of Binary Uniform Sources with SI
  - SI Available at Both Encoder and Decoder
  - WZ Coding: SI Available only at the Decoder
- Rate Distortion Performance
  - Minimum/maximum rate to encode
  - Minimum/maximum rate loss

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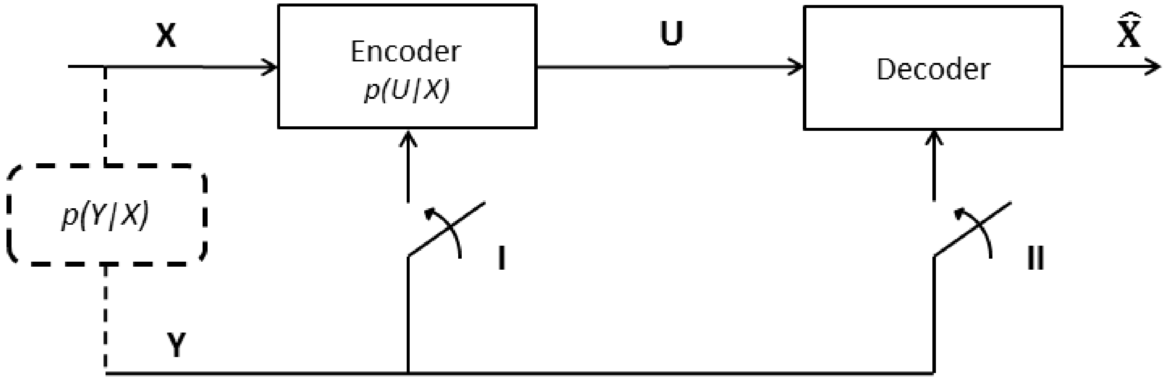
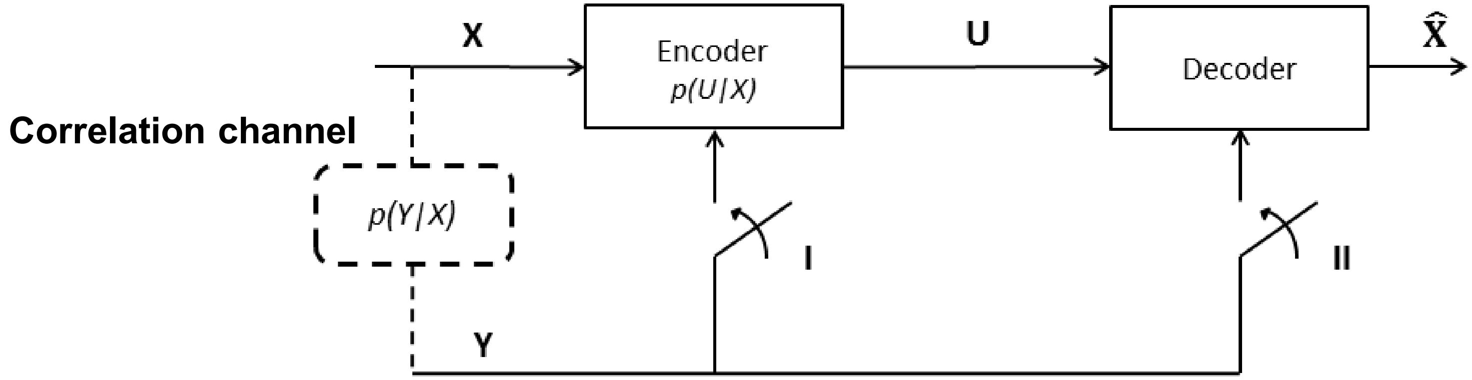
Lossy Source Coding with Side Information



