Unlimited Sampling of Sparse Signals

Recovery of Low-pass Filtered Spikes from Modulo-Folded Samples

Ayush Bhandari† Felix Krahmer‡ Ramesh Raskar†

†Massachusetts Institute of Technology • ‡ Technical University of Munich

Contact: ayush@mit.edu

Poster presented by: Chris Gilliam and Beth Jelfs

Summary

Key Takeaways

- Shannon's Sampling Theorem is fundamental to the fields of signal processing, communications and information theory.
- A practical problem in realizing this theorem is that analog-to-digital converters (ADCs) are finite dynamic range devices while the sampling theorem makes no assumptions on the dynamic range.
- Recently, we introduced the concept of **Unlimited Sampling** [1].

  This unique approach circumvents the clipping or saturation problem in conventional analog-to-digital converters (ADCs).
- We do so by considering a radically different ADC architecture, the **Self-reset ADC** [2], which computes modulo or folded samples.
- The **Unlimited Sampling Theorem** proves that a bandlimited signal can be perfectly recovered from modulo samples.

  The sampling rate is purely dependent on the signal bandwidth and is independent of the ADC threshold.
- By capitalizing on the Unlimited Sampling Theorem, in this work, we study the problem of recovery of a continuous-time sparse signal from low-pass filtered, modulo samples.

Setup for Sparse Signals

We are interested in recovery of low-pass filtered spikes from modulo/folded samples. For this purpose, we will be working with the model:

$$g(t) = \sum_{k=0}^{K-1} c_k \psi(t - t_k) \equiv (s_K * \psi)(t) \qquad (2)$$

where ψ is a bandlimited function and s_K is a continuous time, K–sparse, τ-periodic signal,

$$s_K(t) = \sum_{m\in\mathbb{Z}} \sum_{k=0}^{K-1} c_k \delta(t - t_k - m\tau), \quad t_{k+1} > t_k. \qquad (3)$$

Problem Formulation

Let ψ be a given π–bandlimited, low-pass filter and s_K be the sparse signal defined in (3). Furthermore, let $\{y_n\}_{n=0}^{N-1}$ be the *modulo samples* of g defined in (2). What are conditions for perfect recovery of s_K from $\{y_n\}_{n=0}^{N-1}$?

Our basic strategy for recovering s_K from y_n can be summarized as,

$$y_n \xrightarrow{\text{Unfolding}} g_n \xrightarrow{\text{Sparse Recovery}} s_K(t).$$

This approach relies on extracting unfolded, contiguous sample sequence g_n of size $2K+1$ from which $s_K(t)$ is estimated using high-resolution frequency estimation [3].

Unlimited Sampling of Bandlimited Functions

- Let $T > 0$ be the sampling rate and $g(t)$ be a π-bandlimited function.
- In Unlimited Sampling framework, we sample g using non–linear principle:

  $$y_n = \mathrm{mod}_\lambda(g(nT)), \quad n \in \mathbb{Z}, \quad T > 0$$
- Such folded samples are acquired using a version of the Self-reset ADC [2].
- Even if $g(t) \gg \lambda$, $y_n \in [0, \lambda)$.

  This is not the case with conventional ADC which clips g.

Unlimited Sampling Theorem [1]

Let $g(t)$ be a π-bandlimited function and $\{y_n\}_n$ be the modulo samples of $g(t)$ with sampling rate T. Then, a sufficient condition for recovery of g from $\{y_n\}_n$ up to additive multiples of 2λ is,

$$0 < T\pi e \leq \frac{1}{2}. \qquad (1)$$

In [1], the Unlimited Sampling theorem is complemented with a constructed algorithm which is based on the principle of consistent reconstruction.

A Sparse Sampling Theorem

- Fundamentally different from the bandlimited case [1], for sparse signal recovery (3) , we will be working with finite number of samples.
- Of course we expect that N will be larger than $2K+1$ but the number of samples should still be finite.
- For this purpose, we prove the following **Local Reconstruction Theorem**.

