# Rethinking Super-resolution: The Bandwidth Selection Problem

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## Summary

**Key Takeaways**
- Super-resolution is the art of recovering spikes from their low-pass projections.
- Over the last decade specifically, several significant advancements linked with mathematical guarantees and recovery algorithms have been made.
- Most super-resolution algorithms rely on a two-step procedure: deconvolution followed by high-resolution frequency estimation.
- However, for this to work, exact bandwidth of low-pass filter must be known; an assumption that is central to the mathematical model of super-resolution.
- On the flip side, when it comes to practice, smoothness rather than bandlimit- edness is a much more applicable property.
- Since smooth pulses decay quickly, one may still capitalize on the existing super-resolution algorithms provided that the essential bandwidth is known.
- This problem has not been discussed in literature and is the theme of our work.
- We propose a bandwidth selection criterion which works by minimizing a proxy of estimation error that is dependent of bandwidth.

## Setup for Super-resolution of Sparse Signals

Given \( N \) time-domain, sampled measurements, \( y(nT) \) of the continuous signal

\[
y(t) = \sum_{k=0}^{K-1} c_k \phi(t - t_k),
\]

the super-resolution problem seeks to recover the \( 2K \) unknowns \( \{c_k, t_k\}^{K-1} \) assuming that: (A1) \( K \) and \( \Phi \) are known; and (A2) \( \phi \) is bandlimited (its Fourier transform is compactly supported). The notion of sparsity naturally finds its way in the super-resolution problem because \( y(t) = (\phi \ast s)(t) \) where \( s \) is a continuous-time, \( K \)-sparse signal

\[
s(t) = \sum_{k=0}^{K-1} s_k \delta(t - t_k), \quad t_k \in [0, \tau].
\]

## Recovery Strategy

Typical recovery procedure in the super-resolution problem exploits the structure of sparse signal. This is done in two steps:

1. **Deconvolution**
   - Here \( \hat{s}(nu) \) is estimated by using
     \[
     \hat{s}(nu) = \hat{\phi}(nu) = \sum_{k=0}^{K-1} c_k e^{-jmu(t_k)}, \quad nu \in [-\Omega, \Omega]
     \]
   - where \( \Omega \) is the bandwidth of \( \phi \).

2. **Parameter Estimation**
   - Once \( \hat{s}(nu) \) is computed, its parametric/sinusoidal form is then used for estimating unknown \( \{c_k, t_k\}^{K-1} \) using high resolution spectral estimation methods, fitting approaches or recently developed convex optimization based approaches.

## Super-resolution is Sensitive to Bandwidth

### Bandwidth Affects Reconstruction

Varying \( \Omega \) arbitrarily, leads to the following scenarios.

- When \( \Omega \) is such that \( N < 2K \), the parameter estimation by fitting will fail as the system is under-determined.
- Gradually increasing \( \Omega \) such that \( 2Kw_0 \leq \Omega \leq \Omega_0 \) leads to over-sampling and hence to performance enhancement of the spectral estimation methods.
- Understandably, when \( \Omega \) approaches the heuristically chosen \( \Omega_0 \), the deconvolution step becomes ill-posed.

## Towards a Bandwidth Selection Principle

Typically, in practice, \( \phi \) is smooth and the selection criterion for bandwidth parameter \( \Omega \) is unclear. Consider the case of noisy measurements \( m(t) = y(t) + \epsilon(t) \) where \( \epsilon(t) \) is bounded noise. Dividing \( \tilde{m}(\omega) \) by \( \phi \) (i.e. deconvolving), we obtain

\[
\tilde{m}(\omega) / \phi(\omega) = \sum_{k=0}^{K-1} -c_k e^{-j\omega t_k} + \tilde{\epsilon}(\omega), \quad |\omega| \leq \Omega
\]

\[
|\tilde{\epsilon}(\omega)| \leq \eta \left(\min_{t\in[0,T]} |\phi(\omega)|\right)^{-1}.
\]

The bandwidth selection criterion is given by \( \Omega_{opt} = \arg \min_{\Omega} G(\Omega, \mathcal{D}) \).

In the above, \( G(\Omega, \mathcal{D}) \) upper-bounds a quantity linearized condition number \( \kappa(\Omega) \),

\[
\sup_{\mathcal{D}\in\mathcal{D},D(0,K-1)} \kappa(\Omega) \leq G(\Omega, \mathcal{D}), \quad \mathcal{D} = \{c_k, t_k\}^{K-1} \in \mathbb{R}^{2K}. \]

More precisely, \( \kappa(m) \) is the \( m \)-th norm of the matrix \( J^m \), where \( J \) is the Jacobian matrix representing \( \tilde{s}(nu) \), and \( (\cdot)^m \) is the Moore-Penrose pseudo-inverse.

Theorem: Suppose that \( \forall \mathcal{D} \in \mathcal{D} \subset \mathbb{R}^{2K} \), the amplitudes are bounded: \( 0 < A_1 \leq |c_k| \leq A_2 \), and the minimal distance \( M_0 = \min_{k \neq k'} |t_k - t_{k'}| \geq 2\Delta > 0 \) is also bounded, then exist constants \( C_1, C_2, \ldots \) depending on \( A_1, A_2, K \), such that the following bounds hold.

- **Well-separated Regime**
  If \( \Delta > C_1 / \Omega_0 \), then \( \kappa(\Omega) \leq C_2 / \Omega_0, \quad \ell = 1, 2, \ldots, 2K - 1 \).

- **Single Cluster Regime**
  If \( M_0 < 2\pi / \Omega_0 \), then \( \kappa(\Omega) \leq C_3 / (\Omega_0 \Delta^{2K-2}) \).

### Optimal Bandwidth Computation