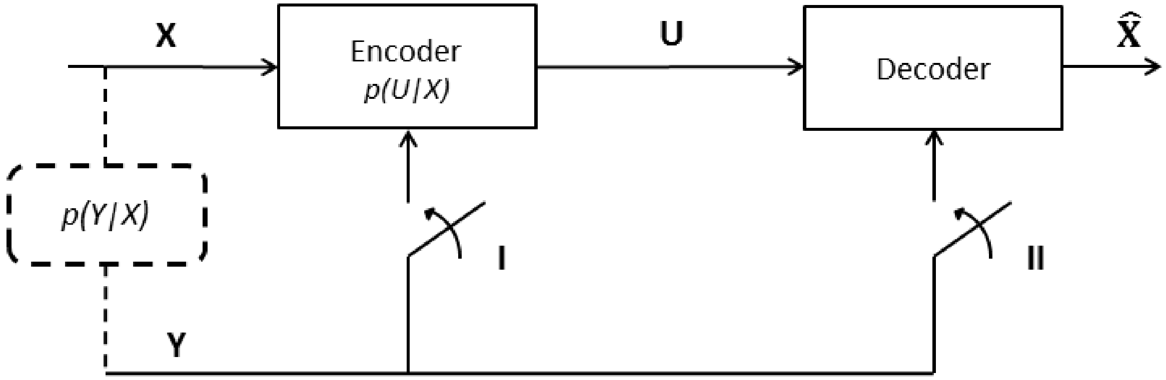
- Original source: ***X***
- Encoded source: ***U***
- Reconstructed source: $\widehat{X}$
- Side information: source ***Y***
  - correlated with the input
  - may be available at the encoder and/or decoder

Correlation Channel



- The side information can be seen as a noisy version of the source
- **Correlation channel**: virtual channel that describes the correlation between source and side information

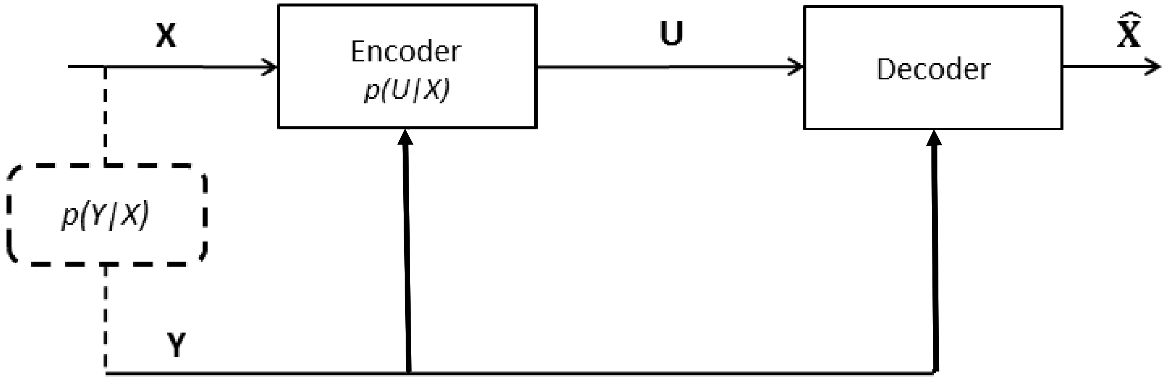
Rate Distortion Function



R(d) function: **minimum encoding rate** in the presence of side information s.t. **E[d(X, $\hat{X}$)]<d**

- Non-negative
- Non-increasing
- Convex

Predictive Coding

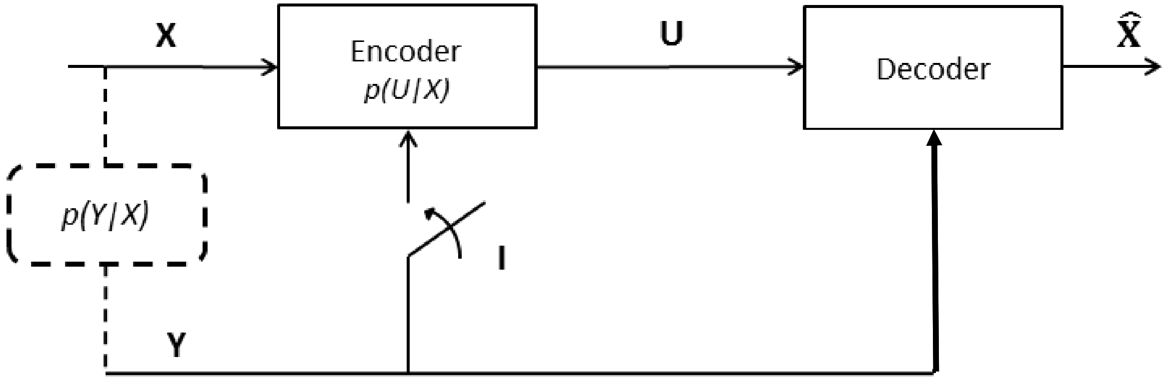


- Side information available at both encoder and decoder: joint encoding of the two sources
- General expression for R(d) bound known [Berger71]:

$$R_{X|Y}(d) = \inf_{p(\hat{x}|x,y):\ E\left[d(X,\hat{X})\right]\leq d} I(X;\hat{X}|Y)$$

Berger, T. "Rate-Distortion Theory." *Encyclopedia of Telecommunications* (1971).