Local Reconstruction Theorem

Let g be a π–bandlimited function with $\|g\|_\infty \leq \beta_g$ and $\{y_n\}_{n=0}^{N-1}$ be the modulo samples of $y(t)$ with sampling rate T. Then a sufficient condition for recovery of N' contiguous samples of g from the y_n (up to additive multiples of 2λ) is that

$$T \leq \frac{1}{2\pi e} \quad \text{and} \quad N \geq N' + 7\frac{\beta_g}{\lambda}. \qquad (4)$$

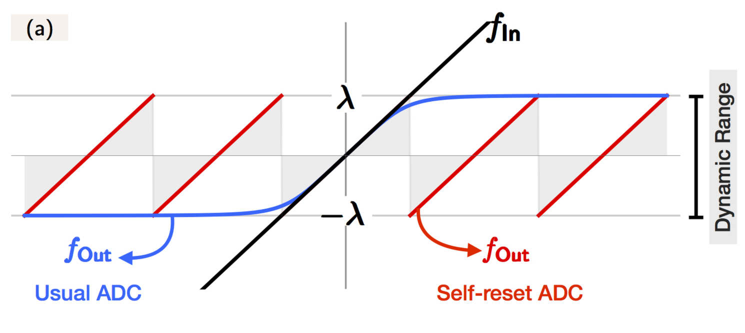
Noting that $\|g\|_\infty = \|s_K * \psi\|_\infty \leq \|\psi\|_\infty \|s_K\|$ (Young's Inequality) and by using $N' = 2K+1$ in the above theorem, we obtain the sufficiency condition for recovery of s_K from N modulo samples.

References

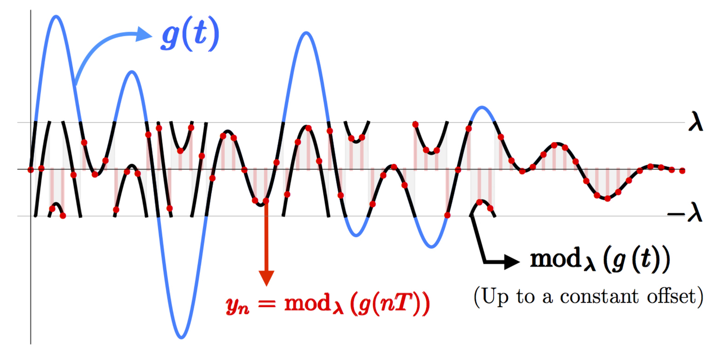
[1] A. Bhandari, F. Krahmer and R. Raskar, "On Unlimited Sampling," Proc. of SampTA, 2017.

[2] J. Rhee and Y. Joo, "Wide dynamic range CMOS image sensor with pixel level ADC," Electron. Lett., 2003.

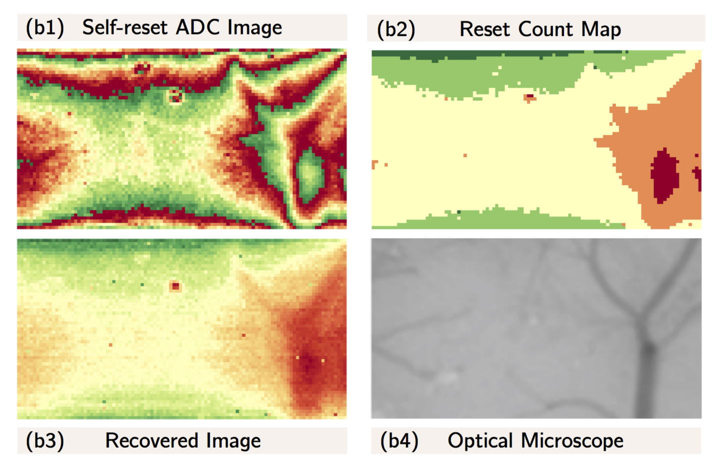
Unlimited Sampling in Action



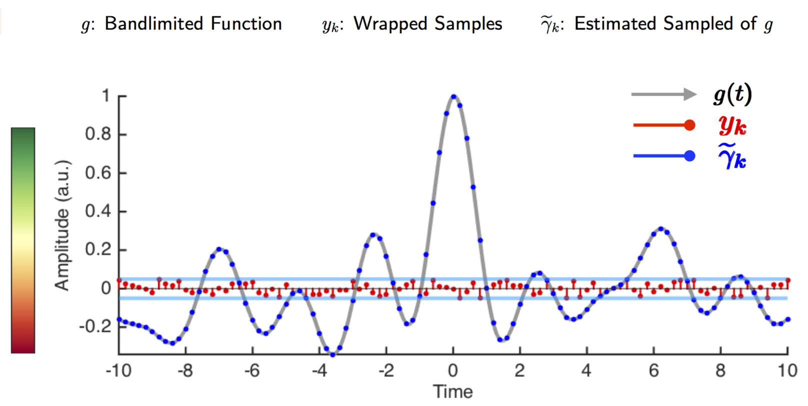
Usual ADC compared with self-reset ADC. (a) Whenever the input signal f_{In} voltage exceeds a certain threshold λ, the output signal f_{Out} in any conventional ADC saturates to λ and this results in clipping. In contrast, whenever $|f_{In}| > \lambda$, the self-reset ADC folds f_{In} such that f_{Out} is always in the range $[-\lambda, \lambda]$. In this way, the self-reset configuration circumvents clipping but introduces discontinuities.