Wyner-Ziv Coding



- Side information available only at the decoder
- General expression for R(d) bound [Wyner,Ziv76]:

$$R_{WZ}(d) = \inf_{p(u|x)p(\hat{x}|u,y):\ E[d(X,\hat{X})] \leq d} I(X;U|Y)$$

Wyner, A.D., Ziv., J. "The rate-distortion function for source coding with side information at the decoder." *IEEE Transactions on Information Theory* (1976)

Rate Loss

- In general, WZ coding comes with a rate penalty when compared to predictive coding [Zamir96]:
  - Less than 0.5 bps for continuous sources and MSE distortion
  - Less than 0.22 bps for binary sources and Hamming distortion

- No rate loss cases:
  - Gaussian source with Gaussian correlation and MSE distortion [Wyner,Ziv76]
  - Binary uniform source with the erasure correlation channel and Hamming distortion [Perron09]

Zamir, R. "The rate loss in the Wyner-Ziv problem." *IEEE Transactions on Information Theory* (1996)

Perron, E., Diggavi, S., and Telatar, E. "Lossy source coding with Gaussian or erased side-information." *ISIT (2009).*

Symmetric vs. Asymmetric Correlation

- In literature, correlation models are usually assumed symmetric
- Recent results show increased interest in asymmetric models:
  - Rate distortion bound for predictive coding in the case of the Z-channel correlation [Steinberg07]
  - LDPCA codes for practical distributed source coding [Varodayan06]
  - In DVC, asymmetric correlation models outperform symmetric correlations [Deligiannis12]

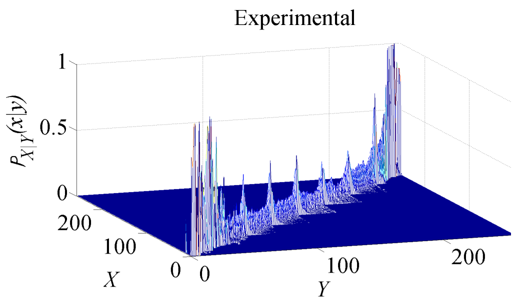
Steinberg, Y. "Coding and common reconstruction." *IEEE Transactions on Information Theory* (2009)

Varodayan, D., Aaron, A. and Girod, B. “Rate-adaptive codes for distributed source coding,” *Signal Processing (*2006).

Deligiannis, N., et al. "Side-information-dependent correlation channel estimation in hash-based distributed video coding." *IEEE Transactions on Image Processing* (2012)

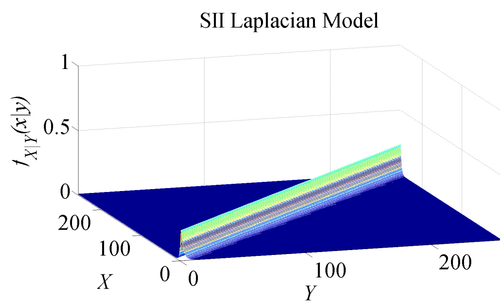
Symmetric vs. Asymmetric Correlation

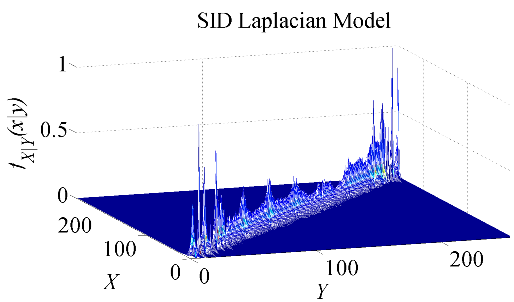
For Laplacian correlations, asymmetric correlation models outperform symmetric correlations [Deligiannis12]



$$SII\ model\colon\ f_{X|Y}(x|y) = \frac{1}{\sigma\sqrt{2}} e^{-\frac{\sqrt{2}|x-y|}{\sigma}}$$

$$SID\ model\colon\ f_{X|Y}(x|y) = \frac{1}{\sigma(y)\sqrt{2}} e^{-\frac{\sqrt{2}|x-y|}{\sigma(y)}}$$





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Binary Distributed Source Coding

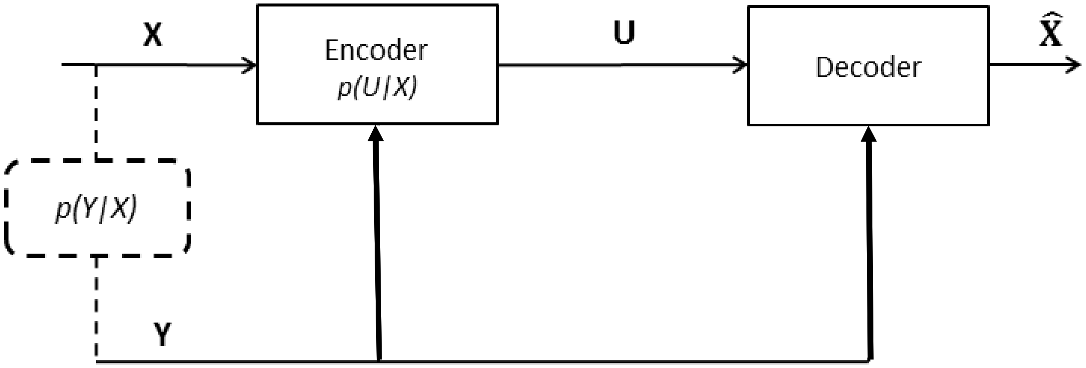
Our DSC problem is defined by:

- The source X is **binary uniform**, i.e., *Bernoulli (0.5)*
- The correlation between the source X and the side information Y is given by a **binary asymmetric** channel
- The distortion metric is the **Hamming distance**, i.e., $d(X, \hat{X}) = X \oplus \hat{X}$

Question: what is the **R(d) function**?

SI Available at Encoder and Decoder

$p(X|Y) = \begin{pmatrix} 1-a* & a* \\ b* & 1-b* \end{pmatrix}$



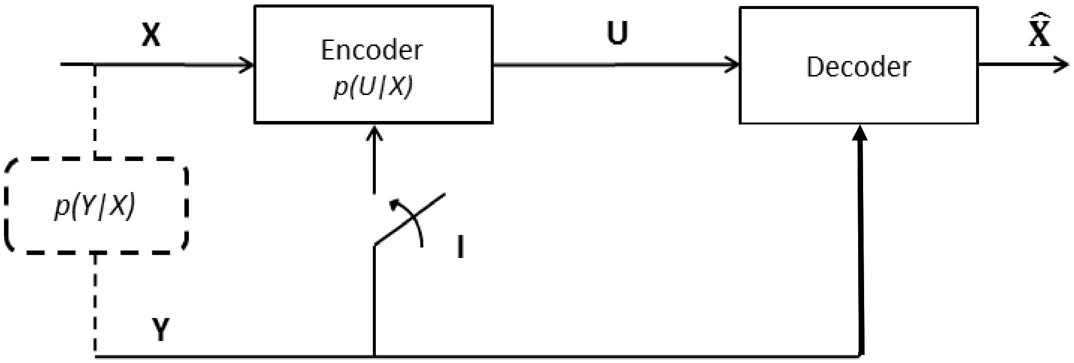
$$R_{X|Y}(d) = \inf_{p(\hat{x}|x,y):\ E\left[d(X,\hat{X})\right] \le d} I(X;\hat{X}|Y),$$

$$R_{X|Y}(d) = \begin{cases} p(Y=0)\cdot\left[H(a^*) - H(d)\right] + \\ \qquad p(Y=1)\cdot\left[H(b^*) - H(d)\right], \quad \text{if } d \le b^* \\ p(Y=0)\cdot\left[H(a^*) - H\left(\frac{d-\frac{a}{2}}{p(Y=0)}\right)\right], \\ \qquad \text{if } b^* \le d \le D_{max} \\ 0, \qquad \text{if } d \ge D_{max} \end{cases}$$