Unlimited Sampling Architecture

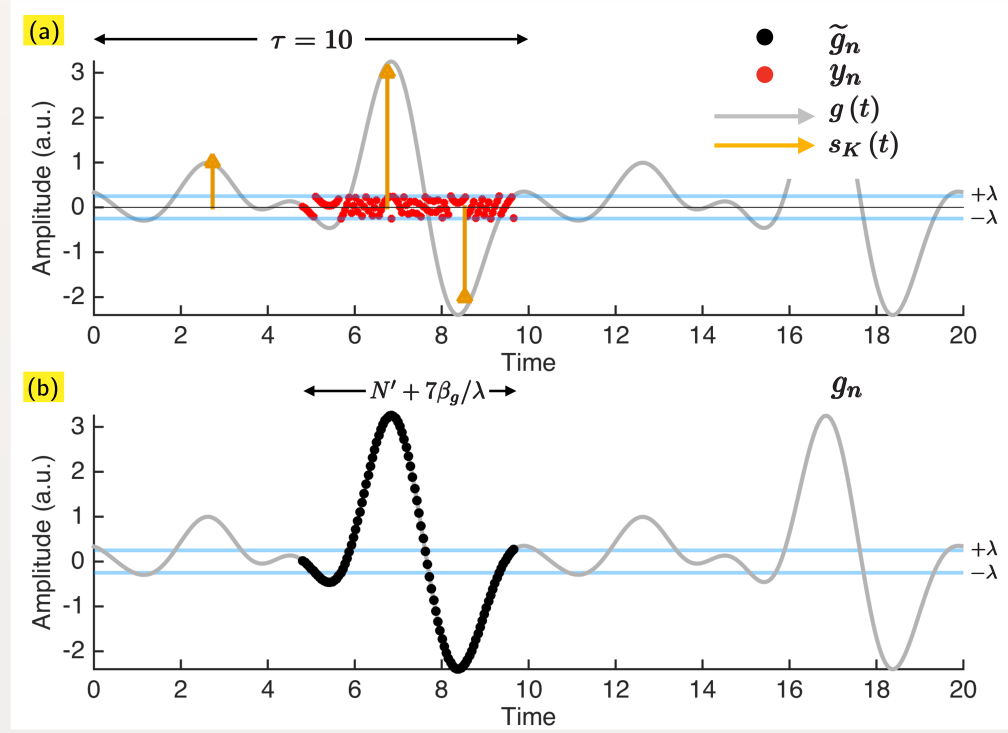


(b) Images obtained with prototype self-reset ADC. (b1) Image obtained with a self-reset ADC shows folded amplitudes. (b2) For each pixel, the "reset count map" shows the number of times the image amplitude has undergone folding. (b3) Unfolded image based on reset count map. (b4) Image obtained using an optical microscope.



Reconstruction from folded samples.
Maximum signal amplitude exceeds the threshold λ by a factor 20 times.

Local Reconstruction: Example



Sparse signal recovery via local reconstruction of modulo samples with $\beta_g = 3.2511$ and $\lambda = 0.25$. (a) We plot K–sparse signal $s_K(t)$ with $K=3$ and $\tau = 10$, the low-pass filtered signal $g = s_K * \psi$ where $\psi(t) = \mathrm{sinc}(t)$ as well as modulo samples y_n with $T = 0.0485$. (b) Using our algorithm, we estimate unfolded samples $\tilde{g}_n$ from $N = 99$ modulo samples of y_n. For this purpose $L = 3$. The reconstruction is observed to be exact (upto machine precision). Given $2K+1$ of $\tilde{g}_n$, the spikes are estimated using standard sparse recovery methods [3].

References Continued

[3] M. Vetterli, P. Marziliano, and T. Blu, "Sampling signals with finite rate of innovation," *IEEE Trans. Signal Process.*, 2002.