SI Available only at Decoder

$$p(U|X) = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}$$

$$p(Y|X) = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}$$



- The problem does not have an analytical solution
- Express both rate and distortion as functions of *(p,q)*
- Numerically find minimum achievable rate values

SI Available only at Decoder

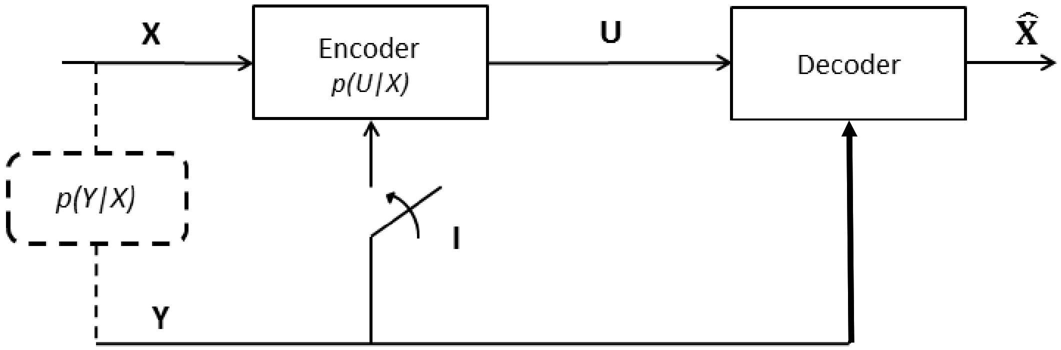
- General solution:

$$R_{WZ}(d) = \inf_{p(u|x)p(\hat{x}|u,y):\ E[d(X,\hat{X})]\leq d} I(X;U|Y),$$

- Binary WZ solution:

$$\begin{aligned} I(X;U|Y) \triangleq\ & R^*_{WZ}(p,q) = H(U|Y) - H(U|X) \\ &= \frac{(1-a+b)}{2} \cdot H\left(\frac{(1-a)(1-p)+bq}{1-a+b}\right) \\ &+ \frac{(a+1-b)}{2} \cdot H\left(\frac{a(1-p)+(1-b)q}{a+1-b}\right) \\ &\qquad - \frac{1}{2} \cdot \left[H(p) + H(q)\right]. \end{aligned}$$

SI Available only at Decoder



Best choice at decoder: $\hat{x} = f(u, y) = \underset{x}{\operatorname{argmax}}\ p(x|u, y)$

Solution: $$\mathcal{D}(p,q) = (bq + \min((1-a)p, b(1-q)) + \min(a(1-p), (1-b)q) + ap)/2.$$

- if $0 \le d < \frac{a}{2\cdot(1-b)}$, $\hat{X} = U$ *and* $d = \frac{p+q}{2}$
- if $\frac{a}{2\cdot(1-b)} \le d \le \frac{b}{2\cdot(1-a)}$, $\hat{X} = U \cup Y$ *and* $d = \frac{bq+(1-a)p+a}{2}$
- if $\frac{b}{2\cdot(1-a)} \le d \le \frac{a+b}{2}$, $\hat{X} = U \cap Y$ *and* $d = \frac{(1-b)q+ap+b}{2}$

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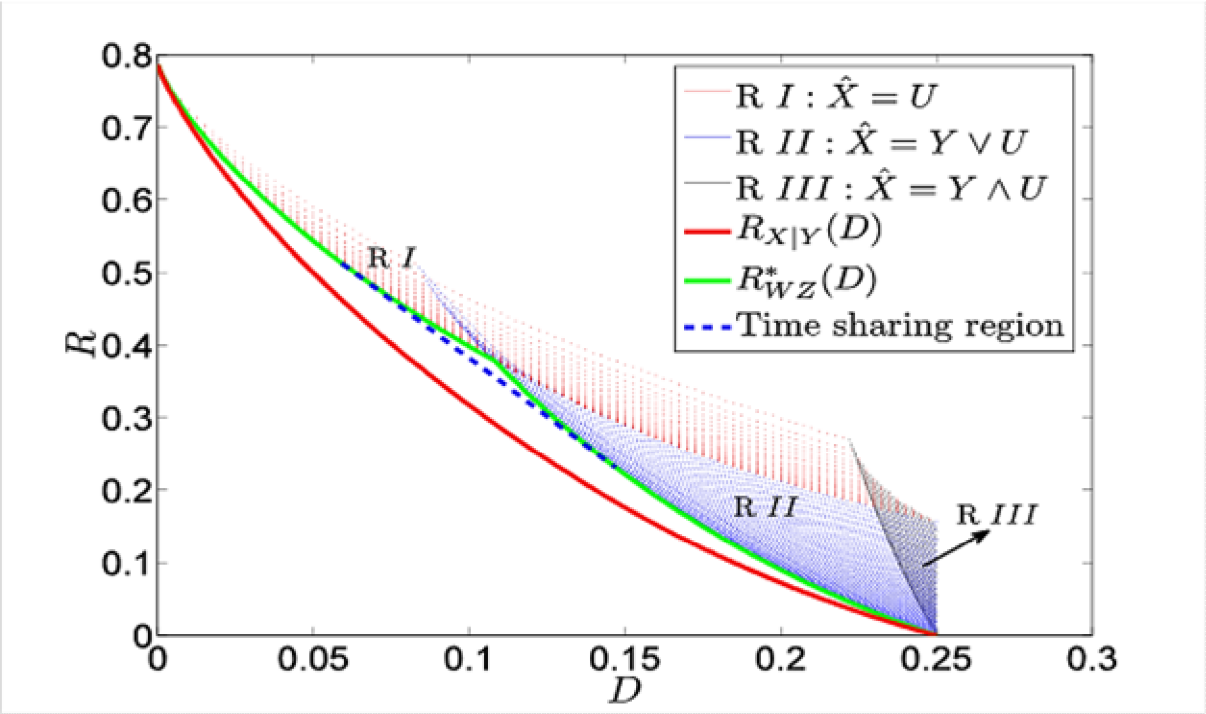
R(D) function for BAC correlation

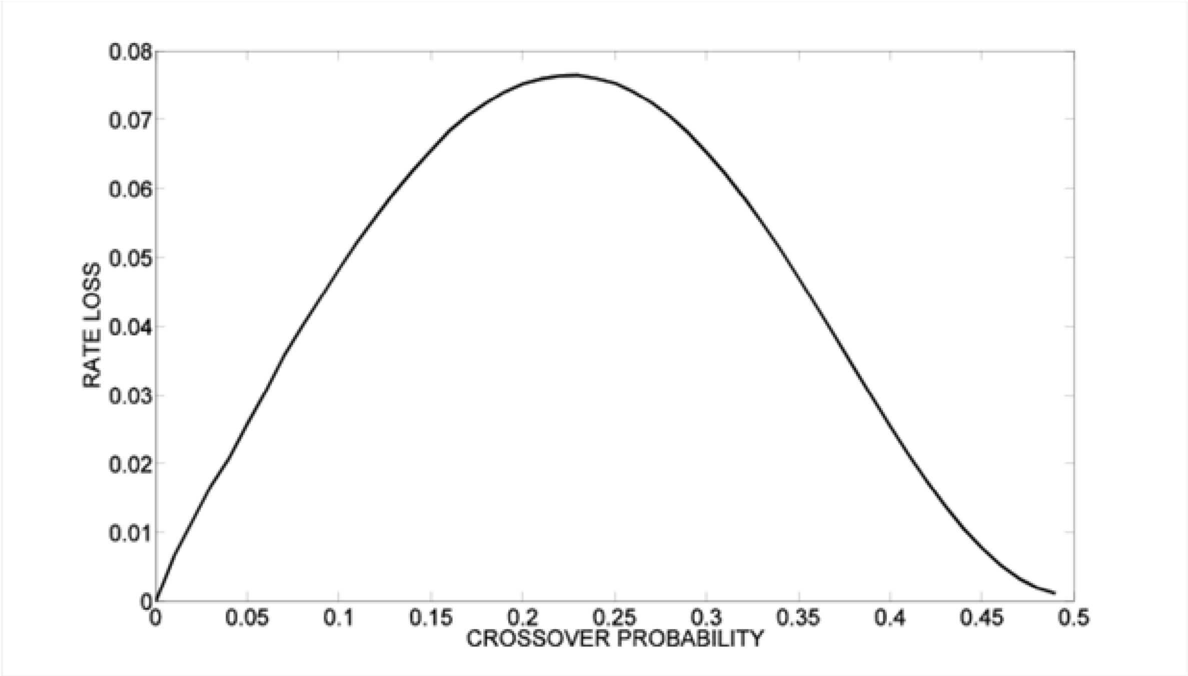
- Possible reconstruction functions:

$$\widehat{\boldsymbol{X}} = \boldsymbol{U},\ \widehat{\boldsymbol{X}} = \boldsymbol{U} \cup \boldsymbol{Y} \text{ and } \widehat{\boldsymbol{X}} = \boldsymbol{U} \cap \boldsymbol{Y}$$



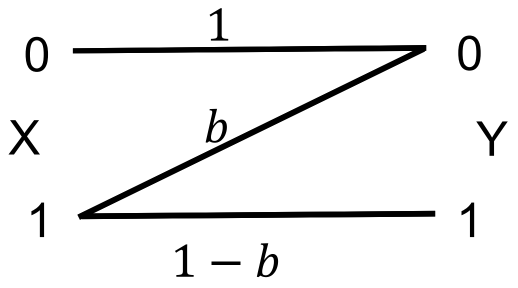
Maximum Rate Loss

The maximum rate loss: symmetric correlation channels

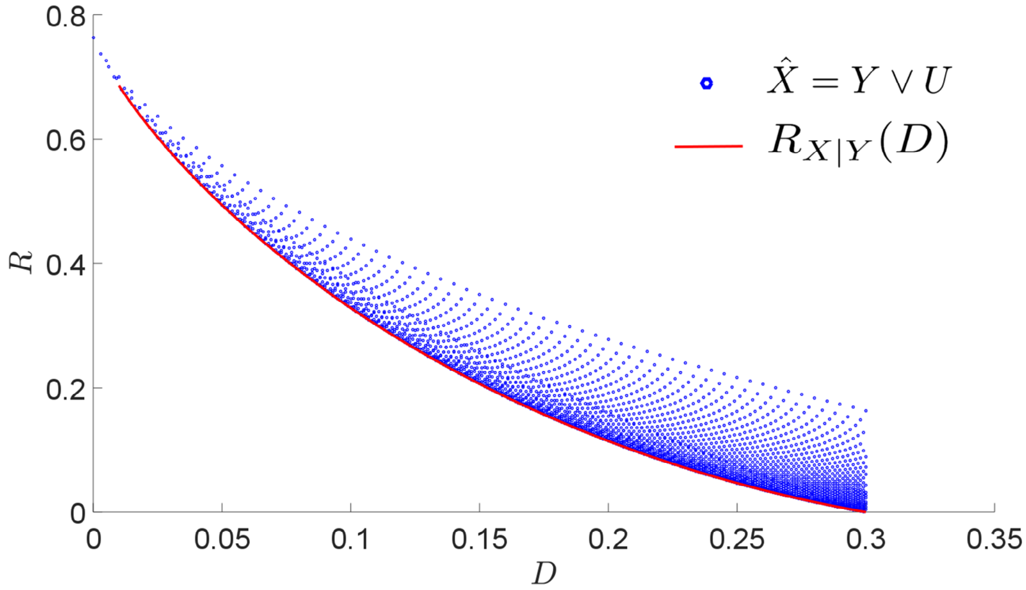


The maximum value of the rate-loss:
$\Delta R = 0.0765\ bps\ for\ a = b = 0.227$

Minimum Rate Loss



The Z-channel correlation, has no rate loss [Deligiannis14]



Deligiannis, N., et al. "The No-Rate-Loss Property of Wyner-Ziv Coding in the Z-Channel Correlation Case." *IEEE Communications Letters* (2014)

Conclusions

For WZ coding of binary uniform sources, given a specific distortion level:

- Highest rate to encode and highest rate loss – symmetric correlations
- Lowest rate to encode and no rate loss – Z-channel

Bound on rate loss:

- 0.22 bps [Zamir96] is not tight
- 0.0765 bps for uniform sources and symmetric correlation

Thank you for your attention!